Advanced Studies in Contemporary Mathematics
35 (2025), No. 1, pp. 5 - 16

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2025.35.1.5

ON THE DEGREE OF APPROXIMATION OF SIGNALS(FUNCTIONS) USING $(E, l)(E, l)(C, 1)$ TRIPLE PRODUCT MEANS OF FOURIER SERIES

Jitendra Kumar Kushwaha*
Department of Mathematics and Statistics,
DDU Gorakhpur University, Gorakhpur
Laxmi Rathour†
Department of Mathematics
National Institute of Technology
Chaltlang, Aizawl 796012
Mizoram, India
Lakshmi Narayan Mishra ‡
Department of Mathematics
Vellore Institute of Technology,Vellore-632014,
Tamil Nadu, India
Vishnu Narayan Mishra§
Department of Mathematics,
Indira Gandhi National Tribal University,Lalpur,
Amarkantak-484887, M.P.
Radha Vishwakarma¶
Department of Mathematics and Statistics,
DDU Gorakhpur University, Gorakhpur

Abstract: In this paper, we have established a very interesting result for the degree of approximation of signals by $(E, l)(E, l)(C, 1)$ triple product summability Method. The results presented in this paper is the generalization of many known and Unknown results.

Mathematics Subject Classification: 2000
Keywords and Phrases: Signals (functions), $(C, 1)$ means, (E, l) means, $(E, l)(C, 1)$ Product Summability Method, $(E, l)(E, l)(C, 1)$ Triple Product Summability Method, Lip$(\zeta(t), r)$ class, Fourier Series.

*k.jitendrakumar@yahoo.com
†laxmirathour817@gmail.com
‡lakshminarayanmishra04@gmail.com
§vishnunarayanmishra@gmail.com
¶vishwakarmaradha89@gmail.com

Submission date- 8/10/2024

1 Introduction

Recently, summability theory plays a significant role to study area of Fourier analysis, Wavelet analysis, Fixed point theory and many other fields . The degree of approximation of error of functions belonging to various classes have been determined by various investigators. Recently, , Lal and Kushwaha [11], Nigam and Sharma([5]&[6]), Kushwaha etal [9]. Kushwaha and Vishwakarma [8] and many others are work in this direction of approximation by product summability method of Fourier series. Devaiya and Srivastava [13] and Sonker and Sangwan [12] have determined the approximation of error of function of different type of product means of Fourier series and its conjugate.
Now working in this direction, we have determined the degree of approximation of function by $(E, l)(E, l)(C, 1)$ triple product summability method of series. Therefore, this result will be useful for researchers in future.

2 Definition and Notations

Let $\sum_{\nu=0}^{\infty} u_\nu$ be a given infinite series with sequence of its ν^{th} partial sum $\{s_\nu\}$.
Let $\{\Omega_\eta^{E_l}\}$ denote the sequence of $(E, l) = E_\eta^l$ means of the sequence $\{s_\nu\}$. If the (E, l) transform of $\{s_\nu\}$ is defined as

$$\Omega_\eta^{E_l}(\zeta; x) = \frac{1}{(1+l)^\eta} \sum_{\nu=0}^{\eta} \binom{\eta}{\nu} l^{\eta-\nu} s_\nu \to s \quad \text{as} \quad \eta \to \infty \tag{2.1}$$

Then the series $\sum_{\nu=0}^{\infty} u_\nu$ is said to be summable to the number s by (E, l) method. (Hardy)
Let $\{\Omega_\eta^{C_1}\}$ denote the sequence of $(C, 1) = C_\eta^1$ mean of the sequence $\{s_\nu\}$.
If the $(C, 1)$ transform of $\{s_\nu\}$ is defined as

$$\Omega_\eta^{C_1}(\zeta; x) = \frac{1}{\eta+1} \sum_{\nu=0}^{\eta} s_\nu(\zeta; x) \to s \quad \text{as} \quad \eta \to \infty \tag{2.2}$$

Then the series $\sum_{\nu=0}^{\infty} u_\nu$ is said to be summable to the number $'s'$ by the $(C, 1)$ method. (Cesàro)
Thus if (E, l) transform of $(C, 1)$ transform defines $(E, l)(C, 1)$ transformation and denoted by $E_\eta^l C_\eta^1$ then if

$$\Omega_\eta^{E_l C_1}(\zeta; x) = \frac{1}{(1+l)^\eta} \sum_{\nu=0}^{\eta} \binom{\eta}{\nu} l^{\eta-\nu} \frac{1}{(\eta+1)} \sum_{h=0}^{\eta} s_h \to s \quad \text{as} \quad \eta \to \infty \tag{2.3}$$

where $\Omega_\eta^{E_l C_1}$ denotes the sequence of $E_l C_1$ means that is $(E,l)(C,1)$ product means of the sequence $\{s_\nu\}$.
Then the series $\sum_{\nu=0}^{\infty} u_\nu$ is said to be summable to the number $'s'$ by the $(E,l)(C,1)$ method.
Now again if (E,l) transformation of $(E,l)(C,1)$ transformation defines $(E,l)(E,l)(C,1)$ transformation and denoted by $E_\eta^l E_\eta^l C_\eta^1$ then if

$$\Omega_\eta^{E_l E_l C_1}(\zeta;x) = \frac{1}{(1+l)^\eta}\sum_{\nu=0}^{\eta}\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^\nu}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{(h+1)}\sum_{k=0}^{h} s_k \to s \text{ as } \eta\to\infty \quad (2.4)$$

where $\Omega_\eta^{E_l E_l C_1}$ denotes the sequence of $E_l E_l C_1$ means that is $(E,l)(E,l)(C,1)$ product means of the sequence $\{s_\nu\}$.
Then the series $\sum_{\nu=0}^{\infty} u_\nu$ is said to be summable to the number $'s'$ by the $(E,l)(E,l)(C,1)$ method.
Let $\chi(x)$ be 2π- periodic, Lebesgue integrable function on $[-\pi,\pi]$ then its Fourier series associated with a point x is defined by

$$\chi(x) = \frac{1}{2}a_0 + \sum_{\nu=1}^{\infty}(a_\nu\cos\nu x + b_\nu\sin\nu x) = \sum_{\nu=0}^{\infty}A_\nu, \qquad \nu\in N \qquad (2.5)$$

is called the Fourier series with ν^{th} partial sum $s_\nu(\chi;x)$.
The conjugate series of Fourier series (5) is given by

$$\overline{\chi(x)} = \sum_{k=1}^{\infty}(a_n\sin nx - b_n\cos nx) = \sum_{\nu=0}^{\infty}B_\nu, \qquad \nu\in N \qquad (2.6)$$

We use following notations throughout the paper

$$\chi_x(t) = \chi(x+t)+\chi(x-t)-2\chi(t)$$
$$\overline{\chi_x(t)} = \frac{1}{2}\{\chi(x+t)-\chi(x-t)\}$$

we also write

$$\kappa_\eta(t) = \frac{1}{2\pi(1+l)^\eta}\sum_{\nu=0}^{\eta}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^\nu}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{(h+1)}\left\{\sum_{k=0}^{h}\frac{\sin(k+1/2)t}{\sin t/2}\right\}\right] \quad (2.7)$$

$$\overline{\kappa}_\eta(t) = \frac{1}{2\pi(1+l)^\eta}\sum_{\nu=0}^{\eta}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^\nu}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{(h+1)}\left\{\sum_{k=0}^{h}\frac{\cos(k+1/2)t}{\sin t/2}\right\}\right] \quad (2.8)$$

and L_r-norm is defined by

$$\|\chi\|_r = \left(\int_0^{2\pi}|\chi(x)|^r dx\right)^{1/r}, \qquad r\geq 1$$

and the estimation of errors which is known as degree of approximation of a function ζ given by Zygmund .

$$E_\eta(x) = min\|\Delta_\eta(x)-\chi(x)\|_r$$

where $\Delta_\eta(x)$ is some η^{th} degree trigonometric polynomial. This method of approximation is called the trigonometric Fourier Approximation.
A function $\chi \in Lip\ \alpha$ if

$$|\chi(x+t)-\chi(x)| = O(|t|^\alpha) \quad \text{for} \quad 0<\alpha\leq 1, t>0$$

and the function $\chi \in Lip(\alpha, r)$ if

$$\left(\int_0^{2\pi} |\chi(x+t)-\chi(x)|^r dx\right)^{1/r} = O(|t|^\alpha) \quad \text{for} \quad 0<\alpha\leq 1, t>0.$$

Given a positive increasing function $\zeta(t)$ and an integer $r\geq 1,\ \chi \in Lip\{\zeta(t), r\}$.

$$\left(\int_0^{2\pi} |\chi(x+t)-\chi(x)|^r dx\right)^{1/r} = O\{\zeta(t)\}.$$

If $\zeta(t)=t^\alpha$ then $Lip(\zeta(t), r)$ class coincides with the $Lip(\alpha, r)$ class and if $r\to\infty$ then $Lip(\alpha, r)$ class reduces to $Lip\ \alpha$ class.
A function $\chi \in W(L^r, \zeta(t))$ if

$$\left(\int_0^{2\pi} |\chi(x+t)-\chi(x)|^r \sin^{\beta r}(t/2) dx\right)^{1/r} = O\{\zeta(t)\}, \quad \beta\geq 0, \quad r\geq 1, \quad t>0.$$

where $\chi(t)$ is increasing function of t.
we observe that

$$Lip\alpha \subseteq Lip\,(\alpha, r) \subseteq Lip\,(\zeta(t), r) \subseteq W\,(L^r, \zeta(t)) \quad \text{for} \quad 0<\alpha\leq 1, \quad r\geq 1.$$

3 Main Theorems

The aim of this study is to generalize the theorem of Saxena and Prabhakar [10], Devaiya and Srivastva [13], Sonker and Sangwan [12].

Theorem 3.1: Let $\{p_\eta\}$ is positive sequence which is monotonic and non-increasing with real constants.

$$P_\eta = \sum_{w=0}^{\eta} p_w \to \infty \qquad \text{as} \qquad \eta\to\infty$$

If χ satisfy the condition as below

$$\chi(t) = \int_0^t |\chi(u)|du = o\left[\frac{t}{\beta\left(\frac{1}{t}\right).P_t}\right] \qquad \text{as} \qquad t\to 0 \tag{3.1}$$

Provided β is positive, non-increasing and monotonic function of t

$$\log\eta = O\left[\{\beta(\eta)\}.P_\eta\right], \qquad \text{as} \qquad \eta\to\infty \tag{3.2}$$

the approximation of function at $x = t$ using triple product means of its Fourier series is given by

$$\left|\Omega_\eta^{E^l.E^l.C^1} - \chi_x(x)\right| \quad = \quad O(1) \qquad \text{as} \qquad \eta \to \infty$$

Theorem 3.2: Let $\{p_\eta\}$ is positive sequence which is monotonic and non-increasing with real constants.

$$P_\eta = \sum_{w=0}^{\eta} p_w \to \infty \qquad \text{as} \qquad \eta \to \infty$$

If $\overline{\chi}$ satisfy the conditions as below

$$\overline{\chi}(t) = \int_0^t |\chi(u)|du = o\left[\frac{t}{\beta\left(\frac{1}{t}\right).P_t}\right] \qquad \text{as} \qquad t \to 0 \tag{3.3}$$

Provided β is positive, non-increasing and monotonic function of t

$$\log\eta = O\left[\{\beta(\eta)\}.P_\eta\right], \qquad \text{as} \qquad \eta \to \infty \tag{3.4}$$

then approximation of the function at $x = t$ using triple product mean is given by

$$\left|\overline{\Omega_\eta^{E^l.E^l.C^1}} - \overline{\chi_x(x)}\right| \quad = \quad O(1) \qquad \text{as} \qquad \eta \to \infty$$

Where $\overline{\Omega_\eta^{E^lE^lC^1}}$ denotes $\overline{(E,l)(E,1)(C,1)}$ transform of partial sums of the series (2.6)

4 Lemma

For proving the main theorems the significant lemmas are as given below.

Lemma 4.1 $|\kappa_\eta(t)| = O(\eta) \quad for \quad 0 \le t \le \frac{1}{\eta};\ \sin\eta t \le \eta\sin\eta t$

proof 4.1:

$$\begin{aligned}
|\kappa_\eta(t)| &\le \frac{1}{2\pi(1+l)^\eta}\left|\sum_{\nu=0}^{\eta}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^\nu}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{(h+1)}\left\{\sum_{k=0}^{h}\frac{\sin(\nu+1/2)t}{\sin t/2}\right\}\right]\right| \\
&\le \frac{1}{2\pi(1+l)^\eta}\sum_{\nu=0}^{\eta}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^\nu}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{h+1}\left\{\sum_{k=0}^{h}\frac{(2\nu+1)\sin(t/2)}{\sin t/2}\right\}\right] \\
&\le \frac{1}{2\pi(1+l)^\eta}\sum_{\nu=0}^{\eta}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^\nu}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{(h+1)}.(2h+1)\sum_{k=0}^{h}1\right] \\
&\le \frac{1}{2\pi(1+l)^\eta}\sum_{\nu=0}^{\eta}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^\nu}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{(h+1)}.(2h+1)(h+1)\right] \\
&\le \frac{1}{2\pi(1+l)^\eta}\sum_{\nu=0}^{\eta}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^\nu}\sum_{h=0}^{\nu}\binom{\nu}{h}(2h+1)\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2\pi(1+l)^{\eta}}\sum_{\nu=0}^{\eta}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^{\nu}}(2\nu+1)\sum_{h=0}^{\nu}\binom{\nu}{h}\right]\\
&= \frac{1}{2\pi(1+l)^{\eta}}\sum_{\nu=0}^{\eta}\binom{\eta}{\nu}(2\nu+1)2^{\nu}\\
&= \frac{1}{2\pi(1+l)^{\eta}}(2\nu+1)2^{\nu}\sum_{\nu=0}^{\eta}\binom{\eta}{\nu}\\
&= O(\eta)
\end{aligned}
$$

Lemma 4.2 $|\kappa_\eta(t)| = O(\frac{1}{t}),$ *for* $\frac{1}{\eta} \leq t \leq \pi, t \leq \pi\sin(\frac{t}{2})$ *and* $\sin(\eta t) \leq 1.$

proof 4.2:

$$
\begin{aligned}
|\kappa_\eta(t)| &\leq \frac{1}{2\pi(1+l)^{\eta}}\left|\sum_{\nu=0}^{\eta}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^{\nu}}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{(h+1)}\left\{\sum_{k=0}^{h}\frac{\sin(\nu+1/2)t}{\sin t/2}\right\}\right]\right|\\
&\leq \frac{1}{2\pi(1+l)^{\eta}}\sum_{\nu=0}^{\eta}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^{\nu}}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{(h+1)}\left\{\sum_{k=0}^{h}\frac{1}{t/\pi}\right\}\right]\\
&= \frac{1}{2t(1+l)^{\eta}}\sum_{\nu=0}^{\eta}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^{\nu}}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{(h+1)}(h+1)\right]\\
&\leq \frac{1}{2t(1+l)^{\eta}}\sum_{\nu=0}^{\eta}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^{\nu}}\sum_{h=0}^{\nu}\binom{\nu}{h}l^{\nu-h}\right]\\
&= \frac{1}{2t(1+l)^{\eta}}\sum_{\nu=0}^{\eta}\left[\binom{\eta}{\nu}l^{\eta-\nu}\right]\\
&\leq \frac{1}{2t}\\
&= O\left(\frac{1}{t}\right)
\end{aligned}
$$

Lemma 4.3 $|\overline{\kappa}_\eta(t)| = O\left(\frac{1}{t}\right)$ *for* $\frac{1}{\eta} \leq t \leq \pi, t \leq \pi\sin\left(\frac{t}{2}\right).$

proof 4.3:

$$
\begin{aligned}
|\overline{\kappa}_\eta(t)| &\leq \frac{1}{2\pi(1+l)^{\eta}}\left|\sum_{\nu=0}^{\eta}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^{\nu}}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{(h+1)}\left\{\sum_{k=0}^{h}\frac{\cos(k+1/2)t}{\sin t/2}\right\}\right]\right|\\
&\leq \frac{1}{2\pi(1+l)^{\eta}}\left|\sum_{\nu=0}^{\eta}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^{\nu}}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{(h+1)}\left\{\sum_{k=0}^{h}\frac{\cos\left(k+1/2\right)}{t/\pi}\right\}\right]\right|\\
&\leq \frac{1}{2t(1+l)^{\eta}}\left|\sum_{\nu=0}^{\eta}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^{\nu}}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{(h+1)}Re\left\{\sum_{k=0}^{h}e^{h\left(k+\frac{1}{2}\right)t}\right\}\right]\right|\\
&\leq \frac{1}{2t(1+l)^{\eta}}\left|\sum_{\nu=0}^{\eta}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^{\nu}}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{(h+1)}Re\left\{\sum_{k=0}^{h}e^{hkt}\right\}\right]\right|\left|e^{ht/2}\right|
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{1}{2t(1+l)^{\eta}}\left|\sum_{\nu=0}^{\tau-1}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^{\nu}}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{(h+1)}Re\left\{\sum_{k=0}^{h}e^{hkt}\right\}\right]\right| \\
&+ \frac{1}{2t(1+l)^{\eta}}\left|\sum_{\nu=\tau}^{\eta}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^{\nu}}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{(h+1)}Re\left\{\sum_{k=0}^{h}e^{hkt}\right\}\right]\right| \\
&= J_1 + J_2
\end{aligned}$$

$$\begin{aligned}
\text{Now, } |J_1| &\leq \frac{1}{2t(1+l)^{\eta}}\left|\sum_{\nu=0}^{\tau-1}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^{\nu}}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{(h+1)}Re\left\{\sum_{k=0}^{h}e^{hkt}\right\}\right]\right| \\
&\leq \frac{1}{2t(1+l)^{\eta}}\left|\sum_{\nu=0}^{\tau-1}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^{\nu}}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{(h+1)}.\sum_{k=0}^{h}1\right]\right| \\
&\leq \frac{1}{2t(1+l)^{\eta}}\sum_{\nu=0}^{\tau-1}\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^{\nu}}\sum_{h=0}^{\nu}\binom{\nu}{h}l^{\nu-h} \\
&\leq \frac{1}{2t(1+l)^{\eta}}\sum_{\nu=0}^{\tau-1}\binom{\eta}{\nu} = O\left(\frac{1}{t}\right)
\end{aligned}$$

$$\text{Similarly} \quad |J_2| = O\left(\frac{1}{t}\right)$$

$$\text{therefore} \quad |\overline{\kappa_{\eta}}| = O\left(\frac{1}{t}\right) \tag{4.1}$$

5 Proof of Theorems

Proof 3.1 According to Titchmarsh [3] Let $s_{\eta}(\chi;x)$ is partial sum of Fourier series

$$|s_{\eta}(\chi;x)-\chi(x)| = \frac{1}{2\pi}\int_0^{\pi}\chi(t)\frac{\sin(\eta+1/2)t}{\sin(t/2)}dt$$

The $(E,l)(E,l)(C,1)$ transform of $s_{\eta}(\chi;x)$ is given by

$$\left|\Omega_{\eta}^{E^lE^lC^1}-\chi_x(x)\right| = \frac{1}{2\pi(1+l)^{\eta}}\sum_{\nu=0}^{\eta}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^{\nu}}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{(h+1)}\int_0^{\pi}\chi(t)\left\{\sum_{k=0}^{h}\frac{\sin(\nu+1/2)t}{\sin t/2}\right\}dt\right]$$

$$= \int_0^{\pi}|\chi_x(t)|\,|\kappa_{\eta}(t)|\,dt$$

By using the assumptions of Theorem, it is to be shown that

$$\int_0^{\pi}|\chi_x(t)|\,|\kappa_{\eta}(t)|\,dt = O(1) \qquad \text{as} \qquad \eta\to\infty$$

we set the limit of γ from 0 to π

$$\begin{aligned}
\left|\Omega_\eta^{E^1E^1C^1} - \chi(x)\right| &= \int_0^\pi |\chi(t)|\,|\kappa_\eta(t)|\,dt \\
&= \left[\int_0^{1/\eta} |\chi(t)| + \int_{1/\eta}^{\gamma} |\chi(t)| + \int_{\gamma}^{\pi} |\chi(t)|\right] |\kappa_\eta(t)|\,dt \\
&= I_1 + I_2 + I_3 \qquad (say) \qquad (5.1)
\end{aligned}$$

Using second mean value theorem in second Integral using Lemma 4.1, equation (3.1)& (3.2), we have

$$\begin{aligned}
|I_1| &\leq \int_0^{1/\eta} |\chi(t)|\,|\kappa_\eta(t)|\,dt = O(\eta)\left[\int_0^{1/\eta} |\chi(t)|\,dt\right] = O(\eta)\left[o\left\{\frac{1}{P_\eta.\eta\beta(\eta)}\right\}\right] \\
&= O\left\{\frac{1}{P_\eta.\beta(\eta)}\right\} = O\left\{\frac{1}{\log\eta}\right\} = O(1) \qquad \text{as} \qquad \eta \to \infty \qquad (5.2)
\end{aligned}$$

using Lemma 4.2 equations (3.1) and (3.2) we have,

$$\begin{aligned}
|I_2| &\leq \int_{1/\eta}^{\gamma} |\chi(t)|\,|\kappa_\eta(t)|\,dt = O\left[\int_{1/\eta}^{\gamma} |\chi_x(t)|\left(\frac{1}{t}\right) dt\right] \\
&= O\left[\left\{\frac{1}{t}\chi(t)\right\}_{1/\eta}^{\gamma} + \int_{1/\eta}^{\gamma} \left\{\frac{1}{t^2}\chi(t)\right\} dt\right] \\
&= O\left[o\left\{\frac{1}{P_t.\beta\left(\frac{1}{t}\right)}\right\}_{1/\eta}^{\gamma} + \int_{1/\eta}^{\gamma} o\left(\frac{1}{P_t.t\beta\left(\frac{1}{t}\right)}\right) dt\right]
\end{aligned}$$

Putting $\frac{1}{t} = u$ in second term

$$\begin{aligned}
&= O\left[o\left\{\frac{1}{P_\eta.\beta(\eta)}\right\} + \int_{1/\gamma}^{\eta} o\left(\frac{1}{P_u.u\beta(u)}\right) du\right] \\
&= O\left\{\frac{1}{P_\eta.\beta(\eta)}\right\} + O\left\{\frac{1}{P_\eta.\eta\beta(\eta)}\right\}\int_{1/\gamma}^{\eta} du \\
&= O\left\{\frac{1}{\log\eta}\right\} + O\left\{\frac{1}{\log\eta}\right\}\int_{1/\gamma}^{\eta} du \\
&= O(1) \qquad \text{as} \qquad \eta \to \infty \qquad (5.3)
\end{aligned}$$

By considering summability Regularity condition and taking Riemann-Lebesgue Theorem

$$|I_3| \leq \int_{\gamma}^{\pi} |\chi(t)|\,|\kappa_\eta(t)|\,dt = O(1) \qquad \text{as} \qquad \eta \to \infty \tag{5.4}$$

collecting equations (5.2) (5.3) and (5.4) we get

$$\left|\Omega_\eta^{E^l E^l C^1} - \chi_x(x)\right| = O(1) \qquad \text{as} \qquad \eta \to \infty.$$

Proof 3.2

On using Riemann-Lebesgue Theorem and according to Lal [11], Let $\overline{s}_\eta(\chi; x)$ be partial sum of the series (2.2)

$$\overline{s}_\eta(\chi;x) - \overline{\chi}(x) = \frac{1}{2\pi}\int_0^\pi \chi(t)\left(\frac{\cos(\eta+\frac{1}{2})t}{\sin(t/2)}\right)dt$$

$$\text{or} \quad \overline{s}_\eta(\chi;x) - \left(-\frac{1}{2\pi}\int_0^\pi \cot(t/2)\overline{\chi}(t)\right) = \frac{1}{2\pi}\int_0^\pi \overline{\chi}(t)\frac{\cos\left(\eta+\frac{1}{2}\right)t}{\sin(t/2)}dt$$

$$\text{or} \quad \overline{s}_\eta(\chi;x) - \left(-\frac{1}{2\pi}\int_0^{\frac{1}{\eta}} \cot t/2\overline{\chi}(t)dt - \frac{1}{2\pi}\int_{\frac{1}{\eta}}^{\pi} \cot(t/2)\overline{\chi}(t)dt\right)$$
$$= \frac{1}{2\pi}\left(\int_0^{\frac{1}{\eta}} + \int_{\frac{1}{\eta}}^{\gamma} + \int_{\gamma}^{\pi}\right)\overline{\chi}(t)\frac{\cos\left(\eta+\frac{1}{2}\right)t}{\sin(t/2)}dt$$

$$\overline{s}_\eta(\chi;x) - \left\{-\frac{1}{2\pi}\int_{\frac{1}{\eta}}^{\pi}\cot(t/2)\overline{\chi}(t)dt\right\} = \frac{1}{2\pi}\int_0^{\frac{1}{\eta}}\left[\frac{\cos\left(\eta+\frac{1}{2}\right)t}{\sin t/2} - \cot t/2\right]\overline{\chi}(t)dt$$
$$+ \frac{1}{2\pi}\int_{\frac{1}{\eta}}^{\gamma}\frac{\cos\left(\eta+\frac{1}{2}\right)}{\sin(t/2)}\overline{\chi}(t)dt + \frac{1}{2\pi}\int_{\gamma}^{\pi}\frac{\cos\left(\eta+\frac{1}{2}\right)t}{\sin(t/2)}\overline{\chi}(t)dt$$

taking $\overline{(E,l)(E,l)(C,1)}$ transform, we get

$$\left|\overline{\Omega_\eta^{E^l E^l C^1}} - \overline{\chi(x)}\right| = \frac{1}{2\pi(1+l)^\eta}\sum_{\nu=0}^{\eta}\left[\binom{\eta}{\nu}\frac{l^{\eta-\nu}}{(1+l)^\nu}\sum_{h=0}^{\nu}\binom{\nu}{h}\frac{l^{\nu-h}}{h+1}\int_0^\pi\sum_{k=0}^{h}A\right]$$

$$\text{where,}\quad A = \left\{\int_0^{\frac{1}{\eta}}\left(\frac{\cos\left(\eta+\frac{1}{2}\right)t-\cos t/2}{\sin t/2}\right)+\int_{\frac{1}{\eta}}^{\gamma}\frac{\cos\left(\eta+\frac{1}{2}\right)t}{\sin t/2}+\int_{\gamma}^{\pi}\frac{\cos\left(\eta+\frac{1}{2}\right)t}{\sin t/2}\right\}\overline{\chi}(t)dt$$

$$\begin{aligned} &= O(1)+\int_{1/\eta}^{\gamma}|\chi(t)|\,|\overline{\kappa}_\eta(t)|\,dt+\int_{\gamma}^{\pi}|\chi(t)|\,|\overline{\kappa}_\eta(t)|\,dt \\ &= O(1)+L_1+L_2 \quad (say) \end{aligned} \tag{5.5}$$

Consider Lemma 5.3, equations (3.2) and (3.3)

$$\begin{aligned} |L_1| &\le \int_{1/\eta}^{\gamma}|\chi|\,|\overline{\kappa}_\eta(t)|\,dt = O\left[\int_{1/\eta}^{\gamma}\frac{1}{t}|\chi|\,dt\right] = O\left[\left\{\frac{1}{t}\chi\right\}_{1/\eta}^{\gamma}+\int_{1/\eta}^{\gamma}\frac{1}{t^2}\chi dt\right] \\ &= O\left[o\left\{\frac{1}{P_t.\beta\left(\frac{1}{t}\right)}\right\}_{1/\eta}^{\gamma}+\int_{1/\eta}^{\gamma}o\left(\frac{1}{P_t.t\beta\left(\frac{1}{t}\right)}\right)dt\right] \end{aligned}$$

on putting $\frac{1}{t}=u$ in second term,

$$\begin{aligned} &= O\left[o\left\{\frac{1}{P_\eta.\beta(\eta)}\right\}+\int_{1/\gamma}^{\eta}o\left(\frac{1}{P_u.u\beta(u)}\right)du\right] \\ &= O\left\{\frac{1}{P_\eta.\beta(\eta)}\right\}+O\left\{\frac{1}{P_\eta.\eta.\beta(\eta)}\right\}\int_{1/\gamma}^{\eta}1.du \\ &= O\left\{\frac{1}{\log\eta}\right\}+O\left\{\frac{1}{\log\eta}\right\}=O(1) \qquad \text{as} \qquad \eta\to\infty \end{aligned} \tag{5.6}$$

using regularity condition in method of summability and Riemann-Lebesgue Theorem, we have

$$|L_2| \le \int_{\gamma}^{\pi}|\chi|\,|\overline{\kappa}_\eta(t)|\,dt = O(1) \qquad \text{as} \qquad \eta\to\infty \tag{5.7}$$

Collecting equations (5.5),(5.6) and (5.7) we have

$$\left|\overline{\Omega_\eta^{E^lE^lC^1}}-\overline{\chi(x)}\right| = O(1) \qquad \text{as} \qquad \eta\to\infty$$

6 Corollaries

Some Corollary are given, which are derived from Theorems

Corollary 6.1. If we take $l = 1$ in Theorem (3.1), then triple product summability $(E,l)(E,l)(C,1)$ reduce to $(E,1)(E,1)(C,1)$, then

$$\left|\Omega_\eta^{E_1E_1C_1} - \chi(x)\right| = O(1), \quad \text{as} \quad \eta \to \infty$$

Corollary 6.2. If we take $l = 1$ and $(C,1) = 1$ in Theorem (3.1), then triple product summability $(E,l)(E,l)(C,1)$ reduce to $(E,1)(E,1)$, then

$$\left|\Omega_\eta^{E_1E_1} - \chi(x)\right| = O(1), \quad \text{as} \quad \eta \to \infty$$

Corollary 6.3. If we take $l = 1$ in Theorem (3.2), then triple product summability $\overline{(E,l)(E,l)(C,1)}$ reduce to $\overline{(E,1)(E,1)(C,1)}$, then

$$\left|\overline{\Omega_\eta^{E_1E_1C_1}} - \overline{\chi}(x)\right| = O(1), \quad \text{as} \quad \eta \to \infty$$

Corollary 6.4. If we take $l = 1$ and $(C,1) = 1$ in Theorem (3.2) then triple summability $\overline{(E,l)(E,l)(C,1)}$ reduce to $\overline{(E,1)(E,1)}$, then

$$\left|\overline{\Omega_\eta^{E_1E_1}} - \overline{\chi}(x)\right| = O(1), \quad \text{as} \quad \eta \to \infty$$

7 Conclusion

In this paper, we have estimated degree of approximation by using double Euler product of Cesáro means which becomes $(E,l)(E,l)(C,1)$ triple product mean. This result is new one which generalizes several results known in this field. Therefore, it will be very useful for new researchers in comming days.

References

[1] A. Zygmund, *Trigonometric Series*, Vol.1, 2^{nd} Rev.ed., Cambridge Univ. Press, Cambridge, New York, 1959.

[2] B.E. Rhaodes ,*"On the degree of approximation of functions belonging to Lipschitz class by Hausdroff means of its Fourier series"*, Tamkang J. Math .34 (2003) , no. 3,245-247.

[3] E.C. Titchmarch,*"The Theory of functions"*. Oxford Uni Press; 1952.

[4] G.H. Hardy, *Divergent series,* 1^{st} , Oxford University press , 1949.

[5] Hare Krishna Nigam and Kusum Sharma, *Degree of Approximation of a class of functions by $(C,1)(E,q)$ means of Fourier series,* IAENG International Journal of Applied Mathematics, 41:2, IJAM -41-2-07.

[6] H.K. Nigam , Kusum Sharma, *On $(E,1)(C,1)$ Summability of Fourier Series and its conjugate series*,International Journal of Pure and Applied Mathematics, Volume 82 No.3 2013 ,365-375 , ISSN: 1311 - 8080(printed Version) url:http://www.ijpam.eu.

[7] J.K. Kushwaha, *"Approximation of functions by* $(C,2)(E,1)$ *product summability method of Fourier series"*.Ratio Mathematica ,Volume 38, 2020,pp.341-348.

[8] Jitendra Kumar Kushwaha, Radha Vishwakarma *"Estimation of errors of signals(functions) by* $(C,2)(E,\delta)$ *product means of Fourier series"*. Tuijin Jishu/Journal of Propulsion Technology,ISSN: 1001-4055, vol.44 No. 6(2023).

[9] Jitendra Kumar Kushwaha, Laxmi Rathour, Lakshmi Narayan Mishra, Krishna Kumar,*Degree of Approximation of signals(functions) by* $(C,2)(E,l)$ *Product means of Conjugate series of Fourier Series*, Tuijin Jishu/Journal of Propulsion Technology , ISSN: 1001-4055,Vol. X No. Y(20–).

[10] K. Saxena , M. Prabhakar *"A study of double Euler summability method of Fourier series and it's conjugate series "*. Int. Jour . Sci. Innovative Math Res. 2016; 4(1): 46-52.

[11] Shyam Lal, J.K. Kushwaha,*"Degree of approximation of Lipschitz function by product summability method"*, International Mathematical forum, Vol.4, no. 43(2009)2101-2107.

[12] Smita Sonker and Paramjeet Sangwan, *Approximation of Fourier and its conjugate series by Triple Euler product summability* , Journal of Physics: Conference Series, 1770(2021)012003 doi: 10.1088/1742-6596/1770/1/012003.

[13] Sachin Devaiya and Shailesh Kumar Srivastava, *Approximation of Functions and Conjugate of Functions using Product mean* $(E,q)(E,q)(E,q)$, Palestine Journal of Mathematics, vol.11 (Special Issue 1)(2022),29-37.