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A STUDY ON DEGENERATE HYPERBOLIC FUNCTIONS

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Abstract. Recently, degenerate hyperbolic functions have been widely studied as a degenerate versions of the hyperbolic functions. According to their definitions, this paper focuses on a few higher order degenerate hyperbolic functions to explore their properties. Specifically, we calculate some integral problems. Mainly comprising Volkenborn and Fermionic p-adic integrals. we also find some relational identities between these integrals and some polynomials and numbers.

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1. Introduction

In[1], Kim-Kim-Kim introduced degenerate hyperbolic functions as

$$\begin{aligned}\sinh_\lambda(x:a)&=\frac{e_\lambda^x(a)-e_\lambda^{-x}(a)}{2},\\ \cosh_\lambda(x:a)&=\frac{e_\lambda^x(a)+e_\lambda^{-x}(a)}{2},\\ \tanh_\lambda(x:a)&=\frac{e_\lambda^x(a)-e_\lambda^{-x}(a)}{e_\lambda^x(a)+e_\lambda^{-x}(a)}.\end{aligned}\tag{1}$$

The degnerate exponential fucition is given by

$$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!},\quad (\text{see}[1,2,4]),\tag{2}$$

where $(x)_{0,\lambda}=1$, $(x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$, $(n\geq 1)$.

Let p be a fixed odd prime number. In this paper, $\mathbb{Z}_p,\mathbb{Q}_p$ and $\mathbb{C}_p$ are the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$ accordingly. The p-adic norm is denoted as $|p|_p=\frac{1}{p}$. The Fermionic p-adic integral of f on $\mathbb{Z}_p$ was introduced by Kim [5,6] as

$$\begin{aligned}\int_{\mathrm{Z}_p}f(x)d\mu_{-1}(x)&=\lim_{N\to\infty}\sum_{x=0}^{p^N-1}f(x)\mu_{-1}(x+p^N\mathrm{Z}_p)\\ &=\lim_{N\to\infty}\sum_{x=0}^{p^N-1}f(x)(-1)^x,\quad (\text{see}[1,5,6,7]).\end{aligned}\tag{3}$$

From (3), we get

$$(4)\qquad \int_{\mathbb{Z}_p} f(x+1)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = 2f(0), \quad (\text{see}[1,5,6]).$$

The Volkenborn integral on $\mathbb{Z}_p$ is defined by

$$(5)\qquad \begin{aligned}\int_{\mathbb{Z}_p} f(x)d\mu(x) &= \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)\mu(x+p^N\mathbb{Z}_p)\\ &= \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x), \quad (\text{see}[1,5,6,7]).\end{aligned}$$

From (5), we obtain

$$(6)\qquad \int_{\mathbb{Z}_p} f(x+1)d\mu(x) - \int_{\mathbb{Z}_p} f(x)d\mu(x) = f'(0).$$

From (6), we observe that

$$(7)\qquad \int_{\mathbb{Z}_p} e_\lambda^x(t)d\mu(x) = \frac{\frac{1}{\lambda}log(1+\lambda t)}{e_\lambda(t)-1} = \sum_{n=0}^{\infty}\beta_{n,\lambda}\frac{t^n}{n!}, \quad (\text{see}[1,2]),$$

where $\beta_{n,\lambda}$ are called fully degenerate Bernoulli numbers.

From (4), we get

$$(8)\qquad \int_{\mathbb{Z}_p} e_\lambda^x(t)d\mu_{-1}(x) = \frac{2}{e_\lambda(t)+1} = \sum_{n=0}^{\infty}\epsilon_{n,\lambda}\frac{t^n}{n!}, \quad (\text{see}[1,2]),$$

where $\epsilon_{n,\lambda}$ are called degenerate Euler numbers.

The generating function of the Stirling numbers of the first kind is given by

$$(9)\qquad \frac{1}{k!}(log(1+t))^k = \sum_{n=0}^{\infty}S_1(n,k)\frac{t^n}{n!}, \quad (\text{see}[3]).$$

2. Degenerate hyperbolic functions

In this paper, we investigate the integrals of degenerate hyperbolic functions. Specifically, we calculate Volkenborn and Fermionic p-adic integrals of the higher-order r degnerate hyperbolic functions. Consider hyperbolic functions of higher-order $r \in \mathbb{Z}$ which are given by

$$(10)\qquad \begin{aligned}\sinh_\lambda^r(x:a) &= \left(\frac{e_\lambda^x(a)-e_\lambda^{-x}(a)}{2}\right)^r,\\ \cosh_\lambda^r(x:a) &= \left(\frac{e_\lambda^x(a)+e_\lambda^{-x}(a)}{2}\right)^r,\\ \tanh_\lambda^r(x:a) &= \left(\frac{\sinh_\lambda(x:a)}{\cosh_\lambda(x:a)}\right)^r.\end{aligned}$$

From (2), we note that

$$e_\lambda^{kx}(a) = e_{\frac{\lambda}{k}}^{x}(ka) = \sum_{n=0}^{\infty}(x)_{n,\frac{\lambda}{k}} k^n \frac{a^n}{n!}. \tag{11}$$

When $k = 0$, equation (11) becomes $e_\lambda^0(a) = 1$.

From (10), we observe

$$\begin{aligned} \int_{\mathbb{Z}_p} \sinh_\lambda^r(x:a)d\mu(x) &= \int_{\mathbb{Z}_p} \left(\frac{e_\lambda^x(a) - e_\lambda^{-x}(a)}{2}\right)^r d\mu(x) \\ &= \frac{1}{2^r}\int_{\mathbb{Z}_p}\sum_{k=0}^{r}\binom{r}{k}(-1)^k e_\lambda^{(r-2k)x}(a)d\mu(x) \\ &= \frac{1}{2^r}\sum_{k=0}^{r}\binom{r}{k}(-1)^k \int_{\mathbb{Z}_p} e_\lambda^{(r-2k)x}(a)d\mu(x) \\ &= \frac{1}{2^r}\sum_{k=0}^{r}\binom{r}{k}(-1)^k \sum_{n=0}^{\infty}\beta_{m,\frac{\lambda}{r-2k}}(r-2k)^n\frac{a^n}{n!}. \end{aligned} \tag{12}$$

Thus, we obtain the following theorem.

Theorem 2.1. *For $n, k, \geq 0$ and $r \in \mathbb{Z}$, we have*

$$\int_{\mathbb{Z}_p} \sinh_\lambda^r(x:a)d\mu(x) = \frac{1}{2^r}\sum_{n=0}^{\infty}\sum_{k=0}^{r}\binom{r}{k}(-1)^k\beta_{n,\frac{\lambda}{r-2k}}(r-2k)^n\frac{a^n}{n!}.$$

By (10), we have

$$\begin{aligned} \int_{\mathbb{Z}_p} \cosh_\lambda^r(x:a)d\mu(x) &= \int_{\mathbb{Z}_p} \left(\frac{e_\lambda^x(a) + e_\lambda^{-x}(a)}{2}\right)^r d\mu(x) \\ &= \frac{1}{2^r}\int_{\mathbb{Z}_p}\sum_{k=0}^{r}\binom{r}{k} e_\lambda^{(r-2k)x}(a)d\mu(x) \\ &= \frac{1}{2^r}\sum_{k=0}^{r}\binom{r}{k} \int_{\mathbb{Z}_p} e_\lambda^{(r-2k)x}(a)d\mu(x) \\ &= \frac{1}{2^r}\sum_{k=0}^{r}\binom{r}{k} \sum_{n=0}^{\infty}\beta_{m,\frac{\lambda}{r-2k}}(r-2k)^n\frac{a^n}{n!}. \end{aligned} \tag{13}$$

Thus, we get the following theorem.

Theorem 2.2. *For $n, k, \geq 0$ and $r \in \mathbb{Z}$, we have*

$$\int_{\mathbb{Z}_p} \cosh_\lambda^r(x:a)d\mu(x) = \frac{1}{2^r}\sum_{n=0}^{\infty}\sum_{k=0}^{r}\binom{r}{k}\beta_{n,\frac{\lambda}{r-2k}}(r-2k)^n\frac{a^n}{n!}.$$

From (10), Theorem 2.1 and Theorem 2.2, we have

$$
\begin{aligned}
(14)\quad \int_{\mathbb{Z}_p} \tanh_\lambda^r(x:a)d\mu(x) &= \int_{\mathbb{Z}_p} \left(\frac{\sinh_\lambda^r(x:a)}{\cosh_\lambda^r(x:a)}\right) d\mu(x) \\
&= \int_{\mathbb{Z}_p} \left(\frac{e_\lambda^x(a)-e_\lambda^{-x}(a)}{2}\right)^r \left(\frac{e_\lambda^x(a)+e_\lambda^{-x}(a)}{2}\right)^{-r} d\mu(x) \\
&= \int_{\mathbb{Z}_p} \sum_{m=0}^{r} \binom{r}{m} e_\lambda^{(r-2m)x}(a) \sum_{l=0}^{r} \binom{r+l-1}{l}(-1)^l e_\lambda^{(-r-2l)x}(a)d\mu(x) \\
&= \sum_{m=0}^{r}\sum_{l=0}^{r} \binom{r}{m}\binom{r+l-1}{l}(-1)^l \int_{\mathbb{Z}_p} e_\lambda^{(-2l-2m)x}(a)d\mu(x) \\
&= \sum_{m=0}^{r}\sum_{l=0}^{r} \binom{r}{m}\binom{r+l-1}{l}(-1)^l \sum_{n=0}^{\infty} \beta_{n,\frac{\lambda}{-2l-2m}}(-2l-2m)^n \frac{a^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{r}\sum_{l=0}^{r} \binom{r}{m}\binom{r+l-1}{l}(-1)^l \beta_{n,\frac{\lambda}{-2l-2m}}(-2l-2m)^n \frac{a^n}{n!}.
\end{aligned}
$$

On the other hand,

(15)

$$
\begin{aligned}
\int_{\mathbb{Z}_p} \tanh_\lambda^r(x:a)d\mu(x) &= \int_{\mathbb{Z}_p} \left(1-\frac{2}{e_\lambda^{2x}(a)}\right)^r d\mu(x) \\
&= \int_{\mathbb{Z}_p} \sum_{k=0}^{r} \binom{r}{k} \left(\frac{-2}{e_\lambda^{2x}(a)+1}\right)^k d\mu(x) \\
&= \int_{\mathbb{Z}_p} \sum_{k=0}^{r} \binom{r}{k}(-2)^k \sum_{m=0}^{k} \binom{k}{m} e_\lambda^{-2kx}(a)d\mu(x) \\
&= \sum_{k=0}^{r} \binom{r}{k}(-2)^k \sum_{m=0}^{k} \binom{k}{m} \sum_{n=0}^{\infty} \beta_{n,\frac{\lambda}{-2k}}(-2k)^n \frac{a^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{r}\sum_{m=0}^{k} (-2)^{n+k}k^n \binom{r}{k}\binom{k}{m} \beta_{n,\frac{\lambda}{-2k}} \frac{a^n}{n!}.
\end{aligned}
$$

By comparing the coefficients on both sides of (14) and (15), we arrive at the following result.

Theorem 2.3. *For $n, k \geq 0$, $r \in \mathbb{Z}$, we have*

$$
\sum_{m=0}^{r}\sum_{l=0}^{r} \binom{r}{m}\binom{r+l-1}{l}(-1)^l \beta_{n,\frac{\lambda}{-2l-2m}}(-2l-2m)^n = \sum_{k=0}^{r}\sum_{m=0}^{r} (-2)^{n+k}k^n \binom{r}{k}\binom{k}{m} \beta_{n,\frac{\lambda}{-2k}}.
$$

Now, we observe hyperbolic sine function

$$\begin{aligned}
(16) \qquad \sinh_\lambda(x:a) &= \left(\frac{e_\lambda^x(a)-e_\lambda^{-x}(a)}{2}\right)\\
&= \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{log(1+\lambda a)}{\lambda}\right)^k - \left(-\frac{log(1+\lambda a)}{\lambda}\right)^k\frac{x^k}{k!}\\
&= \sum_{k=0}^{\infty}\left(\frac{log(1+\lambda a)}{\lambda}\right)^{2k+1}\frac{x^{2k+1}}{(2k+1)!}\\
&= \sum_{k=0}^{\infty}\frac{1}{\lambda^{2k+1}}\sum_{n=2k+1}^{\infty}S_1(n,2k+1)\frac{(\lambda a)^n}{n!}x^{2k+1}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n-1}{2}]}\lambda^{n-2k-1}S_1(n,2k+1)x^{2k+1}\frac{a^n}{n!}.
\end{aligned}$$

From (10), we note that

$$(17) \qquad \frac{d}{dx}\sinh_\lambda(x:a) = \frac{1}{\lambda}log(1+\lambda a)\cosh_\lambda(x:a).$$

From (16) and (17), we have

$$\begin{aligned}
(18) \qquad \frac{d}{dx}\sinh_\lambda(x:a) &= \frac{d}{dx}\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n-1}{2}]}\lambda^{n-2k-1}S_1(n,2k+1)x^{2k+1}\frac{a^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n-1}{2}]}(2k+1)\lambda^{n-2k-1}S_1(n,2k+1)x^{2k}\frac{a^n}{n!}.
\end{aligned}$$

On the other hand in (17)

(19)

$$\begin{aligned}
\frac{1}{\lambda}log(1+\lambda a)\cosh_\lambda(x:a) &= \cosh_\lambda(x:a)(e_\lambda(a)-1)\sum_{m=0}^{\infty}\beta_{m,\lambda}\frac{a^m}{m!}\\
&= \cosh_\lambda(x:a)\left(\sum_{l=0}^{\infty}(1)_{l,\lambda}\frac{a^l}{l!}\sum_{m=0}^{\infty}\beta_{m,\lambda}\frac{a^m}{m!} - \sum_{n=0}^{\infty}\beta_{n,\lambda}\frac{a^n}{n!}\right)\\
&= \cosh_\lambda(x:a)\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}(1)_{n-m,\lambda}\beta_{m,\lambda} - \beta_{n,\lambda}\right)\frac{a^n}{n!}.
\end{aligned}$$

Thus, by comparing the coeffients on both sides of (18) and (19), we get the following theorem.

Theorem 2.4. *For $n,k \geq 0$, we have*

$$\left(\sum_{m=0}^{n}\binom{n}{m}(1)_{n-m,\lambda}\beta_{m,\lambda} - \beta_{n,\lambda}\right)\cosh_\lambda(x:a) = \sum_{k=0}^{[\frac{n-1}{2}]}(2k+1)\lambda^{n-2k-1}S_1(n,2k+1)x^{2k}.$$

Next, we observe degenerate hyperbolic consine function

(20)

$$
\begin{aligned}
&\cosh_\lambda(x:a) = \left(\frac{e_\lambda^x(a)+e_\lambda^{-x}(a)}{2}\right)\\
&= \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{log(1+\lambda a)}{\lambda}\right)^k + \left(-\frac{log(1+\lambda a)}{\lambda}\right)^k \frac{x^k}{k!}\\
&= \sum_{k=0}^{\infty}\left(\frac{log(1+\lambda a)}{\lambda}\right)^{2k} \frac{x^{2k}}{(2k)!}\\
&= \sum_{m=0}^{\infty}\left(\sum_{l_1=1}^{\infty}(l_1-1)!(-\lambda)^{l_1-1}\frac{a^{l_1}}{l_1!}\right)\cdots\left(\sum_{l_{2m}=1}^{\infty}(l_{2m}-1)!(-\lambda)^{l_{2m}-1}\frac{a^{l_{2m}}}{l_{2m}!}\right)\frac{x^{2m}}{2m!}\\
&= \sum_{m=0}^{\infty}\sum_{n=2m}^{\infty}\sum_{l_1+\cdots+l_{2m}=n}\frac{(l_1-1)!\cdots(l_{2m}-1)!(-\lambda)^{n-2m}}{l_1!\cdots l_{2m}!}\frac{a^n}{n!}\frac{x^{2m}}{2m!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{l_1\cdots l_{2m}=n}\sum_{m=0}^{[\frac{n}{2}]}\binom{n}{l_1,\cdots,l_{2m}}(l_1-1)!\cdots(l_{2m}-1)!(\lambda)^{n-2m}\frac{x^{2m}}{2m!}\right)\frac{a^n}{n!}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\cosh_\lambda(x:a) &= \frac{e_\lambda^x(a)+e_\lambda^{-x}(a)}{2}\\
&= \frac{1}{2}\sum_{n=0}^{\infty}\left((x)_{n,\lambda}+(-1)^n < x >_{n,\lambda}\right)\frac{a^n}{n!}.
\end{aligned}
\tag{21}
$$

By comparing coefficients on both sides of (20) and (21), we have the following theorem.

Theorem 2.5. *For $n, m \geq 0$, we have*

$$
(x)_{n,\lambda}+(-1)^n < x >_{n,\lambda} = 2\sum_{l_1\cdots l_{2m}=n}\sum_{m=0}^{[\frac{n}{2}]}\binom{n}{l_1,\cdots,l_{2m}}(l_1-1)!\cdots(l_{2m}-1)!\lambda^{n-2m}\frac{x^{2m}}{2m!}.
$$

Now, we observe hyperbolic functions on fermionic integral

$$(22)\quad \int_{\mathbb{Z}_p} \sinh_\lambda^r(x:a)d\mu_{-1}(x) = \int_{\mathbb{Z}_p} \left(\frac{e_\lambda^x(a) - e_\lambda^{-x}(a)}{2}\right)^r d\mu_{-1}(x)$$
$$= \frac{1}{2^r}\int_{\mathbb{Z}_p} \sum_{k=0}^{r} \binom{r}{k}(-1)^k e_\lambda^{(r-2k)x}(a)d\mu_{-1}(x)$$
$$= \frac{1}{2^r}\sum_{k=0}^{r} \binom{r}{k}(-1)^k \int_{\mathbb{Z}_p} e_\lambda^{(r-2k)x}(a)d\mu_{-1}(x)$$
$$= \frac{1}{2^r}\sum_{k=0}^{r} \binom{r}{k}(-1)^k \sum_{n=0}^{\infty} \epsilon_{m,\frac{\lambda}{r-2k}}(r-2k)^n \frac{a^n}{n!}.$$

Thus, we have following theorem.

Theorem 2.6. *For $n, r \geq 0$, we have*

$$\int_{\mathbb{Z}_p} \sinh_\lambda^r(x:a)d\mu_{-1}(x) = \frac{1}{2^r}\sum_{n=0}^{\infty}\sum_{k=0}^{r} \binom{r}{k}(-1)^k \epsilon_{m,\frac{\lambda}{r-2k}}(r-2k)^n \frac{a^n}{n!}.$$

By (10), we have

$$(23)\quad \int_{\mathbb{Z}_p} \cosh_\lambda^r(x:a)d\mu_{-1}(x) = \int_{\mathbb{Z}_p} \left(\frac{e_\lambda^x(a) + e_\lambda^{-x}(a)}{2}\right)^r d\mu_{-1}(x)$$
$$= \frac{1}{2^r}\int_{\mathbb{Z}_p} \sum_{k=0}^{r} \binom{r}{k} e_\lambda^{(r-2k)x}(a)d\mu_{-1}(x)$$
$$= \frac{1}{2^r}\sum_{k=0}^{r} \binom{r}{k} \int_{\mathbb{Z}_p} e_\lambda^{(r-2k)x}(a)d\mu_{-1}(x)$$
$$= \frac{1}{2^r}\sum_{k=0}^{r} \binom{r}{k} \sum_{n=0}^{\infty} \epsilon_{m,\frac{\lambda}{r-2k}}(r-2k)^n \frac{a^n}{n!}.$$

Thus, we have following theorem.

Theorem 2.7. *For $n, r, k \geq 0$, we have*

$$\int_{\mathbb{Z}_p} \cosh_\lambda^r(x:a)d\mu_{-1}(x) = \frac{1}{2^r}\sum_{n=0}^{\infty}\sum_{k=0}^{r} \binom{r}{k} \epsilon_{m,\frac{\lambda}{r-2k}}(r-2k)^n \frac{a^n}{n!}.$$

From degenerate hyperbolic cosine function, we get

$$(24)\quad \cosh_\lambda(x:a) = \frac{e_\lambda^x(a) + e_\lambda^{-x}(a)}{2}$$
$$= \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{log(1+\lambda a)}{\lambda}\right)^n + \left(-\frac{log(1+\lambda a)}{\lambda}\right)^n \frac{x^n}{n!}$$
$$= \sum_{n=0}^{\infty} \left(\frac{log(1+\lambda a)}{\lambda}\right)^{2n} \frac{x^{2n}}{2n!}.$$

Replacing in (24) x by $\cosh_\lambda(x:a)$ and observing then Fermionic integral, we get

$$\begin{aligned}(25)\qquad &\int_{\mathbb{Z}_p}\cosh_\lambda(\cosh_\lambda(x:a):a)d\mu_{-1}(x)\\
&=\int_{\mathbb{Z}_p}\sum_{r=0}^{\infty}\left(\frac{log(1+\lambda a)}{\lambda}\right)^{2r}\frac{\cosh_\lambda(x:a)^{2r}}{2r!}d\mu_{-1}(x)\\
&=\sum_{r=0}^{\infty}\left(\frac{log(1+\lambda a)}{\lambda}\right)^{2r}\frac{1}{2r!}\int_{\mathbb{Z}_p}\cosh_\lambda(x:a)^{2r}d\mu_{-1}(x)\\
&=\sum_{r=0}^{\infty}\left(\frac{log(1+\lambda a)}{\lambda}\right)^{2r}\frac{1}{2r!}\sum_{l=0}^{\infty}\sum_{k=0}^{2r}\binom{2r}{k}\epsilon_{l,\frac{\lambda}{2r-2k}}(2r-2k)^l\frac{a^l}{l!}\\
&=\sum_{r=0}^{\infty}\frac{1}{\lambda^{2r}}\sum_{m=2r}^{\infty}S_1(m,2r)\frac{(\lambda a)^m}{m!}\sum_{l=0}^{\infty}\sum_{k=0}^{2r}\binom{2r}{k}\epsilon_{l,\frac{\lambda}{2r-2k}}(2r-2k)^l\frac{a^l}{l!}\\
&=\sum_{m=0}^{\infty}\sum_{r=0}^{[\frac{m}{2}]}\lambda^{m-2r}S_1(m,2r)\frac{a^m}{m!}\sum_{l=0}^{\infty}\sum_{k=0}^{2r}\binom{2r}{k}\epsilon_{l,\frac{\lambda}{2r-2k}}(2r-2k)^l\frac{a^l}{l!}.\end{aligned}$$

Thus, we obtain the following theorem.

Theorem 2.8. *For $n,r,m\geq 0$, we have*

$$\int_{\mathbb{Z}_p}\cosh_\lambda(\cosh_\lambda(x:a):a)d\mu_{-1}(x)$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{r=0}^{[\frac{m}{2}]}\sum_{k=0}^{2r}\binom{n}{m}\binom{2r}{k}\lambda^{m-2r}S_1(m,2r)\epsilon_{n-m,\frac{\lambda}{2r-2k}}(2r-2k)^{n-m}\frac{a^n}{n!}.$$

By replacing x by $\sinh_\lambda(x:a)$ in (16) and then taking Fermionic integral, we get

$$\begin{aligned}(26)\qquad &\int_{\mathbb{Z}_p}\sinh_\lambda(\sinh_\lambda(x:a):a)d\mu_{-1}(x)\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n-1}{2}]}\lambda^{n-2k-1}S_1(n,2k+1)\int_{\mathbb{Z}_p}\sinh_\lambda^{2k+1}(x:a)\frac{a^n}{n!}.\end{aligned}$$

From Theorem 2.6 and (26), we have the following theorem.

Theorem 2.9. *For $n,r,l\geq 0$, we have*

$$\int_{\mathbb{Z}_p}\sinh_\lambda(\sinh_\lambda(x:a):a)d\mu_{-1}(x)$$

$$= \frac{1}{2^{(2r+1)}} \sum_{n=0}^{\infty} \sum_{l=0}^{n} \sum_{k=0}^{2r+1} \sum_{k=0}^{[\frac{l-1}{2}]} \binom{n}{l} \binom{2r+1}{k} (-1)^k \lambda^{l-2k-1}$$
$$\times S_1(l, 2k+1) \epsilon_{n-l, \frac{\lambda}{2r-2k+1}} (2r-2k+1)^{n-l} \frac{a^n}{n!}.$$

3. Conclusion

In this paper, we considered the integrals of degenerate hyperbolic functions. By means of the definition of three kinds of degenerate hyperbolic functions, as well as Volkenborn and Fermionic p-adic integrals, we introduced the degenerate hyperbolic functions of higher order r. And we derived some new results and identities of these types. In more deail, we calculate the Volkenborn and the Fermionic p-adic integrals of the degenerate hyeprbolic sine, cosine and tangent functions of higher order r. Furthermore, we established the relationship between the degenerate fully Bernoulli and Euler polynomials. Our results provide some new ideas and approaches.

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