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THE CONTINUUM HYPOTHESIS

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Abstract. Power sets of generalized natural numbers are sufficient to prove that cardinality of $2^{\mathbb{N}}$ and the real numbers coincide. The cardinality of geometric sets is combined with inequality on the measures of curves in a planar domain to prove the equivalence of the cardinalities $\aleph_0^n$, $n \geq 3$, with $\aleph_0^2$. A set $\hat{C}$ is defined to be the set of real numbers with a decimal expansion such that the n^{th} place can be evaluated after a countably infinite number of arithmetic operations. A projection from the set of real numbers to $\hat{\mathrm{C}}$ exists. The cardinality of $\hat{\mathrm{C}}$ is demonstrated to be $\aleph_0^2$, which must be equated with cardinality of the continuum. It is proven that there is no other cardinality between $\aleph_0$ and $2^{\aleph_0}$.

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1. Introduction

The continuum hypothesis concerns the existence of sets with cardinality between $\aleph_0$, that of the natural numbers, and $2^{\aleph_0}$, the cardinality of the real numbers. It may be recalled that the set of rational numbers can be placed in one-to-one correspondence with $\mathbb{N}$. Since the real numbers may be defined by the limiting values of convergent Cauchy sequences of rational numbers, the continuum hypothesis would be valid if there do not exist any sets with a cardinality greater than $card(\mathbb{Q})$ and less than $card(\mathbb{R})$.

It may be demonstrated that geometry distinguishes between sets of equal cardinality. Furthermore, a combination of analytic and geometric methods can yield relations between cardinalities. The ideal boundaries of Riemann surfaces. for example, are known to be described by a countable collection of points, a Cantor sets of ends or a continuum. The comparison between the cardinalities of countable and Cantor sets then may be refined through geometrical models such as the categories of ideal boundaries of Riemann surfaces.

The characterization of Riemann surfaces of by the ideal boundaries is given by the limit $lim_{n\rightarrow\infty} r_n = \infty$ on the Cantor set $E(p_1p_2p_3...)$, where the unit interval is divided into p_n equal segments and the central interval is deleted, and r_n is the Robin constant of $E(p_1p_2...p_n)$ that equals the inverse of the capacity. Two conditions that can be satisfied by the Cantor set are $\sum_{\nu=1}^{\infty} \frac{log\ p_\nu}{2^\nu} = \infty$ or

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$\prod_{\nu=1}^{\infty}\left(1-\frac{1}{p_\nu}\right)^{2^{-\nu}}=0$ [1]. While the first equation is not necessary, the second relation is required for the capacity to be zero. These constraints cannot be satisfied if $p_\nu = p$ for all ν. The cardinality of the set satisfying the second condition is less than or equal to *card* $\mathbb{N} = \aleph_0$. The latter estimate follows from the decrease in the number of divisions of the subintervals in the sets $E(p_1...p_\nu)$ for large ν, and the inclusion of sets of ends for finite ν with $\nu < \nu_{max}$.

The relation between the quasiconformal extension of Riemann surfaces with ideal boundaries having dimension between 0 and 1 and the continuum hypothesis is elucidated. The set of values of the the harmonic dimension of the boundary, equal to the cardinality of the set of minimal Martin boundary points, in the range $[1, 2^{\aleph_0}]$, may be enumerated in a investigation of the validity of the continuum hypothesis. [2]. The existence of surface with a set of ends of cardinality $\aleph_0^n$ with $n > 2$ is demonstrated in §2. The equality of the cardinalities $\aleph_0^2$ and $\aleph_0^n$, $n \geq 3$, is proven in §3. The cardinality of all real intervals is derived and found to be equal to $2^{\aleph_0}$. The equivalence of $\aleph_0^2$ and $2^{\aleph_0}$ in the binary decimal system is found to be valid for the continuum.

2. **Surfaces with Ideal boundaries of Non-zero Harmonic Measure**

A relation between the continuum hypothesis and the classification of Riemann surfaces has been conjectured through a proposed equality of the set of harmonic dimensions of O_G surfaces and a set of cardinalities between 1 and $\aleph$ [2]. The set of cardinalities first can be determined for the open planar surfaces in $\hat{\mathbb{C}}$. It is found that there is a bijective correspondence with nonempty compact subsets of $\hat{\mathbb{C}}$ and those that have zero logarithmic capacity or an Evans-Selberg potential will belong to O_G [3]. By the Picard principle for the Martin kernel, it follows that *dim* $(\hat{\mathbb{C}})$ = *card* $\Delta_1(\hat{\mathbb{C}})$ = *card* K [2], where $\Delta_1(R)$ is the minimal Martin boundary of R. The cardinality of K can be deduced from the Cantor-Bendixson Theorem, the K is a closed set in $\mathbb{C}$ and a union of a countable and a perfect set. The Cantor-Bendixson theorem had been proven initially for closed sets in the real line [4][5]. However, it can be extended to any separable completely metrizable topological space, including $\hat{\mathbb{C}}$.

The only Riemann surfaces that can be conformally mapped to a planar region are schlichtartig and have genus zero. However, it is proven that *card* $\Delta_1(R)$ = *card* $\Delta_1(W_R)$, where W_R is a Heins surface with a single boundary component. The cohomology of a neighbourhood of this boundary component will be determined by the boundary curves in the exhaustion of the surface and that neighbourhood can be mapped to a planar region [6]. Therefore, the harmonic dimension of the minimal Martin boundary will be equal to that of a planar region and the previous results regarding the compact set representing the complement to the image of the neighbourhood are relevant. The set of harmonic dimensions belonging to the class of O_G surfaces will not be larger than that for open planar surfaces, $\mathbb{N} \cup \{\aleph_0, \aleph\}$.

The Heins problem also may be extended to the classes of surfaces between O_G and Type II surfaces. The ideal boundaries, irrespective of the harmonic measure, are represented by intermediate sets with cardinalities that would determine the validity of the continuum hypothesis.

It remains to be determined if every transfinite ordinal between $\aleph_0$ and $\aleph$ can be derived through this method, A bijective mapping between the transfinite ordinals $\omega,\ \omega+1, ..., \omega+\alpha, ...$ with a maximum cardinality of $\aleph$ and points sets in the real line is necessary. The completion of power sets of generalized natural numbers provides a method for determining the relation if such sets are perfect.

Theorem 1. The cardinality of the set of ends of Toki's surface is $\aleph_0^2$.

Proof.

The example of Toki's surface in $O_{HD} \backslash O_{HB}$ is constructed by letting

(2.1)

$$\Delta_0 = \Delta \backslash \cup_{n,\nu} \; S^{\nu}_{mn}$$

$$S^{\nu}_{mn} = \{z = re^{i\theta} | -2^{-2\mu} \leq log \; r \leq -2^{-(2\mu+1)}, \; \theta = \nu \cdot 2\pi 2^{-2\mu}, \; \nu = 1, ..., 2^{2\mu}, \; \mu = 2^{m-1}(2n-1)\}$$

and, denoting $\{\Sigma(h)\}_1^{\infty}$, $\{\Sigma'(h)\}_1^{\infty}$ be two sequences of duplicates of Σ_0, such that, for fixed $m = 1, 2, ...$ and $j = 0, 1, ..., \; i = 1, ..., m, \; \Sigma(i + mj)$ is joined with $\Sigma'(i + m + mj)$ for even j and $\Sigma'(i - m + mj)$ for odd j crossing along every slit S^{ν}_{mn}, $n = 1, 2, ...$, by passing to the covering surface [7]. The number of sheets is given by summing over m and j to give $\sum_{j \; even} \sum_{m=1}^{\infty} 1 + \sum_{j \; odd} \sum_{m=1}^{\infty} 1$ with a cardinality equal to $\aleph_0^2$.

□

If the sheets are separated at each of the slits, with rejoining at the other side, the ends of the surface would be related to each value of μ given by n and ν. When n tends to infinity, μ increases linearly for fixed n and the cardinality of the set of ends then increases to

$$lim_{N \to \infty} \sum_{\substack{\mu = 2^{m-1}(2n-1) \\ m,n=1}}^{N} 2^{2\mu} \qquad (2.2)$$

since $2^{m_1 - 1}(2n_1 - 1) = 2^{m_2 - 1}(2n_2 - 1)$ if and only if $m_1 = m_2$ and $n_1 = n_2$, The cardinality of the set defined by $\{\mu\}$ remains $\mathbb{Z}$. Therefore, the sum would be $2^{2^{\aleph_0}}$. A cardinality of $2^{2^{\aleph_0}}$ is larger than that of the real line and equal to that of the sphere [8]. Since the boundary of the unit disk, which is a covering of any Riemann surface of genus $g \geq 2$, only has the cardinality of the real line, it cannot be possible to separate the ends of the sheets at each of the slits labelled by m, n and ν and preserve the topology of a differentiable manifold. The cardinality $\aleph_0^2$ must be compared to that of $\mathbb{N}$, $\aleph_0$, and $2^{\mathbb{N}}$, $\aleph = 2^{\aleph_0}$. This problem is raised also by the evaluation of the cardinality of sets in planar domains.

3. Power Sets of the Generalized Natural Numbers Consisting of Infinitesimals

Although the power set $2^{\mathbb{N}}$ generated by the natural numbers has the same cardinality as the real numbers, these two sets do not coincide. A generalization of the natural numbers is required for such an identification. In nonstandard analysis, a generalized natural number is defined with the inclusion of infinitesimals [9]. The rules of arithmetic operations have been delineated for the generalized natural numbers. The infinitesimals are necessary for a relation with $\mathbb{R}$, because the set of real numbers form a continuum. The power set based on $\mathbb{N}^*$, the set of generalized natural numbers, will have the cardinality of $\mathbb{R}$. Based on the formalism for the integers and rationals, the cardinality of $\mathbb{N}^*$ and a set of generalized rational numbers $\mathbb{Q}^*$, would be equal. The relation between $2^{\mathbb{Q}^*}$ and $\mathbb{R}$ then may be considered within the context of the Cauchy completion of $\mathbb{Q}$ into $\mathbb{R}$.

Cardinality is fixed and defined consistently for each set, and yet, there is a further delineation of the cardinality of subsets of a geometrical manifold. The set of accumulation points of handles on the sphere is an example. The cardinality of this set may be set equal to to a number less than or equal to $2^{\aleph_0}$ [**?**]. A surface in this category can be modelled by a sphere with an infinite set of sequences of handles accumulating at points with rational coordinates, which would require the removal of the rational multiples of 2π for the longitudinal and latitudinal angles.

Theorem 2. The excision of points with rational coordinates with a fixed numerators requires the removal of a set of cardinality of order $2^B card(\mathbb{Z})^B$ and replaced by a semicircular arc of arbitarily infinitesimal radius such that the integral measure of the arcs at the k^{th} stage of the development of the rectifiable path decreases with k. Since the cardinality of the set of points on the arcs is greater than or equal to the cardinality of the set of excised points, it must be a monotonic function of k. Then the cardinality $card(\mathbb{Z}^B) = \aleph_0^B$ for finite $B \geq 3$ may be equated with $\aleph_0^2$.

Proof.

Mapping the sphere to the plane, the unit circle will have rational values of $x = \frac{a}{c}$ or $y = \sqrt{1 - \frac{a^2}{c^2}}$ that may be excised and replaced by semicircular arcs about these points. These arcs actually subtend an angle larger than π because the endpoints must be located on the original circle rather than a straight line. Let the radii obe ϵ_1 such that

$$\left(x - \frac{a}{c}\right)^2 + \left(y - \frac{\sqrt{c^2 - a^2}}{c}\right)^2 = \epsilon_1^2 \qquad (3.1)$$
$$x^2 + y^2 \geq 1$$

on these semicircular paths. Then on each arc γ_{i_0}, there will a countably infinite set of points that can be selected, for example, by the rationality of one of the coordinates (x_{i_0}, y_{i_0}). Labelling the points $\{(x_{i_0 i_1}, y_{i_0 i_1}\}$, where

$$(x - x_{i_0 i_1})^2 + (y - y_{i_0 i_1})^2 = \epsilon_2^2 \qquad (3.2)$$
$$x_{i_0 i_1}^2 + y_{i_0 i_1}^2 \geq (1 + \epsilon_1)^2$$

on the second class of arcs. A countably infinite sequence of semicircular arcs about points with either rational x or y yields a rectifiable path. It can be verified that at the k^{th} stage, a set of points of cardinality $2^\ell card(\mathbb{Z})^k$ is removed. Given that the semicircular arcs would be described by

$$(x - x_{i_0 i_1 \ldots i_{k-1}})^2 + (y - y_{i_0 i_1 \ldots i_{k-1}})^2 = \epsilon_k^2 \qquad (3.3)$$

the neighbourhoods of the excised rational points will be nonoverlapping if $\sum_k \epsilon_k < 1$. This upper bound will be achieved if the number of stages of the sequence is finite or the magnitude $|\epsilon_k|$ decreases more rapidly than $\delta \frac{1}{k^{\frac{1}{2}}}$. Since the cardinality of $\mathbb{Q}$ is $\aleph_0$, the cardinality of the set of excised points which approaches that of the rectifiable curve is

$$\sum_{\ell=1}^{B} 2^k card(\mathbb{Z}^k) - card(\mathbb{Z}) < 2^{B+1} card(\mathbb{Z}^B) \tag{3.4}$$

The value of B may be chosen to be arbitrarily large, and the limit of B tending to infinite is majorized by $2^{\aleph_0}2^{\aleph_0} = 2^{2\aleph_0}$ since $card(\mathbb{Z})^B \leq 2^{\aleph_0}$ for all finite B. Furthermore, the sum of the lengths of the nonintersecting portions of the semicircles at the k^{th} stage has the same order for each k, since it is a path between two circles of radii 1 and $1 + \sum_{\ell=1}^{k} \epsilon_k$ with a tangent having an everywhere positive derivative in the direction of the traversal. Since the excised centers form sets of increasing cardinality $2^k card(\mathbb{Z}^k)$, the nonintersecting portions of the semicircular arces would decrease at each successive stage. Yet, there continue to be sets of excised points, and the union of these sets after B stages has a cardinality between $2^B card(\mathbb{Z}^B)$ and $2^{B+1} card(\mathbb{Z}^B)$.

The ordering of the cardinalities $\aleph_0^n$, $n \geq 2$, depends on the equality of $\aleph_0^2$ with the cardinality of any set of real numbers represented in a dyadic expansion. Irrespective of the uniqueness of the representation of real numbers by decimals of arbitrary length, the cardinalities $\aleph_0^2$ and the cardinality of the real continuum may be examined within the system of binary units. The approximation of real numbers in this system by rational numbers in $\mathbb{Q}$ such that $N\ card\ \mathbb{Q}$ is identified with $card\ Q$ for finite N, while the limit of N tending to infinite yields $card\ \mathbb{R}$, provides an analytic method, separate from the Zermelo-Fraenkel axioms, for determining the relation between the cardinalities $\aleph_0^2$ and $2^{\aleph_0}$.

Logical consistency then would be restored if and only if the cardinalities of the set of excised points or the rectifiable curve are equated at each stage, since $card(\mathbb{Z}^k) \geq card(\mathbb{Z}^{k-1})$. Therefore, the cardinality of $\mathbb{Z}^k$ may be identified with $\aleph_0^2$ for all finite k. Since $\aleph_0^2$ can be designated to be the next cardinality beyond $\aleph_0$, there would be a maximum of one cardinality between $\aleph_0$ and $2^{\aleph_0}$ [8].

It is evident that an enumeration of sets with cardinality less than or equal to $2^{\aleph_0}$ would be given by these rectifiable paths with cardinalities $\aleph_0^B$ for finite B. The radii of the semicircles ϵ_k tend to zero in the limit of B tending to ∞, which, therefore, cannot yield a conclusion regarding the relation between $\aleph_0^2$ and $2^{\aleph_0}$.

□

The cardinality of $\mathbb{Z}^\alpha$, $2 \leq \alpha < 3$, can be equated with that of $\mathbb{Z}^2$. The fractional exponent may be achieved by considering branching of the set of integers into either two or three routes, with the average having a value between these two integers. Then, there is a countable sequence of stages with one extra branch that is embedded identified with another element of $\mathbb{Z}$ embedded in $\mathbb{Z}^2$.

The generalized continuum hypothesis implies the axiom of choice [11]. It is not possible to posit the validity of this hypothesis in the proof. The axiom of choice, however, is included in the ZFC axiomatic system, and it would sufficient to conclude, for example, that the accumulation point of a subsequence of an infinite sequence of rational numbers is a real number. Conversely, every real number is represented as an equivalence class of a convergent sequence of rational numbers.

The cardinality of the set of ends of Toki's surface does not introduce an intermediate value because it can be identified with $\aleph_0^2$. The existence of a conformal conjugation of a Type I surface to a Type II surface [12], with a boundary of non-zero linear measure, is indicative of an equivalence between the cardinality of the sets of ideal boundaries points of Type I surfaces not in O_G, greater than $card\ \mathbb{Z} = \aleph_0$, and $2^{\aleph_0}$.

Theorem 3. There can exist real numbers with an expansion such that the n^{th} decimal place cannot be evaluated in a countable number of arithmetic operations over $\mathbb{Q}$. There is a projection from these numbers to numbers that belong to the set $\hat{\mathrm{C}}$ of countably infinite algorithmic computable real numbers which preserves its cardinality. The cardinality of $\hat{\mathrm{C}}$ can be proven to be $\aleph_0^2$, and it may be equated with $card(\mathbb{R})$, which thereby verifies the continuum hypothesis.

Proof.

The decimal representation characterizes the real numbers as the Cauchy equivalence classes of rational numbers such that a countable number of arithmetical operations over $\mathbb{Q}$ is required to define a real number. A transcendental number such as π is given by an infinite series over $\mathbb{Q}$, since its decimal representation can be found after a finite number of arithmetic operations [13]. The n^{th} decimal place of many transcendental numbers may be given after a countable number of arithmetic operations. The evaluation of a number such as $\tau_1^{\tau_2}$, with τ_1 and τ_2 being independent transcendental numbers over $\mathbb{A}$ that may be computed to the n^{th} decimal place after a countably infinite arithmetic calculation, would require initially an algorithm with uncountable number of arithmetic operations over $\mathbb{Q}$. Suppose that

$$\tau_1 = \sum_{n=0}^{\infty} \frac{a_{1n}}{q^n} \qquad (3.5)$$
$$\tau_2 = \sum_{n=0}^{\infty} \frac{a_{2n}}{q^n}.$$

such that a countably infinite algorithm is required to evaluate the coefficients $a_{1n},\ a_{2n}$ for each finite n. Then

$$\tau_1^{\tau_2} = \left(\sum_{n=0}^{\infty} \frac{a_{1n}}{q^n}\right)^{\sum_{n=0}^{\infty} \frac{a_{2n}}{q^n}} \qquad (3.6)$$
$$= \left(\sum_{n=0}^{\infty} \frac{a_{1n}}{q^n}\right)^{a_{20}} \left(\sum_{n=0}^{\infty} \frac{a_{1n}}{q^n}\right)^{\frac{a_{21}}{q}} ... \left(\sum_{n=0}^{\infty} \frac{a_{1n}}{q^n}\right)^{\frac{a_{2k}}{q^k}} ... .$$

The uncountably infinite number of arithmetical operations over $\mathbb{Q}$ in addition to the $\frac{1}{q^{th}}$ root necessary to determine the decimal expansion of $\tau_1^{\tau_2}$, however, can be reduced to a countably infinite algorithm to evaluate the n^{th} decimal place. It may be recalled that τ_1 and τ_2 belong to the class of transcendental numbers such that a countable number of steps are sufficient to find the n^{th} decimal places, and

$$\left|\tau_1 - \frac{p_1}{10^{N_0}}\right| < \frac{c_1}{(10^{N_0})^{r(\tau_1)}} \qquad (3.7)$$
$$\left|\tau_2 - \frac{p_2}{10^{N_0}}\right| < \frac{c_2}{(10^{N_0})^{r(tau_2)}}$$

with $r(\tau)$ being the Lioville-Roth constant or τ. Suppose that $\tau_1 > 1$. Then $\tau_1^{\tau_2}$ is a monotonically increasing function of τ_2 and $\tau_1^{\lfloor\tau_2\rfloor} < \tau_1^{\tau_2} < \tau_1^{\{\tau_2\}}$. Given that τ_2

is not equal to the $\frac{ln\ \alpha}{ln\ \tau_1}$, where α is an algebraic number, $\tau_1^{\tau_2}$ is a transcendental number and

$$\left|\tau_1^{\tau}{}_2 - \frac{p_{12}}{10^{N_0}}\right| < \frac{c_{12}}{(10^{N_0})^{r(\tau_1^{\tau_2}}} \le \frac{c_{12}}{(10^{N_0})^{2^+}} \tag{3.8}$$

as $r(\tau_1^{\tau_2}) \ge 2^+$. Furthermore,

$$\begin{aligned}\left|\tau_1^{\lfloor\tau_2\rfloor} - \left(\frac{p_1}{10^{N_0}}\right)^{\lfloor\tau_2\rfloor}\right| &= \left|\tau_1 - \frac{p_1}{10^{N_0}}\right| (\lfloor\tau_2\rfloor + 1)(\tau_1 + \epsilon_1)^{\lfloor\tau_2\rfloor - 1} \\ &< \frac{c_1}{(10^{N_0})^{r(\tau_1)}}(\lfloor\tau_2\rfloor + 1)(\tau_1 + \epsilon_1)^{\lfloor\tau_2\rfloor - 1} \\ \left|\tau_1^{\{\tau_2\}} - \left(\frac{p_1}{10^{N_0}}\right)^{\{\tau_2\}}\right| &< \frac{c_1}{(10^{N_0})^{r(\tau_1)}}(\{\tau_2\} + 1)(\tau_1 + \epsilon_1)^{\{\tau_2\} - 1}.\end{aligned} \tag{3.9}$$

where $\epsilon_1 \le \frac{c_1}{(10^{N_0})^{r(\tau_1)}}$. The function $\left|\tau_1^{\tau_2} - \left(\frac{p_1}{(10^{N_0}}\right)^{\tau_2}\right|$ is a montonically increasing function of τ_2 and

(3.10)
$$\frac{c_1}{(10^{N_0})^{r(\tau_1)}}(\lfloor\tau_2\rfloor+1)(\tau_1+\epsilon_1)^{\lfloor\tau_2\rfloor-1} < \left|\tau_1^{\tau_2} - \left(\frac{p_1}{10^{N_0}}\right)^{\tau_2}\right| < \frac{c_1}{(10^{N_0})^{r(\tau_1)}}(\{\tau_2\}+1)(\tau_1+\epsilon_1)^{\{\tau_2\}-1}.$$

Since $r(\tau_1) \ge 2^+$,

$$\left|\tau_1^{\tau_2} - \left(\frac{p_1}{10^{N_0}}\right)^{\tau_2}\right| < \frac{c_{12}^{(1)}}{10^{2N_0}}. \tag{3.11}$$

Similarly,

(3.12)

$$\begin{aligned}\left|\left(\frac{p_1}{10^{N_0}}\right)^{\tau_2} - \left(\frac{p_1}{10^{N_0}}\right)^{\frac{p_2}{10^{N_0}}}\right| &= \Bigg|\left(1 + \frac{\tau_2}{ln\ \left(\frac{p_1}{10^{N_0}}\right)} + \frac{1}{2!}\left(\frac{\tau_2}{ln\ \left(\frac{p_1}{10^{N_0}}\right)}\right)^2 + ...\right) \\ &\quad - \left(1 + \frac{\frac{p_2}{10^{N_0}}}{ln\ \left(\frac{p_1}{10^{N_0}}\right)} + \frac{1}{2!}\left(\frac{\frac{p_2}{10^{N_0}}}{ln\ \left(\frac{p_1}{10^{N_0}}\right)}\right)^2 + ...\right)\Bigg| \\ &< \frac{\left|\tau_2 - \frac{p_2}{10^{N_0}}\right|}{ln\ \left(\frac{p_1}{10^{N_0}}\right)} + \frac{1}{2!}\left|\tau_2 - \frac{p_2}{10^{N_0}}\right| \cdot \frac{2(\tau_2+\epsilon_2)^2}{\left(ln\ \left(\frac{p_1}{10^{N_0}}\right)\right)^2} + ... \\ &= \left|\tau_2 - \frac{p_2}{10^{N_0}}\right| \cdot \sum_{m=1}^{\infty}\frac{1}{m!}m\left(\frac{\tau_2+\epsilon_2)}{ln\ \left(\frac{p_1}{10^{N_0}}\right)}\right)^m \\ &= \left|\tau_2 - \frac{p_2}{10^{N_0}}\right| \cdot \frac{\tau_2+\epsilon_2)}{ln\ \left(\frac{p_1}{10^{N_0}}\right)}\ e^{\frac{\tau_2+\epsilon)}{ln\ \left(\frac{p_1}{10^{N_0}}\right)}} \\ &< \frac{c_2}{(10^{N_0})^{r(\tau_2)}}\frac{\tau_2+\epsilon_2)}{ln\ \left(\frac{p_1}{10^{N_0}}\right)}\ e^{\frac{\tau_2+\epsilon)}{ln\ \left(\frac{p_1}{10^{N_0}}\right)}} \\ &< \frac{c_{12}^{(2)}}{10^{2N_0}}.\end{aligned}$$

with $\epsilon_2 \le \frac{c_2}{(10^{N_0})^{r(\tau_2)}}$. Therefore,

$$\left|\tau_1^{\tau_2} - \left(\frac{p_1}{10^{N_0}}\right)^{\frac{p_2}{10^{N_0}}}\right| < \frac{c_{12}^{(1)}}{10^{2N_0}} + \frac{c_{12}^{(2)}}{10^{2N_0}} = \frac{c_{12}}{10^{2N_0}} \tag{3.13}$$

with $c_{12} = c_{12}^{(1)} + c_{12}^{(2)}$. A similar inequality is valid when $\tau_1 < 1$.

A countably infinite number of arithmetic operations would be required to compute the expansions of τ_1, τ_2 and $\tau_1^{\tau_2}$ to the N_0^{th} decimal place. Since $2N_0 \geq N_0+1$ for a finite integer $N_0 \geq 2$ and $\left(\frac{p_1}{10^{N_0}}\right)^{\frac{p_2}{10^{N_0}}}$ may be evaluated in a finite number of steps, only a countably infinite algorithm is necessary for calculate $\tau_1^{\tau_2}$ to N_0+1 decimal places. By induction, the n^{th} decimal place of $\tau_1^{\tau_2}$ for any finite n can be evaluated after a countably infinite number of arithmetical operations.

There is a surjection from the set of Gödel numbers to the finitely algorithmic computable real numbers C, evaluated to a given precision after a finite number of arithmetical operations and forming a subfield of the real numbers, although the surjective function from $\mathbb{N}$ to C is not computable [14][15]. This class of numbers could be considered to be separate from numbers of the type $\tau_1^{\tau_2}$, since an infinite number of operations is required for an evaluation of the n^{th} decimal place. The class of countably infinite algorithmic computable real numbers, to be denoted $\hat{\mathrm{C}}$, would be defined by the computability of the n^{th} decimal place in their decimal representations in a countable number of arithmetic operations. This set is larger than the set of finitely algorithmic computable real numbers, which has cardinality $\aleph_0$. The transcendental numbers $\tau_1^{\tau_2}$, $\tau_1^{\tau_2^{\tau_3}}$, ... represent the set which maximize the length of the algorithm for their computation. Consequently, any real number would belong to $\hat{\mathrm{C}}$. The projection from the set of real numbers to $\hat{\mathrm{C}}$ does not alter this cardinality.

The relation between the number of arithmetical operations to evaluate the decimal expansions of elements in $\hat{\mathrm{C}}$ and the cardinality of this set remains to be established. When a finite algorithm is sufficient to find n^{th} decimal place in a finite expansion, the number in the set of finite algorithmic computable real numbers, C, can be given after a finite number of steps. If only finite, arbitrarily large decimal expansions are enumerated, the set C would have cardinality $\aleph_0$. By analogy, for a countably infinite algorithmic computable real number, the number of arithmetic calculations for a finite decimal expansion is countable. Enumerating the elements by altering each of the decimal places in an arbitrarily large expansion, the cardinality of this set, $\hat{\mathrm{C}}$, now would be equated with $\aleph_0^2$. By Theorem 3, $\aleph_0^2 = card(\hat{\mathrm{C}}) = card(\mathbb{R}) = 2^{\aleph_0}$, and the continuum hypothesis is proven.

□

It is evident that this theorem confirms the cardinality of $2^{\aleph_0}$ for uncountable projective sets given the validity of the Axiom of Projective Determinacy [16]. The infinite-genus surface is an example of a noncompact, first countable, countably compact space which contains a closed subset homeomorphic to ω_1 with a natural order topology [17]. The condition on countability of the meager open set union representing a compact subset is satisfied, and projective determinancy is valid [18].

The rules of cardinal arithmetic indicatae that the cardinality of the union of two sets of cardinalities α and β can be equated to $max(\alpha, \beta)$. With regard to the product, the selection of the set is critical in the validity of any statement. The decimal representation of a number on the real line consists of an integer part belonging to a set of cardinality $\aleph_0$ and a decimal expansion in a set of cardinality $\aleph_0^2$. Since the second cardinality does enumerate the decimal expansions to an infinite number of places of every real number, it is considered not to be necessary to multiply the two cardinalities $\aleph_0$ and $\aleph_0^2$ to establish the cardinality of the real line, because the cardinality of the continuum is already given by that of the unit interval $[0,1]$, which would be determined by the decimal expansions. It may be noted, however, that the cardinality of the product of sets with a countably infinite number of elements never can be identified with the maximal of the cardinalities of each of the sets, and $\aleph_0\aleph_0 = \aleph_0^2$ is distinguished in set theory from $\aleph_0$.

The value of a real number would be unique with a difference of zero to an infinite number of decimal places. The existence of an alternative numbering method for real numbers yields a logical contradiction. Suppose that one could continue the decimal expansion of a real number beyond $\aleph_0$ places to $\aleph_0^2$ places. If this continuation is hypothesized to be countable, then the decimal expansion again would represent an exact representation of any real number. Further, if this expansion is continued again to $\aleph_0^n$, n$\geq$ 3, a similar result occurs. Therefore, the only way of achieving an ambiguity in the real numbers would be the possibility of an uncountable decimala expansion. However, it must then be concluded that there is an intermediate cardinality $\aleph_0^k$ of an uncountable set. If $\aleph_0^k$ is uncountable, all of the previous powers of $\aleph_0$ would be countably equivalent to $\aleph_0$. Identifying these cardinalities with $\aleph_0$, the diagonal rectangular array for $\mathbb{Q}$ may be used to define the next power of $\aleph_0$, which is $\aleph_0^2$. Therefore, k can be set equal to 2. Then, $\aleph_0^2$ must be a cardinality different from $\aleph_0$.

From the above discussion, a characterization of the real line by a product of sets of cardinalities $\aleph_0$ and $\aleph_0^2$ logically might be interpreted as a set of cardinality $\aleph_0^2$. The standard decimal expansion is sufficient to characterize the real line since the difference between adjoining elements can be reduced to zero within a fraction with a denominator raised to a Liouville-Roth constant tending to infinity. An uncountable decimal expansion would yield differences that are not measurable, and its extent would equal a fraction of the number r less than $\frac{r}{\aleph_0} = 0$. Consequently, the initial decimal expansion provides a representation of real numbers, and any logical exceptions no longer exist. With a countable decimal expansion, $\aleph_0^2$ covers an entire interval within the real number line. The conventional majorization of the power N^K by 2^N for $K < N\frac{ln\ 2}{ln\ N}$, and conversely 2^N by N^M, where $M > N\frac{ln\ 2}{ln\ N}$, is valid for finite N. Since $M \to \infty$, when $N \to \infty$, this majorization will is indicative of sets of cardinality $\aleph_0^n$ being finitely comparable and yet isomorphic to proper subsets having cardinality $2^{\aleph_0}$, which is conventionally interpreted as the cardinality of the continuum. Nevertheless, the unit interval is not strictly covered by the Cantor set, and a union of these sets is required. Therefore, to avoid an inconsistency, $\aleph_0^2$, $\aleph_0^n$ and $2^{\aleph_0}$ all must be defined to be cardinals that represent the real continuum. From the above discussion, a characterization of the entire real line by a product of sets of cardinalities $\aleph_0$ and $\aleph_0^2$ logically might be interpreted as a set of cardinality $\aleph_0\aleph_0^2 = \aleph_0^3$. However, given that both the unit interval and the real line are examples of the continuum, $\aleph_0^2$ and $\aleph_0^n$, $n \geq 3$, are comparable cardinalities. The categorical equivalence of $\aleph_0^2$ and $2^{\aleph_0}$ follows from the characterization of the set $\hat{\mathrm{C}}$ of real numbers with an n^{th} decimal place that can be evaluated with a countably infinite algorithm for all finite n.

4. **The Decidability of Propositions**

The existence of a statement of the validity of a proposition and its negation would be sufficient to demonstrate the inconsistency of a two-valued logical system. An axiomatic system satisfying the Zermelo-Fraenkel axioms and the continuum hypothesis would remain consistent [19]. The independence of the continuum hypothesis from these axioms follows from consistency even with its negation [20][21]. The continuum hypothesis nevertheless represents an example of a proposition which could be proven in a logical system that includes the Zermelo-Fraenkel axioms and the axiom of choice [22]. Given the independence of the continuum hypothesis from the ZFC axioms, the identification of the cardinalities $\aleph_0^2$ and $\aleph_1$ cannot be proven from this axiomatic system. By contrast, the equality of of $\aleph_0^B$, $B \geq 3$, with $\aleph_0^2$, and and the representation of real numbers by a set of cardinality $\aleph_0^2$, distinguishes the geometrical formulation of the completion of the rational numbers into $\mathbb{R}$, that does not introduce paradoxes related to certain ZFC axioms, since sets of cardinality $\aleph_2$ do not occur in the proof of Theorem 2.

The possibility of circumventing the undecidability of certain propositions in an axiomatic system, following Gödel's theorem [23], may be investigated through the introduction of a many-valued logic system characteristic of quantum theory. The description of quantum mechanics through wavefunctions that have a squared absolute value equal to the probability yields the prediction that there exist outcomes of experiments that occur with relative frequencies that are fractional. Statements about the truth or falsity of a physical property would be replaced by propositions on both possibilities being valid a priori.

A proposition stating the truth and falsity of a conjecture about a single outcome remains inconsistent. However, by the predicate calculus in two-valued logic, these propositions could be cast in the form of negations. Propositions in the many-valued logical system [24], which are allowed to have negative sentences, may be formally consistent, by contrast with two-valued logic.

The Riemann surfaces constitute a geometrical representation of a consistent quantum theory. The handles cause a bifurcation in the position coordinates of the string coordinate wavefunction. Surfaces with a set of ends of cardinality $\aleph_0^n$, $n \geq 2$, could be regarded as a manifestation of a logical system with n possible results. The equivalence of the cardinalities $\aleph_0^n$, $n \geq 2$, would be sufficient to restore consistency to the the defining set of axioms.

5. **Conclusion**

The continuum hypothesis is central to axiomatic set theory. The incompleteness of the propositional calculus of a logical system is evident in the absence of a proof of the continuum hypothesis.

A careful study of the cardinality of geometric sets reveals that geometry distinguishes between set theoretic equivalence of cardinalities. It may be demonstrated, for example, that the cardinality of the set of points on the sphere S^n equals $2^{n\aleph_0}$. The distinction between manifolds is reflected in an ordering of the cardinalities. Space-filling Peano curves would appear to represent an intermediate geometric construct between a curve of cardinality $2^{\aleph_0}$ and a planar domain of cardinality $2^{2\aleph_0}$.

A similar conclusion is reached for Riemann surfaces. The enumeration of the harmonic dimensions of the ideal boundary of a surface in the class O_G can be reformulated as the counting of accumulation points of handles on a sphere. A countable number of handles would be consistent with the vanishing of the harmonic measure of the ideal boundary. When the cardinality of the set of ends equals $2^{\aleph_0}$, the Riemann surface can be transformed to a fundamental region to a border arc representing a continuum for the ideal boundary. There is a large set of Riemann surfaces, however, that belong to the classes between O_G and Type II. The class O_{HD} includes Toki's surface, which has a set of ends of cardinality $\aleph_0^2$ in the range $[\aleph_0, 2^{\aleph_0}]$, which is the next cardinality above $\aleph_0$ and $\aleph_0^n$, $n > 2$, is a cardinality that may be identified with $\aleph_0^2$. The geometric sets of Theorem 2, defined by the removal of points with rational coordinates on sequences of semicircular arcs, have cardinalities equal to $\aleph_0^B$, $B \geq 3$m which must be identified with $\aleph_0^2$. There is a maximum of one cardinality between $\aleph_0$ and $2^{\aleph_0}$.

Various degrees of computability of subsets of real numbers have a connection with their cardinalities. The set $\hat{\mathrm{C}}$ is defined to be the real numbers such that their representation can be computed such that the n^{th} decimal place may be found through a countably infinite algorithm. This set consists of many transcendental numbers. Given two elements $\tau_1,\ \tau_2 \in \hat{\mathrm{C}}$, the number of arithmetical operations required for the evaluation the n^{th} decimal place of initially would be hypothesized to be uncountable. The distance from $\tau_1^{\tau_2}$ to a rational power of a rational power may be bounded, however, and, by the principle of induction, it may be demonstrated that only a countably infinite number of steps are necessary to calculate the n^{th} decimal place. Since sequences of transcendental powers of transcendental numbers are the most complex to compute, this result would be valid for all real numbers. There is a projection from $\mathbb{R}$ to $\hat{\mathrm{C}}$, which may be regarded as an isomorphism. By considering finite decimal expansions, and then letting the lengths tend to infinity, it may be established that the cardinality of $\hat{\mathrm{C}}$ is $\aleph_0^2$. The equality of the cardinalities of $\hat{\mathrm{C}}$ and $\mathbb{R}$ follows from the isomorphism between the two sets. All of the cardinalities of the form $\aleph_0^n$, $n \geq 2$, can be identified with $2^{\aleph_0}$. The upper of lower limits for 2^N in terms of powers of N, for every positive integer N greater than or equal to 2, prevents the existence of any cardinalities between $\{\aleph_0^n,\ n = 2,\ 3,\ 4,\ ...\}$ and $2^{\aleph_0}$, and the validity of the continuum hypothesis is established.

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