Advanced Studies in Contemporary Mathematics
35 (2025), No. 1, pp. 1 – 4

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2025.35.1.1

MAXIMAL NORMAL SUBGROUP IN THE KERNEL OF A PSEUDOCHARACTER ON A GROUP

A. I. Shtern

Abstract. We define an object which turns out to be the maximal normal subgroup in the kernel of a pseudocharacter on a group.

§ 1. Introduction

In this note, we introduce an object which turns out to be the maximal normal subgroup in the kernel of a pseudocharacter on a group. For the generalities concerning pseudocharacters, see [1–4]. See also [5] for some applications of the maximal normal subgroup in the kernel of a pseudocharacter.

§ 2. Preliminaries

Lemma. *Let G be a group, let N be a normal subgroup of G, and let π be the canonical epimorphism of G onto G/N. If a pseudocharacter f on G (such that $f(gh) - f(g) - f(h)| \le c$ for $g, h \in G$) vanishes on N, then there exists a pseudocharacter φ on the group G/N such that $f = \psi \circ \varphi$. If G is a topological group, N is closed, and f is continuous, then φ is continuous.*

Proof. Let G be a group, let N be a normal subgroup of G, let $g \in G$, $n \in N$, and let f be a pseudocharacter on G vanishing on N. Let $m \in \mathbb{N}$. Then

$$m|f(gn)-f(g)| = |f(gn)^m)-mf(g)| = |f(g^m(\prod_{k=m-1}^{1} g^{-k}ng^k)n)-f(g^m)| \le c,$$

2010 *Mathematics Subject Classification.* Primary 22A25, Secondary 22E25.

Submitted November 20, 2024.

Key words and phrases. pseudocharacter, kernel of a pseudocharacter, the maximal normal subgroup in the kernel of a pseudocharacter.

since

$$(\prod_{k=m-1}^{1} g^{-k} n g^{k}) n \in N;$$

this implies that $f(gn) = f(g)$ for all $n \in N$ and all $g \in G$. This means that f is constant on every coset of N in G. Define a real-valued function φ on G/N by setting $\varphi(gN) = f(g)$ (since f is constant on the cosets of N, it follows that this definition is correct). The above formula for $m|f(gn)-f(g)|$, $g \in G$, $n \in N$, together with a similar formula for

$$|f(gn)^{-m}) - f(g)^{-m}|,$$

shows that φ is a pseudocharacter on G/N, and that $\varphi = \psi \circ \pi$, where π is the canonical epimorphism of G onto G/N. The continuity assertion for topological groups follows immediately from the last formula.

§ 3. Main Result

Theorem. *Let G be a group, let f be a pseudocharacter on G, and let $N = \ker f$, i.e., $N = \{g \in G : f(g) = 0\}$. Let*

$$M = \{u \in \ker f \mid f(gu) = f(g) \text{ for all } g \in G\}.$$

Then the following assertions hold:

(1) $M^{-1} \subset M$;

(2) $f(ug) = f(g)$ for all $u \in M$ and all $g \in G$;

(3) M contains the products of its elements, i.e., $f(u_1 u_2) = 0$ and $f(gu_1u_2) = f(g)$ for every $u_1, u_2 \in M$ and all $g \in G$;

(4) M is invariant under the inner automorphisms of G;

(5) M is a normal subgroup of G.

Proof. (1)

$$f(g) = f(gu^{-1}u) = f(gu^{-1})$$

for every $u \in M$ and all $g \in G$, which implies that $u^{-1} \in M$ for all $u \in M$.

(2) Let $g \in G$ and $u \in M$. Then

$$f(g) = f(ugu^{-1}) = f(ug)$$

by (1). Therefore, $f(ug) = f(g)$ for all $u \in M$ and all $g \in G$.

(3)

$$f(g(u_1u_2)) = f(gu_1) = f(g)$$

for all $g \in G$ and all $u_1, u_2 \in M$.

(4) Let $g, h \in G$ and $u \in M$. Then

$$\begin{aligned} f(ghuh^{-1}) &= f(hh^{-1}ghh^{-1}huh^{-1}) = f(hh^{-1}ghuh^{-1}) = f(h^{-1}ghu) \\ &= f(h^{-1}gh) = f(g). \end{aligned}$$

(5) M contains the inverses (1) and products (3) and is inner invariant (4).

This completes the proof of the theorem.

§ 4. Discussion

Since, for every normal subgroup $N \subset \ker f$, we have $f(gu) = f(g)$ for every $g \in G$ and all $n \in N$, it follows that $N \subset M$ for every normal subgroup $N \subset \ker f$. This immediately implies the following assertion.

Corollary. *Let G be a group, f a pseudocharacter on G. The normal subgroup M is the maximal normal subgroup contained in $\ker f$.*

It is of interest to find conditions under which the above pseudocharacter φ on G/M (see the lemma) has trivial kernel.

Acknowledgments

I thank Professor Taekyun Kim for the invitation to publish this paper in the Advanced Studies of Contemporary Mathematics.

Funding

The research was supported by the Scientific Research Institute for System Analysis of the National Research Centre "Kurchatov Institute" according to the project FNEF-2024-0001.

References

1. A. I. Shtern, Quasireprezentations and pseudorepresentations, Funktsional. Anal. i Prilozhen. **25** (2), 70–73 (1991) [Funct. Anal. Appl. **25** (2), 140–143 (1991)].
2. A. I. Shtern, Quasisymmetry. I. Russ. J. Math. Phys. **2** (3), 353–382 (1994).
3. A. I. Shtern, A version of van der Waerden's theorem and a proof of Mishchenko's conjecture on homomorphisms of locally compact groups, Izv. Ross. Akad. Nauk Ser. Mat. **72** (1), 183–224 (2008) [Izv. Math. **72** (1), 169–205 (2008)].

4. A. I. Shtern, Finite-dimensional quasirepresentations of connected Lie groups and Mishchenko's conjecture, Fundam. Prikl. Mat. **13** (7), 85–225 (2007) [J. Math. Sci. (N. Y.) **159** (5), 653–751 (2009)].
5. A. I. Shtern, *Center of a Pseudocharacter*, Proc. Jangjeon Math. Soc. **28** (2025), no. 1 (to appear).

Moscow Center for Fundamental and Applied Mathematics, Moscow, 119991 Russia
Department of Mechanics and Mathematics,
Moscow State University,
Moscow, 119991 Russia
Federal State Institution
Scientific Research Institute for System Analysis of the National Research Centre "Kurchatov Institute",
Moscow, 117312 Russia
E-mail: aishtern@mtu-net.ru, rroww@mail.ru