Advanced Studies in Contemporary Mathematics 35 (2025), No. 1, pp. 39 - 48
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2025.35.1.39

On one application of the maximum principle to the model of economic growth

D. Dolgy

ABSTRACT. We consider the model of economic growth from the point of view of the optimal control problem. We formulate conditions of the existence of the optimal solution and, on the base of Pontryagin's maximum principle, find its structure.

2010 MATHEMATICS SABJECT CLASSIFICATION: 34K35, 47N70, 93C15.

KEYWORDS AND PHRASES. Optimal control problem, Pontryagin's maximum principle, model of economic growth.

1. Introduction

Optimal control theory began to take shape as a mathematical discipline in the 1950s. The motivation for its development were the actual problems of automatic control, satellite navigation, aircraft control, chemical engineering and a number of other engineering problems.

Optimal control is regarded as a modern branch of the classical calculus of variations, which is the branch of mathematics that emerged about three centuries ago at the junction of mechanics, mathematical analysis and the theory of differential equations. The calculus of variations studies problems of extreme in which it is necessary to find the maximum or the minimum of some numerical characteristic (functional) defined on the set of curves, surfaces, or other mathematical objects of a complex nature.

The development of the calculus of variations is associated with the names of some famous scientists: Bernoulli, Euler, Newton, Lagrange, Weierstrass, Hamilton and others. Optimal control problems differ from variation problems by the additional requirements imposed on sought solution, and these requirements are sometimes difficult and even impossible to fit applying for solving the methods of the calculus of variations. The need for practical methods resulted in further development of variation calculus, which ultimately led to the formation of the modern theory of optimal control. This theory, absorbed all previous achievements in the calculus of variations, and it was enriched with new results and new content. The central results of the theory – the Pontryagin's maximum principle and the dynamic programming method of Bellman – became well known in the scientific and engineering community, and these are now widely used in various academic fields [1, 2, 3, 7, 11].

In this paper, we consider one very interesting application of the methods of optimal control theory for solution of some economical problem known as model of economic growth or Ramsey–Cass–Koopmans model [4, 5, 6, 8, 10, 12]. This model serves as a powerful tool for analyzing the dynamics of consumption and capital accumulation in an economy. It elucidates the complex interplay between current and future preferences and provides a framework for evaluating the impacts of different economic policies. In this paper, we derive the basic conditions for existence of the optimal solution in the model of economic growth and, using Pontryagin's maximum principle, find its structure.

2. Specifying of the problem

Optimal control problems are classified on several types: the simplest problem, the two point minimum time problem, the general problem, the problem with intermediate states, the common problem, etc. [1, 7, 11]

We consider one of them – the simplest problem (*S-problem*). According to [1], it consists of minimizing a terminal functional on a set of processes $x(t)$, $u(t)$ of a controlled system with fixed left end of a trajectory and fixed initial t_0 and terminal t_1 times. This problem has the form

$$J = \Phi(x(t_1)) \to \min ,$$

$$\dot{x} = f(x,u,t), \;\; x(t_0) = x^0, \;\; u \in U, \;\; t \in [t_0, t_1] , \tag{1}$$

where a scalar function $\Phi(x)$ belongs to the class $C_1(R^n \to R)$. Here $x \in R^n$ is state variable, $u \in U \subset R^r$ is control variable. Time interval $I = [t_0, t_1]$ is fixed. We assume that the function $f(x,u,t)$ is defined and continuous on $R^n \times U \times I$ and has continuous partial derivatives $f_x(x,u,t)$ on that set. We consider a set of piecewise-continuous functions $u(t)$ with the range in compact U as the class of controls. A process $x(t),\ u(t)$ is regarded to be *optimal* if for any other process $\tilde{x}(t),\ \tilde{u}(t)$, the following inequality is true

$$\Phi(x(t_1)) \le \Phi(\tilde{x}(t_1)).$$

The *S*-problem consists in determining of the optimal process.

Necessary conditions of optimality in S-problem are given by Pontryagin's maximum principle [1, 11].

Theorem. Let process $x(t),\ u(t)$ be optimal in S-problem. Then there exists a solution $\psi(t)$ of conjugate initial-value problem

$$\dot{\psi} = -H(\psi, x(t), u(t), t),\ \psi(t_1) = -\Phi_{x(t_1)}(x(t_1)) \tag{2}$$

such that for any $t_0 \le t \le t_1$

$$H(\psi(t), x(t), u(t), t) = \max_{u \in U} H(\psi(t), x(t), u, t). \tag{3}$$

Here $H(\psi, x, u, t) = \psi^T f(x, u, t)$ is Hamilton function (Hamiltonian). We will consider the application of the maximum principle for solution of one important problem known as Ramsey model of economic growth.

The Ramsey model, formulated by Francis Ramsey in 1928, is a cornerstone of economic theory that analyzes optimal resource allocation and consumption over time. It serves as a framework for understanding how economic agents can maximize their welfare while considering constraints related to resources and time.

The Ramsey model illustrates the trade-off between current and future consumption. A high return on capital encourages higher savings and investments, leading to greater future consumption. Conversely, a low return incentivizes more immediate consumption. This dynamic highlights the balancing act that agents face when deciding how much to consume today versus saving for tomorrow.

This article explores the Ramsey model as an optimal control problem, highlighting its key components, conditions for optimality, and applications.

The Ramsey model is built on several fundamental assumptions:

1. Infinite Time Horizon: Agents plan their consumption and investment decisions with an infinite time perspective.

2. Intertemporal Utility: Economic agents derive utility from consumption that varies over time.
3. Capital Accumulation: Investments in capital influence future production capabilities.
4. Population Growth: An increasing population affects overall consumption and savings.
5. Technological Progress: Advances in technology can enhance resource efficiency.

Let us proceed to formal constructions.

The model is a closed economy. In the usual setup, time is continuous starting, for simplicity, at $t = 0$ and continuing forever. By assumption, the only productive factors are capital K and labour L, both required to be nonnegative. The labour force, which makes up the entire population, is assumed to grow at a constant rate n, i.e. $\dot{L} = nL$, implying that $L = L_0 e^{nt}$ with initial level $L_0 > 0$ at $t = 0$. Finally, let Y denote aggregate production and C denote aggregate consumption.

The variables that the Ramsey–Cass–Koopmans model ultimately aims to describe are $c = \dfrac{C}{L}$, the per labor consumption, as well as $k = \dfrac{K}{L}$, the so-called capital intensity. It does so by first connecting capital accumulation, written $\dot{K} = \dfrac{dK}{dt}$ in Newton's notation, with consumption C, describing a consumption-investment trade-off. More specifically, since the existing capital stock decays by depreciation rate δ (assumed to be constant), it requires investment of current-period production output Y. Thus, $\dot{K} = Y - \delta K - cL$. The relationship between the productive factors and aggregate output is described by the aggregate production function $Y = F(K, L)$.

If we ignore the problem of how consumption is distributed, then the rate of utility U is a function of aggregate consumption. That is, $U = U(C, t)$. To avoid the problem of infinity, we exponentially discount future utility at a discount rate $\rho \in (0, \infty)$. A high $\rho \in (0, \infty)$ reflects high impatience. The social planner's problem is maximizing the social welfare function $U_0 = \int_0^\infty e^{-\rho t} U(C, t) dt$.

To transform Ramsey model to the classical simplest problem of optimal control, we introduce some assumptions. First, we consider fixed time interval $[0,T]$ and suppose that population does not change through the time. The basic variables of the closed economy are: $Y(t)$ - aggregate production function; $K(t)$ - capital; $I(t)$ - investments in economy; $C(t)$ - aggregate consumption. The relationship between these variables is described by

$$Y(t) = I(t) + C(t)\,,\ Y(t) = F(K(t))\,,\ \dot{K}(t) = I(t)\,.$$

Assuming that $U(C,t) = C(t)$, $Y(t) = a \cdot K(t)$, $a > 0$ (a is return on investment ratio) and $K(0) = K_0 > 0$ we arrive at the problem of optimal control

$$U_0 = \int_0^T e^{-\rho t} C(t)dt \to \max\,,$$

$$\dot{K}(t) = aK(t) - C(t)\,,$$

$$K(0) = K_0\,,$$

$$0 \le C(t) \le aK(t)\,,\ \ t \in [0,T]\,.$$

Here $C(t)$ is regarded as control variable.

We denote by $u = C(t)$, $x = K(t)$ control and state variables accordingly to form the simplest problem

$$U_0 = \int_0^T e^{-\rho t} u dt \to \max\,,\ \dot{x} = ax - u\,,\ x(0) = x_0\,,\ 0 \le u \le ax\,,\ t \in [0,T]\,. \tag{4}$$

3. Solution of the simplest problem

To solve problem (4) we need to transform mixed constraint $0 \le u \le ax$ to homogeneous one. We introduce new control v as $v = \dfrac{u}{ax}$ and state variables $x_1 = x$, $\dot{x}_2 = e^{-\rho t} u$ with initial condition $x_2(0) = 0$ and, finally, obtain

$$-x_2(T) \to \min$$

$$\begin{cases} \dot{x}_1 = ax_1(1-v) \\ \dot{x}_2 = e^{-\rho t}x_1 v \end{cases} \quad (5)$$

$$x_1(0) = x_0,\; x_2(0) = 0$$

$$0 \le v \le 1,\; t \in [0,T].$$

We form Hamiltonian $H(\psi, x, u, t) = ax_1(1-v)\psi_1 + e^{-\rho t}x_1 v\psi_2$ and conjugate initial-value problem

$$\begin{cases} \dot{\psi}_1 = -a(1-v)\psi_1 - e^{-\rho t}v\psi_2 \\ \dot{\psi}_2 = 0 \end{cases};\; \begin{cases} \psi_1(T) = 0 \\ \psi_2(T) = 1 \end{cases}.$$

From the latter, we get $\psi_2(t) = 1,\; 0 \le t \le T$. Therefore, the first differential equation in the conjugate system becomes $\dot{\psi}_1 = -a(1-v)\psi_1 - e^{-\rho t}v$.

We begin to construct optimal control in backwards. Using maximum principle, we get

$$H(\psi, x, u, t) = ax_1(1-v)\psi_1 + e^{-\rho t}x_1 v = ax_1\psi_1 + x_1\left(-a\psi_1 + e^{-\rho t}\right)v \to \max_{0\le v\le 1},\; 0 \le t \le T.$$

Let $t = T$. Then the extreme problem

$$x_1(T)\left(-a\psi_1(T) + e^{-\rho T}\right)v(T) \to \max_{0\le v\le 1}$$

has a solution $v(T) = 1$ since $x_1(t) > 0$ for all $0 \le t \le T$ and $\psi_1(T) = 0$, $e^{-\rho t} > 0$.

We introduce the function $p(t) = -a\psi_1(t) + e^{-\rho t}$. Then the condition (3) of the maximum principle can be written as

$$p(t)v(t) \to \max_{0\le v\le 1},\; 0 \le t \le T. \quad (6)$$

Since Hamiltonian $H(\psi, x, u, t)$ is continuous function [1] then, by [9], we conclude that $v(t) = 1$ on $t \in (\tau, T]$ (Fig. 1).

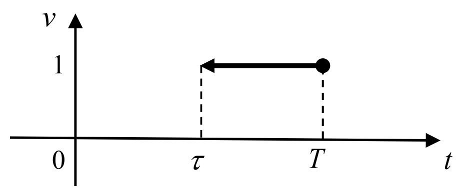

Fig.1. Optimal control on interval $t \in (\tau, T]$.

Here τ is the possible breakpoint of control $v(t)$.

We derive the conditions for existence of τ. From (6), we get

$$p(\tau) = -a\psi_1(\tau) + e^{-\rho\tau} = 0 .$$

Therefore, $\psi_1(\tau) = \frac{1}{a}e^{-\rho\tau}$. Thus, from $\dot{\psi}_1 = -a(1-v)\psi_1 - e^{-\rho t}v$ and $v(t) = 1$, on $(\tau, T]$ we have

$$\dot{\psi}_1 = -e^{-\rho t} \text{ and } \psi_1(t) = \tfrac{1}{\rho}e^{-\rho t} + const .$$

Applying the initial condition $\psi_1(T) = 0$, we obtain

$$\psi_1(t) = \tfrac{1}{\rho}e^{-\rho t} - \tfrac{1}{\rho}e^{-\rho T} .$$

At $t = \tau$, we have

$$\psi_1(\tau) = \tfrac{1}{a}e^{-\rho\tau} = \tfrac{1}{\rho}e^{-\rho\tau} - \tfrac{1}{\rho}e^{-\rho T} .$$

From the last equality, we get

$$e^{-\rho(T-\tau)} = \left(1 - \frac{\rho}{a}\right)$$

and, finally,

$$\tau = T + \frac{1}{\rho}\ln\left(1 - \frac{\rho}{a}\right).$$

Thus, a breakpoint τ of optimal control exists if

$$\rho < a \text{ and } T + \frac{1}{\rho}\ln\left(1 - \frac{\rho}{a}\right) > 0 . \qquad (7)$$

Assume that conditions (7) hold. Then there are two possibilities to the left from breakpoint τ:

Case 1. Interval Δ, where $p(t) = 0$.

Case 2. Some interval, where $p(t) < 0$.

We examine both cases. In Case 1, we have $p(t) = 0$. Therefore $\dot{p}(t) = 0$ and

$$\dot{p}(t) = -a\dot{\psi}_1(t) - \rho e^{-\rho t} = -a\left(-a(1-v)\psi_1 - e^{-\rho t}v\right) - \rho e^{-\rho t} =$$
$$a^2(1-v)\psi_1 + ae^{-\rho t}v - \rho e^{-\rho t} = a^2(1-v)\tfrac{1}{a}e^{-\rho t} + ae^{-\rho t}v - \rho e^{-\rho t} =$$
$$(a-\rho)e^{-\rho t} = 0.$$

Since $e^{-\rho t} \neq 0$, we get $a - \rho = 0$ and $a = \rho$. The latter contradicts to (7). This means that Case 1 does not hold.

Thus, we have some interval on which $p(t) < 0$ (Case 2). From (4), we get $v(t) = 0$. It remains to show that there is not any other breakpoint τ_1 of optimal control on interval $(0, \tau)$. Indeed,

$$\dot{\psi}_1 = -a(1-v)\psi_1 - e^{-\rho t}v = -a\psi_1 \text{ and } \psi_1(t) = const \cdot e^{-at}.$$

Taking into account

$$\psi_1(\tau) = \tfrac{1}{a} e^{-\rho\tau},$$

we obtain

$$\psi_1(t) = \tfrac{1}{a} e^{-a(t-\tau)-\rho\tau}.$$

Consider

$$\dot{p}(t) = -a\dot{\psi}_1 - \rho e^{-\rho t} = a^2\psi_1 - \rho e^{-\rho t} = a e^{-a(t-\tau)-\rho\tau} - \rho e^{-\rho t} =$$
$$e^{-\rho t}\left(a e^{(\rho - a)(t-\tau)} - \rho\right) > 0.$$

Consequently, function $p(t)$ is growing on $(0, \tau)$ and the point τ_1 does not exist. Thus, finally, we arrive at the structure of optimal control (Fig. 2)

$$v(t) = \begin{cases} 0, & 0 \le t < \tau \\ 1, & \tau \le t \le T. \end{cases} \tag{8}$$

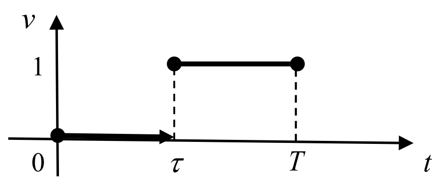

Fig.2. Optimal control on interval $t \in [0, T]$.

And, consequently, optimal trajectory from (5) is

$$\begin{cases} x_1(t) = x_0 e^{at} \\ x_2(t) = 0 \end{cases}, \quad 0 \le t < \tau \quad \text{and} \quad \begin{cases} x_1(t) = x_0 e^{a\tau} \\ x_2(t) = -\frac{x_0 e^{a\tau}}{\rho} e^{\rho t} + \frac{x_0 e^{a\tau}}{\rho} e^{(a-\rho)\tau} \end{cases}, \quad \tau \le t < T.$$

Remarks:

1. If the level of consumption must be greater then zero, say $C(t) = \beta > 0$, the structure of optimal control remains the same as (8), that is

$$v(t) = \begin{cases} \tilde{\beta}, & 0 \le t < \tau \\ 1, & \tau \le t \le T. \end{cases}, \text{ where } \tilde{\beta} = \tfrac{\beta}{ax}.$$

2. If condition (7) does not hold, that is $\rho \ge a$, the investment in economy is not effective. In this case, aggregate production function $Y(t)$ is totally aggregate consumption $C(t)$ and optimal control has the form $v(t) = 1,\ \ \tau \le t \le T$.

Conclusion

We discussed very important question of the theory of optimal control - the application in the economic problems. In particular, we considered the model of economic growth known as model of Ramsey, reduced it to the simplest problem of optimal control and, using Pontryagin's maximum principle, derived the optimal solution. We showed the structure of the most effective distribution of the aggregate production in the closed economic region for various its input conditions. Despite its limitations, Ramsey model remains an essential component of economic theory and practice, offering insights into optimal resource allocation over time.

Acknowledgments

This work was supported by Research Fund of Kwangwoon University in 2024 and Far Eastern Federal University (project No. DVFU-24-04-1.01-0006).

References

[1] Aschepkov L.T., Dolgy D.V., Kim Taekyun, Agarwal Ravi P., Optimal Control. Springer International Publishing AG 2016. 209p. DOI 10.1007/978-3-319-49781-5

[2] Ashchepkov, L. T., Analytical synthesis of an amplitude-constrained controller//Autom. Remote Control 83, No. 7, 1050-1058 (2022).

[3] Aschepkov L.T., Dolgy D.V., Control of the linear multi-step systems in conditions of uncertainty// Far Eastern Mathematical Collection. Issue 4. 1997, p.95-104.

[4] Blanshar O.J., Lectures on Microeconomics, Moscow, Publishing House RANHiGS, 2014, 680 p.

[5] Cass D., Optimum growth in Aggregative model of capital accumulation//The review of Economic Studies. 1965, Vol. 32, #3, pp.233-240.

[6] De Long J.B., Productivity growth, convergence, and welfare: comment//The American economic review. 1988, Vol. 78, #5, pp.1138-1154.

[7] Dolgy D.V., Optimal control, Vladivostok, DVFU. 2004, 53 p.

[8] Koopmans T.C., On the concept of optimal economic growth//Pontificae Academiae Scientiarum Scripta varia//The econometric approach to development planning. 1965, Vol. 28, pp.225-300.

[9] Kudryavtsev L.D., Course of mathematical analysis, Moscow, Nauka, 1981, 687 p.

[10] Newbery D.M., Ramsey model//The new Palgrave dictionary of economics/Macmillan publishers Ltd., UK, 2018, pp.11172-11178.

[11] Pontryagin L.S., Boltiansky V.G., Gamkrelidze R.V., Mishchenko E.F., Mathematical theory of optimal processes, Moscow. Nauka, 1983, 393 p.

[12] Ramsey F.P., A mathematical theory of saving//The economic journal. 1928, Vol.38, #152, pp. 543-559.

KWANGWOON GLOCAL EDUCATION CENTER, KWANGWOON UNIVERSITY, SEOUL 139-701, REPUBLIC OF KOREA & INSTITUTE OF MATHEMATICS AND COMPUTER TECHNOLOGIES, DEPARTMENT OF MATHEMATICS, FAR EASTERN FEDERAL UNIVERSITY, VLADIVOSTOK, RUSSIA

E-mail address: d_dol@mail.ru