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ON THE CONSTRUCTION AND PROPERTIES OF FRAMES USING INCIDENCE MATRIX OF GRAPHS AND THEIR SPECTRA

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Abstract. Frames are considered to be redundant counterparts of bases for vector spaces. This redundant structure favours frames to be rich in both theory and applications. In recent studies on frames, graph theory is one of the significant tools to analyze the properties of different types of frames. In graph theory, we associate a graph with different types of matrices, of which signless Laplacian matrix contributes significantly in exploring the properties of a graph. In this paper, given a graph G, we propose a method to construct a frame from its incidence matrix such that its frame graph is the line graph of a derived graph of G. We analyze various properties of the frame constructed as above, its dual, etc. Further, we investigate the existence of frames with constrained frame bounds, using the properties of the associated graph and its signless Laplacian spectrum.

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1. Introduction

In a finite-dimensional Hilbert space $\mathcal{H}$, a finite sequence of vectors ϕ is a *frame* if and only if ϕ spans $\mathcal{H}$. Research on frame theory was initiated by Duffin and Schaeffer in 1952, in their study on non-harmonic Fourier series [6]. In recent years, frames in finite-dimensional spaces have received much attention from both pure and applied mathematics, as they possess a rich redundant structure when compared to bases for vector spaces. Frames are considered to be more general than orthonormal bases (ONB), yet retain most of the significant properties of ONBs. Further, each vector in a Hilbert space $\mathcal{H}$ can have more than one representation as a linear combination of elements in a frame for $\mathcal{H}$. Most real-life applications demand such flexible structures. For instance, based on the fact that each vector in a vector space has a unique representation as a linear combination of its basis elements, while transmitting sensitive information (as signals) across a communication channel, each signal is treated as a vector in a vector space and is encoded using the coefficients in its representation as a linear combination of basis elements. The signals thus transmitted are decoded easily using dot products of suitable vectors at the receiving end. This results in efficient and secured transmission of information. However, if one of the coefficients is lost during transmission, then it is not possible to decode or retrieve that particular segment of information. This problem is overcome with the help of redundancy in a frame structure. A frame being a spanning set (not necessarily independent) helps in the successful retrieval of information, even if a piece of information is lost during transmission. Perhaps, the orthogonality and linear independence conditions on an ONB restrict its usage in

most of the applications and this limitation is overcome with the help of frames. For basic notions and terminologies in frame theory, the reader may refer to [8]. Of the several studies on frames, the recent approaches relating to frame theory and graph theory have paved a new research avenue for many researchers. For basic notions and terminologies in graph theory, the reader may refer to [14]. Graph Theory is used as a significant tool to study the problems arising in frame theory and vice versa. Some of the significant relations that exist between frames and graphs are discussed in [1, 3, 7, 9, 11, 12, 13]. Particularly, in [1], the authors associate a graph to a frame as follows: *Given a frame ϕ, a graph $G(\phi)$ is associated with ϕ by considering each element of ϕ as a vertex in $G(\phi)$ and two vertices are adjacent in $G(\phi)$ if and only if the corresponding frame elements are non-orthogonal. A simple graph G is called a frame graph if there exists a frame ϕ such that $G(\phi) \cong G$.*
Similarly, with each graph G, we can associate one or more frames using the matrices associated with G. In [1], the authors have discussed two different constructions of frames for a given graph. Motivated by the significance of frame theory in both theoretical and application domains, this paper focuses on exploring further relationships between frames and graphs. Here, we introduce a new construction of the frame for a given graph. The rest of the paper is organized as follows: Section 2 deals with the preliminaries on frames and graphs, necessary for further discussions in this paper. Section 3 discusses the construction of the frame from the new **vertex-edge** matrix. Section 4 discusses the potentials of the frames obtained for different classes of graphs, wherein the frame potentials are expressed in terms of graph parameters. Section 5 exhibits the properties of graphs of the dual frames constructed from special classes of graphs. Finally, Section 6 depicts the existence of frames with given optimal frame bounds using graph theoretic approaches, the intervals containing the lower and upper frame bounds of the frames constructed from different graph classes. Further, it deals with the properties of frames constructed from graphs with restricted conditions, using Laplacian, signless Laplacian matrices, and their eigenvalues.

2. Preliminaries

Throughout this paper, we use $\mathbb{K}$ to denote the field of real numbers, $\mathcal{H}^n$ to denote an n-dimensional Hilbert space over $\mathbb{K}$, ϕ_P, and ϕ_C to denote respectively, a Parseval frame and a Canonical frame of ϕ, A^* to denote the conjugate transpose of matrix (or operator) A. Further, all Hilbert spaces discussed in this paper are considered over the real field.

A finite sequence of vectors, $\phi = \{f_i\}_{i=1}^m$ is called a *frame* for an n-dimensional Hilbert space $\mathcal{H}$ over a field $\mathbb{K}$ if $m \geq n$ and there exist constants A and B such that $0 < A \leq B < \infty$ and

$$A\|x\|^2 \leq \sum_{i=1}^{m} |\langle x, f_i\rangle|^2 \leq B\|x\|^2, \quad \forall\, x \in \mathcal{H}. \tag{1}$$

The constants A and B are called *frame bounds* of ϕ. The supremum taken over all lower frame bounds and the infimum taken over all upper frame bounds of ϕ are respectively the *optimal lower and optimal upper frame bounds* of ϕ. If $A = B$, then ϕ is called a *tight frame* and when $A = B = 1$, ϕ is called a *Parseval frame.* Inequality (1) is equivalent to the condition: $span\{f_i\}_{i=1}^m = \mathcal{H}$ [8].

The *analysis operator*, denoted by F is defined as $F : \mathcal{H}^n \longrightarrow \mathcal{H}^m$ such that $F(x) = (\langle x, f_i\rangle)_{i=1}^m$. The *synthesis operator* F^* is defined as $F^* : \mathcal{H}^m \longrightarrow \mathcal{H}^n$ such that $F^*(x_i)_{i=1}^m = \sum_{i=1}^m x_i f_i$. The *frame operator*, S is defined as $S = F^*F$ where $S(x) = \sum_{i=1}^m \langle x, f_i\rangle f_i$ and the *Grammian operator* G_ϕ is defined as $G_\phi = FF^*$, where $G_\phi(x_i)_{i=1}^m = (\langle \sum_{i=1}^m x_i f_i, f_i\rangle)_{i=1}^m$. The frame operator S is invertible, positive definite and Hermitian. Its smallest and largest eigenvalues are equal to the optimal lower and optimal upper frame bounds, respectively. The frame ϕ is a *Parseval frame* if and only if $S = I$ and is a *tight frame* if and only if $S = \lambda I$, where $\lambda > 0$. For a frame $\phi = \{f_i\}_{i=1}^m$ in $\mathcal{H}^n$, the *frame potential* is defined by

$$FP(\phi) = \sum_{i,j=1}^m |\langle f_i, f_j\rangle|^2. \tag{2}$$

For each frame $\phi = \{f_i\}_{i=1}^m$, there exist two more frames, namely, $\phi_C = \{S^{-1}(f_i)\}_{i=1}^m$ and $\phi_P = \{S^{-\frac{1}{2}}(f_i)\}_{i=1}^m$ called *the canonical dual frame* and the *Parseval frame* of ϕ, respectively.

A *graph* G is an ordered pair (V, E), where V (or $V(G)$) is a non-empty set of elements called *vertices* and E (or $E(G)$) is a set of elements called *edges*, where each edge is an ordered or unordered pair of vertices. A graph G is *directed* if $E(G)$ is a collection of ordered pairs of vertices and is *undirected* otherwise. Let $u, v \in V(G)$. If $e \in E(G)$ and $e = (u, v)$, then u and v are *end vertices* of e and are said to be *adjacent vertices*. If u and v are adjacent, then we write $u \sim v$. Two *edges are adjacent* if they share a common end vertex. An edge $e = (u, u)$ is called a *loop*, and a pair of edges $e_1,\ e_2$ where $e_1 = e_2 = (u, v)$ are called *parallel edges*. A graph having no loops and parallel edges is called a *simple graph*. All graphs considered in this paper are simple and undirected unless stated otherwise. The number of vertices and edges of G are respectively referred to as the *order* and *size* of G. Throughout this paper, the symbols p and q denote respectively, the order and size of a graph G under discussion. A graph of order p and size q is called a (p, q)-graph.
The *line graph* of G is denoted by $L(G)$ and is defined as the graph with vertex set $E(G)$, wherein two elements $e_1, e_2 \in E(G)$ are adjacent in $L(G)$ if and only if they are adjacent in G. An edge with v as an endpoint is said to be *incident on* v and the number of edges incident on v is called the *degree* of v, denoted by $deg(v)$. The minimum and maximum degrees in G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A graph G is said to be *regular* if $\delta(G) = \Delta(G)$ and in particular, k-regular if $\delta(G) = \Delta(G) = k$. For two vertices $u, v \in V$, the set $N(u, v) = \{w \in V : w \sim u \text{ and } w \sim v\}$ is called the *common neighbor set* of u and v. A set of mutually non-adjacent vertices in G is called an *independent* set of G. If S is an independent set of G, then the *pairwise common neighborhood* of S is denoted by $CN(S)$. That is,

$$CN(S) = \bigcup_{u,v \in S} N(u, v). \tag{3}$$

A *walk* of a graph is an alternating sequence of vertices and edges. A *trail* is a walk with non-repeated edges and a *path* is a trail with distinct vertices. A path on

n-vertices is denoted by P_n. A path with the same initial and terminal vertices is called a *cycle* and a cycle on n-vertices is denoted by C_n. A graph having no cycle is *acyclic*. A graph G is *connected* if there exists a path between every pair of vertices in G and is *disconnected*, otherwise. A *tree* is a connected acyclic graph. A graph H is a *subgraph* of G is $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A maximal connected subgraph of G is called a *component* of G. A graph is *complete* if there is an edge between every pair of vertices and a complete graph on n-vertices is denoted by K_n. A graph G is *bipartite* if $V(G)$ is the disjoint union of two independent sets, say X and Y. Here, X and Y are called *partite sets* of G, and (X, Y) is called a bipartition of G. A bipartite graph G with bipartition (X, Y) is *complete bipartite* if each vertex in X is adjacent to every vertex in Y and vice versa. A complete bipartite graph with bipartition (X, Y), where $|X| = m$ and $|Y| = n$ is denoted by $K_{m,n}$. A closed trail that includes all edges of G is called an *Euler trail* of G and a graph having an Euler trail is said to be *Eulerian*. A graph G is *planar* if it can be drawn on a plane without intersecting edges. Two graphs G and H are isomorphic, written as $G \cong H$, if there exists a bijection $f : V(G) \longrightarrow V(H)$ such that $(u, v) \in E(G)$ if and only if $(f(u), f(v)) \in E(H)$. If G is of order p, then the matrix $Q = (q_{ij})_{p\times p}$, defined by

$$q_{ij} = \begin{cases} deg(v_i), & \text{if } i = j \\ 1, & \text{if } (v_i, v_j) \in E \\ 0, & \text{otherwise} \end{cases}$$

is called the *signless Laplacian matrix* of G. Note that $Q(G)$ is symmetric, positive semi-definite and it can be written as $Q(G) = D(G) + A(G)$, where $D(G)$ is the diagonal matrix whose i^{th} diagonal entry is the degree of i^{th} vertex of G and $A(G)$ is the adjacency matrix of G, whose diagonal entries are zero and an $(ij)^{th}$ off-diagonal entry is 1 if the i^{th} and j^{th} vertices are adjacent in G and is 0, otherwise. Similarly, if G is a graph with no isolated vertex (vertex of degree zero), then the matrix $L = I - D^{-\frac{1}{2}}AD^{\frac{1}{2}}$ is called the *normalized Laplacian matrix* of G [4]. The set $H(G)$ is defined as $H(G) = \{M = (m_{ij})_{p\times p} : m_{ij} \neq 0 \iff (v_i, v_j) \in E(G)$ and M is a Hermitian matrix$\}$. We refer to a matrix whose rows and columns are indexed respectively, by vertices and edges of a graph as a *vertex-edge* matrix. The *incidence matrix* of G, denoted by $B(G)$ is a vertex-edge matrix defined as $B(G) = (b_{ij})_{p\times q}$, where $b_{ij} = 1$, if j^{th} edge is incident on the i^{th} vertex and is 0, otherwise.

Let G_1 be a (p_1, q_1)-graph and G_2 be a (p_2, q_2)-graph, where $p_1 \cap p_2 = \emptyset$. Then, the graph operations on G_1 and G_2 are defined as follows:

(1) The *Union* of G_1 and G_2, denoted by $G_1 \cup G_2$ is the graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.
(2) The *Join* of G_1 and G_2, denoted by $G_1 \vee G_2$ is the graph with $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and
$E(G_1 \vee G_2) = E(G_1 \cup G_2) \cup \{(x, y) : x \in V(G_1) \text{ and } y \in V(G_2)\}$.
(3) The *Cartesian Product* of G_1 and G_2, denoted by $G_1 \Box G_2$ is the graph with $V(G_1 \Box G_2) = V(G_1) \times V(G_2)$, where $(u_1, v_1) \sim (u_2, v_2)$ in $G_1 \Box G_2$ if and only if $u_1 = u_2$ and $v_1 \sim v_2$ in G_2 (or) $u_1 \sim u_2$ in G_1 and $v_1 = v_2$.
(4) The *Composition* of G_1 and G_2 given by $G_1 \circ G_2$ is the graph with $V(G_1 \circ G_2) = V(G_1) \times V(G_2)$, where $(u_1, v_1) \sim (u_2, v_2)$ in $G_1 \circ G_2$ if and only if $u_1 \sim u_2$ in G_1 (or) $u_1 = u_2$ and $v_1 \sim v_2$ in G_2.

(5) The *Tensor Product* of G_1 and G_2, denoted by $G_1 \otimes G_2$ is the graph with $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$, where $(u_1, v_1) \sim (u_2, v_2)$ in $G_1 \otimes G_2$ if and only if $u_1 \sim u_2$ in G_1 and $v_1 \sim v_2$ in G_2.
(6) The *Normal Product* of G_1 and G_2, denoted by $G_1 * G_2$ is the graph with $V(G_1 * G_2) = V(G_1) \times V(G_2)$ and $E(G_1 * G_2) = E(G_1 \Box G_2) \cup E(G_1 \otimes G_2)$.

In this paper, we establish a relationship between graphs and frames by constructing a frame from a given graph using the Construction 3.3 discussed in Section 3. Further, the main focus of this paper is to study the properties of the frame so obtained using the properties of the graph from which it was derived and vice versa.

3. Construction of a frame for a graph

Lemma 3.1. [8] *Let $\mathcal{H}$ be a Hilbert space and let $\{f_i\}_{i=1}^m \subset \mathcal{H}$. Let F be the analysis operator for $\{f_i\}_{i=1}^m$. Then the following are equivalent:*
(1) *$\{f_i\}_{i=1}^m$ is a frame for $\mathcal{H}$.*
(2) *The frame operator $S = F^*F$ is an invertible operator on $\mathcal{H}$.*

Definition 3.2. [1] *Let $\mathcal{H}$ be a finite-dimensional Hilbert space with an inner product $\langle .,. \rangle$ defined on it. Then, to each frame ϕ of $\mathcal{H}$, we associate a simple graph $G(\phi)$ defined as follows: $V(G(\phi)) = \phi$ and $a \sim b$ in $G(\phi)$ if and only if $\langle a, b \rangle \neq 0$. A simple graph G is called a **frame graph** in $\mathcal{H}$ if there exists a frame ϕ in $\mathcal{H}$ such that $G(\phi) \cong G$. A frame graph is called a **tight frame graph** in $\mathcal{H}$ if the associated frame is a tight frame in $\mathcal{H}$. Note that the graph $G(\phi)$ itself is a frame graph.*

In the discussions to follow, by a frame graph associated with a frame ϕ, we mean the graph $G(\phi)$ defined as above.

Construction 3.3. *Let G be a given (p, q)-graph. Define G' to be the graph obtained from G by adding a loop at each vertex in G. Let $B(G')$ denote the incidence matrix of G'. For convenience, we order the edges of G' in such a way that $B(G')$ is expressed as follows: $B(G') = (b'_{ij})_{p \times (p+q)} = [I|B(G)]$, where I is the $p \times p$ identity matrix corresponding to the p-loop edges in G' and $B(G)$ is the incidence matrix of G. Clearly, $B(G')$ has full row rank. Therefore by Lemma 3.1, the columns of $B(G')$ form a frame in $\mathbb{R}^p$. We call that collection of columns as ϕ.*

In this way, given a graph G, we associate a frame ϕ in the Hilbert space $\mathbb{R}^p$, whilst given a frame ϕ a ,graph is associated with ϕ using $G(\phi)$. This facilitates studying the properties of frames in terms of graphs and vice versa.

Next, we prove that the frame graph of ϕ defined as above is isomorphic to the line graph of G' and we explore further properties of ϕ.

4. Frame Potentials

Theorem 4.1. *Let G be a (p, q)-graph. Then there exists a frame ϕ in $\mathbb{R}^p$ with the following properties:*

(1) $G(\phi) \cong L(G')$.

(2) $FP(\phi) = p + 6q + \sum\limits_{u\in V} [deg(u)]^2$. **(Frame Potential Formula)**

(3) *If Q is the signless Laplacian matrix of G, then the operator S defined by $S = I + Q$ is a frame operator.*

Proof. Let G' be the graph obtained from G by adding a loop at each vertex in G and $B(G') = (b_{ij})_{p\times(p+q)}$ be the vertex-edge incidence matrix of G'. Let b_i denote the i^{th} column of $B(G')$. Clearly, as $B(G')$ has full row rank, $\phi = \{b_j\}_1^{p+q}$ is a frame in $\mathbb{R}^p$. Without loss of generality, let b_i, for all $1 \le i \le p$ denotes the frame vectors corresponding to the p-loop edges in G' and b_{j+p}, for all $1 \le j \le q$ denotes the frame vectors corresponding to the remaining q non-loop edges in G'.
Proof of (1): Let $E(G') = V(L(G')) = \{e_1, e_2, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}\}$. Without loss of generality, let b_i be the column vector of $B(G')$ corresponding to e_i. Clearly, $V(G(\phi)) = \{b_1, b_2, \ldots, b_{p+q}\}$.
Now, define a mapping $f : V(G(\phi)) \longrightarrow V(L(G'))$ as $f(b_i) = e_i$, for all i $(1 \le i \le p+q)$. Clearly, f is bijective. Next, we show that f preserves adjacency. Let $e_i, e_j \in E(G')$. Suppose $e_i \sim e_j$ in $L(G')$, then they share a common end vertex in G' and hence, their corresponding column vectors b_i and b_j in $B(G')$ are not orthogonal. Consequently, b_i and b_j are adjacent in $G(\phi)$.
Similarly, if a pair of vertices b_i and b_j are adjacent in $G(\phi)$, then their corresponding frame elements, that is, columns in $B(G')$ are not orthogonal. This in turn implies that the edges corresponding to these two columns, namely, e_i and e_j have a common end vertex. That is, e_i and e_j are adjacent in $L(G')$. Therefore, e_i and e_j are adjacent in $L(G')$ if and only if b_i and b_j are adjacent in $G(\phi)$. Hence, $G(\phi) \cong L(G')$.
Proof of (2):

$$\begin{aligned}
FP(\phi) &= \sum_{i=1}^{p+q}\sum_{j=1}^{p+q} |\langle b_i, b_j\rangle|^2 \\
&= \sum_{i=1}^{p}\sum_{j=1}^{p} |\langle b_i, b_j\rangle|^2 + \sum_{i=1}^{p}\sum_{j=p+1}^{p+q} |\langle b_i, b_j\rangle|^2 + \sum_{i=p+1}^{p+q}\sum_{j=1}^{p} |\langle b_i, b_j\rangle|^2 \\
&\quad + \sum_{i=p+1}^{p+q}\sum_{j=p+1}^{p+q} |\langle b_i, b_j\rangle|^2 \\
&= \sum_{i=1}^{p}\sum_{j=1}^{p} |\langle b_i, b_j\rangle|^2 + 2\sum_{i=1}^{p}\sum_{j=p+1}^{p+q} |\langle b_i, b_j\rangle|^2 + \sum_{i=p+1}^{p+q}\sum_{j=p+1}^{p+q} |\langle b_i, b_j\rangle|^2 \\
&= p + 2\sum_{u\in V} deg(u) + 4q + \sum_{u\in V} \Big(deg(u)\Big)\Big(deg(u) - 1\Big) \\
&= p + 4q + 4q + \sum_{u\in V} \Big(deg(u)^2 - deg(u)\Big) \\
&= p + 8q + \sum_{u\in V} deg(u)^2 - 2q \\
&= p + 6q + \sum_{u\in V} [deg(u)]^2.
\end{aligned}$$

Proof of (3): Let Q be the signless Laplacian matrix of G. Then, $B(G)B^*(G) = Q$ [5]. Hence, $S = [I|B(G)][I|B(G)]^* = I + B(G)B^*(G) = I + Q$. Clearly, S is invertible. Therefore, it follows from Lemma 3.1, S is the frame operator in $\mathcal{H}$.

□

Example 4.2. *In this example, Theorem 4.1 is illustrated for the graph $K_{1,4}$.*

(1) Let $G \cong K_{1,4}$. Then clearly, $G(\phi) \cong L(G')$.
(2) Assuming ϕ to be the set of all columns of $B(G')$, we get $FP(\phi) = 49$.
(3) Let $S = B(G')B(G')^*$. Then, it follows from Lemma 3.1 that S can be written as $S = B(G')B(G')^* = I + Q$, where Q is the signless Laplacian matrix of $G = K_{1,4}$.

Corollary 4.3. *Let G be a (p,q) graph and ϕ be the frame constructed from $B(G')$. Then the following conditions are equivalent:*

(1) *G is a maximal planar.*
(2) $FP(\phi) = 19p + \sum_{u \in V} (deg(u))^2 - 36$.

Corollary 4.4. *Let G be a (p,q)-graph. Then the following assertions hold:*

(1) *If G is a k-regular, then $FP(\phi) = p(k^2 + 3k + 1)$.*
(2) *If G is connected, then $FP(\phi) \geq 7p - 6 + \sum_{u \in V} deg(u)^2$.*
(3) *If $G \cong K_{m,n}$ and $m > n$, then $FP(\phi) = 6mn + m + n + mn^2 + nm^2$.*
(4) *If G is a tree, then $FP(\phi) = 7P + \sum_{u \in V} deg(u)^2 - 6$.*
(5) *If G is a connected non-trivial Eulerian graph, then $FP(\phi) \geq 6q + p + 4p$.*
(6) *If G has a perfect matching, then $FP(\phi) \geq 6q + 2 + \sum_{u \in V} deg(u)^2$.*
(7) *If G is planar, then $FP(\phi) \leq 19p + \sum_{u \in V} deg(u)^2 - 36$.*

Corollary 4.5. *Let G_1 be a (p_1,q_1)-graph and G_2 be a (p_2,q_2)-graph. Let ϕ_1, ϕ_2 and ϕ be frames constructed from $B(G_1')$, $B(G_2')$ and $B(G')$, where G is the graph constructed from different graph operations. Then the frame potential of frames corresponding to such graph operations is mentioned in Table 1.*

TABLE 1. Exact frame potential for some graph operations

Graph (G)	**Frame potential of G ($FP(\phi)$)**
$G_1 \cup G_2$	$FP(\phi_1) + FP(\phi_2)$
$G_1 \vee G_2$	$FP(\phi_1) + FP(\phi_2) + 4(p_1q_2 + p_2q_1) + p_1p_2(p_1 + p_2 + 6)$
$G_1 \square G_2$	$6(q_1p_2 + q_2p_1) + p_1p_2 + \sum_{(u_i,v_j) \in V} deg(u_i)^2 + \sum_{(u_i,v_j) \in V} deg(v_i)^2 + 8q_1q_2$
$G_1 \circ G_2$	$6(p_1q_2 + p_2^2q_1) + p_1p_2 + p_2^2 \sum_{(u_i,v_j) \in V} deg(u_i)^2 + \sum_{(u_i,v_j) \in V} deg(v_j)^2 + 8p_2q_1q_2$
$G_1 \otimes G_2$	$12q_1q_2 + p_1p_2 + \sum_{(u_i,v_j) \in V} deg(u_i)^2 deg(v_j)^2$
$G_1 * G_2$	$FP(\phi_1) + FP(\phi_2) - p_1p_2 + 2 \sum_{(u_i,v_j) \in V} (deg(u_i) + deg(v_j))(deg(u_i)deg(v_j))$

Theorem 4.6. *Let G_1 and G_2 be two graphs. Let ϕ_1 and ϕ_2 be the frames constructed from $B(G_1')$ and $B(G_2')$ respectively. If $G_1 \cong G_2$, then $FP(\phi_1) = FP(\phi_2)$.*

Proof. The result follows trivially from the definition of Graph isomorphism and the frame potential formula. □

Example 4.7. *The converse of the above theorem is not true. For example, consider the graphs in Figure 1. Both have frame potential* 56*, but they are non-isomorphic.*

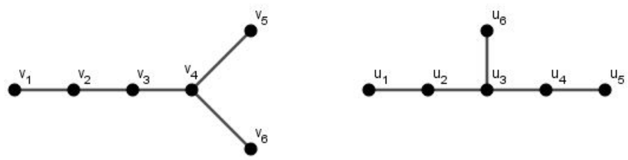

FIGURE 1. Two non-isomorphic graphs with equal frame potential

5. DUAL FRAME

Theorem 5.1. *Let ϕ be the frame corresponding to $B(G')$, where $G \cong K_p$, $p > 1$ with size q. Let ϕ_C be the canonical dual frame of ϕ. Then $G(\phi_C) \cong K_{p+q}$.*

Proof. Let $G \cong K_p$ and $\phi = \{b_i\}_{i=1}^{p+q}$ be the corresponding frame obtained from G' and $B(G') = (b_{ij})_{p\times(p+q)}$ be the vertex-edge incidence matrix of G'. Let b_i be the i^{th} column of $B(G')$. Without loss of generality, let b_i $(1 \le i \le p)$ denote the frame elements corresponding to the p-loops in G' and b_{j+p} $(1 \le j \le q)$ denote the frame elements corresponding to the remaining q non-loop edges in G'. Let $S = (f_{ij})$ be the frame operator corresponding to ϕ and $S^{-1} = (g_{ij})$ be the inverse of S. Now, define the frame operator S and its inverse S^{-1} as follows:

$$(S)_{ij} = \begin{cases} p, & \text{if } i = j. \\ 1, & \text{if } i \neq j. \end{cases} \quad \text{and} \quad (S^{-1})_{ij} = \begin{cases} \frac{2p-2}{(2p-1)(p-1)}, & \text{if } i = j. \\ \frac{-1}{(2p-1)(p-1)}, & \text{if } i \neq j. \end{cases}$$

Let $\phi_C = \{S^{-1}(b_i)\}_{i=1}^{p+q}$ be the canonical dual frame of ϕ. Without loss of generality, assume that $S^{-1}(b_i)$ denotes the canonical dual frame vectors corresponding to p-loops for all i $(1 \le i \le p)$, where $S^{-1}(b_i)$ is the i^{th} column of S^{-1} and for all $1 \le j \le q$, $S^{-1}(b_{j+p})$ denotes the canonical dual frame vectors corresponding to the remaining q non-loop edges in G'. Since $S^{-1}(b_{j+p})$ is the canonical dual frame vectors corresponding to non-loop edges in G', that column can be written as linear combination of two $(S^{-1}(b_i))'^s$, where $S^{-1}(b_i)$ is the canonical dual frame vector corresponding to the loops. Next, we prove that $G(\phi_C) \cong K_{p+q}$.

(1) If both $S^{-1}(b_i)$ and $S^{-1}(b_j)$ are frame vectors corresponding to the loop edges, then
$$\langle S^{-1}(b_i), S^{-1}(b_j)\rangle = \frac{-3p+2}{(2p-1)^2(p-1)^2}.$$

(2) If both $S^{-1}(b_i)$ and $S^{-1}(b_j)$ are frame vectors corresponding to loop and non-loop edge respectively, then
$$\langle S^{-1}(b_i), S^{-1}(b_j)\rangle = \begin{cases} \frac{4p^2-10p+5}{(2p-1)^2(p-1)^2}, & \text{if } \langle b_i, b_j\rangle \neq 0. \\ \frac{-6p+4}{(2p-1)^2(p-1)^2}, & \text{if } \langle b_i, b_j\rangle = 0. \end{cases}$$

(3) If both $S^{-1}(b_i)$ and $S^{-1}(b_j)$ are frame vectors corresponding to the non-loop edges, then

$$\langle S^{-1}(b_i), S^{-1}(b_j)\rangle = \begin{cases} \frac{4p^2-16p+9}{(2p-1)^2(p-1)^2}, & \text{if } \langle b_i, b_j\rangle \neq 0 \text{ and } i \neq j. \\ \frac{-12p+8}{(2p-1)^2(p-1)^2}, & \text{if } \langle b_i, b_j\rangle = 0 \text{ and } i \neq j. \end{cases}$$

The above inner product values are non-zero in all three cases. This implies that all the loop and non-loop edges are mutually adjacent to each other. Hence, $G(\phi_C) \cong K_{p+q}$. □

Theorem 5.2. *Let $G \cong K_{1,p}$ and ϕ be the frame corresponding to $B(G')$ and ϕ_C be the canonical dual frame of ϕ. Then $G(\phi_C) \cong K_{2p+1}$.*

Proof. Let $G \cong K_{1,p}$ and $\phi = \{b_i\}_{i=1}^{2p+1}$ be the corresponding frame which is obtained from G' and $B(G') = (b_{ij})_{(p+1)\times(2p+1)}$ be the vertex-edge matrix in G'. Let $V_1 = \{v_1\}$ and $V_2 = \{u_1, u_2 \dots u_p\}$ be the partite sets corresponding to $K_{1,p}$. Let b_i be the i^{th} column of $B(G')$. Without loss of generality, let $S_1 = \{b_1\}$ be the frame vector corresponding to the loop v_1, $S_2 = \{b_i\}$ $(2 \leq i \leq p+1)$ denote the frame elements corresponding to the remaining p-loops in G' and $S_3 = b_{j+p}$, $(1 \leq j \leq p)$ denote the frame elements corresponding to the remaining p non-loop edges in G'. Let $S = (f_{ij})$ be the frame operator corresponding to the frame ϕ and $S^{-1} = (g_{ij})$ be the inverse of S. Now, we define the frame operator S and its inverse S^{-1} as follows:

$$(S)_{ij} = \begin{cases} p+1, & \text{if } v_i = v_j. \\ 1, & \text{if } v_i \in V_1\,, u_j \in V_2. \\ 2, & \text{if } u_i = u_j. \\ 0, & \text{if } u_i \neq u_j. \end{cases} \text{ and } (S^{-1})_{ij} = \begin{cases} \frac{2^p}{2^{p-1}(p+2)}, & \text{if } v_i = v_j. \\ \frac{-2^{p-1}}{2^{p-1}(p+2)}, & \text{if } v_i \in V_1\,, u_j \in V_2. \\ \frac{2^{p-2}(p+3)}{2^{p-1}(p+2)}, & \text{if } u_i = u_j. \\ \frac{2^{p-2}}{2^{p-1}(p+2)}, & \text{if } u_i \neq u_j. \end{cases}$$

Let $\phi_1 = \{S^{-1}(b_i)\}_{i=1}^{2p+1}$ be the canonical dual frame of ϕ. Without loss of generality, let $S^{-1}(b_i)$, for $1 \leq i \leq p$ denote the canonical dual frame vectors corresponding to p-loops, where $S^{-1}(b_i)$ is the i^{th} column of S^{-1} and $S^{-1}(b_{j+p})$, for $1 \leq j \leq p$ denote the canonical dual frame vector corresponding to the remaining q non-loop edges in G'. Since $S^{-1}(b_{j+p})$ is the canonical dual frame corresponding to non-loop edges in G' and that column can be written as a linear combination of two $(S^{-1}(b_i))$, where $S^{-1}(b_i)$ is the canonical dual frame vector corresponding to the loops. Next, we prove that $G(\phi_C) \cong K_{2p+1}$.

(1) If both $S^{-1}(b_i)$ and $S^{-1}(b_j)$ are frame vectors corresponding to the loops, then

$$\langle S^{-1}(b_i), S^{-1}(b_j)\rangle = \begin{cases} \frac{2^{2p-3}(-2p-6)}{2^{2p-2}(p+2)^2}, & \text{if } b_i \in S_1\,, b_j \in S_2. \\ \frac{2^{2p-4}(2p^2+13p+20)}{2^{2p-2}(p+2)^2}, & \text{if } b_i \neq b_j \in S_2. \end{cases}$$

(2) If both $S^{-1}(b_i)$ and $S^{-1}(b_j)$ are frame vectors corresponding to the loop and non-loop edges, then

$$\langle S^{-1}(b_i), S^{-1}(b_j)\rangle = \begin{cases} \frac{2^{2p-2}}{2^{2p-2}(p+2)^2}, & \text{if } b_i \in S_1\,, b_j \in S_3. \\ \frac{-2^{2p-4}(p+4)}{2^{2p-2}(p+2)^2} \ (or) \ \frac{2^{2p-4}(p^2+3p)}{2^{2p-2}(p+2)^2}, & \text{if } b_i \in S_2\,, b_j \in S_3. \end{cases}$$

(3) If both $S^{-1}(b_i)$ and $S^{-1}(b_j)$ are frame vectors corresponding to the non-loop edges, then

$$\langle S^{-1}(b_i), S^{-1}(b_j)\rangle = \frac{-2^{2p-4}p}{2^{2p-2}(p+2)^2}.$$

The above inner product values are non-zero in all three cases, which implies that all the loop and non-loop edges are mutually adjacent to each other. Hence, $G(\phi_C) \cong K_{2p+1}$. □

In the following theorem, we find the canonical dual frame graph corresponding to the frame vectors of $K_{m,n}$, for $m > 1$.

Theorem 5.3. *For $m > 1$, let $G \cong K_{m,n}$, ϕ be the frame corresponding to $B(G')$ and let ϕ_C be the canonical dual frame of ϕ. Then $G(\phi_C) \cong K_{m+n+mn}$.*

Proof. Let $G \cong K_{m,n}$, $\phi = \{b_i\}_{i=1}^{m+n+mn}$ be the frame corresponding to G' and $B(G') = (b_{ij})_{(m+n)\times(m+n+mn)}$ be the vertex-edge incidence matrix of G'. Let $V_1 = \{v_1, v_2 \dots v_m\}$ and $V_2 = \{u_1, u_2 \dots u_n\}$ be the partite sets corresponding to $K_{m,n}$. Let b_i be the i^{th} column of $B(G')$. Without loss of generality, let b_i $(1 \leq i \leq m+n)$ denote the frame vectors corresponding to the $(m+n)$-loops in G' and b_{j+m+n}, for $1 \leq j \leq mn$ denote the frame vectors corresponding to the remaining mn non-loop edges in G'. Let $S = (f_{ij})$ be the frame operator corresponding to the frame ϕ and $S^{-1} = (g_{ij})$ be the inverse of S. Now, we define S and its inverse S^{-1} as follows:

$$(S)_{ij} = \begin{cases} n+1, & \text{if } v_i = v_j. \\ 0, & \text{if } v_i \neq v_j. \\ 1, & \text{if } v_i \in V_1, u_j \in V_2. \\ m+1, & \text{if } u_i = u_j. \\ 0, & \text{if } u_i \neq u_j. \end{cases} \quad \text{and}$$

$$(S^{-1})_{ij} = \begin{cases} \frac{(m+1)(m+2n+1)}{(n+1)(m+1)(n+m+1)}, & \text{if } v_i = v_j. \\ \frac{n(m+1)}{(n+1)(m+1)(n+m+1)}, & \text{if } v_i \neq v_j. \\ \frac{-(n+1)(m+1)}{(n+1)(m+1)(n+m+1)}, & \text{if } v_i \in V_1, u_j \in V_2. \\ \frac{(n+1)(2m+n+1)}{(n+1)(m+1)(n+m+1)}, & \text{if } u_i = u_j. \\ \frac{m(n+1)}{(n+1)(m+1)(n+m+1)}, & \text{if } u_i \neq u_j. \end{cases}$$

Let $\phi_C = \{S^{-1}(b_i)\}_{i=1}^{m+n+mn}$ be the canonical dual frame of ϕ. Without loss of generality, assume that for $1 \leq i \leq m+n$, $S^{-1}(b_i)$ denote the canonical dual frame vectors corresponding to $m+n$ loop edges, where $S^{-1}(b_i)$ is the i^{th} column of S^{-1} and $S^{-1}(b_{j+m+n})$, for $1 \leq j \leq mn$ denote the canonical dual frame vectors corresponding to the remaining mn non-loop edges in G'. Since $S^{-1}(b_{j+p})$ is the canonical dual frame vectors corresponding to the non-loop edges in G', that column can be written as a linear combination of two $(S^{-1}(b_i))$, where $S^{-1}(b_i)$ is the canonical dual frame vectors correspond to the loops. Hence, corresponding to $S^{-1}(b_i)$ we obtain a complete graph of order $m+n+mn$. Thus, the result follows as in Theorem 5.1. □

Theorem 5.4. *Let G be the complete graph of order p with $p > 1$ and size q and let ϕ be the frame corresponding to $B(G')$. If ϕ_P is the Parseval frame of ϕ, then $G(\phi_P) \cong K_{p+q}$.*

Proof. Let $G \cong K_p$ and $\phi = \{b_i\}_{i=1}^{p+q}$ be the frame corresponding to $B(G')$. Let $B(G') = (b_{ij})_{p\times(p+q)}$ be the vertex-edge incidence matrix of G' and b_i be the i^{th} column of $B(G')$. Without loss of generality, for all $1 \le i \le p$, let b_i denote the frame vectors corresponding to the p-loops in G' and b_{j+p}, for $1 \le j \le q$ denote the frame vectors corresponding to the remaining q non-loop edges in G'. Let $S = (f_{ij})$ be the frame operator corresponding to ϕ and $S^{\frac{-1}{2}} = (g_{ij})$ be the square root of the inverse of S. Now, we define S and $S^{\frac{-1}{2}}$ as follows:

$$(S)_{ij} = \begin{cases} p, & \text{if } i = j. \\ 1, & \text{if } i \neq j. \end{cases} \quad \text{and} \quad (S^{\frac{-1}{2}})_{ij} = \begin{cases} \frac{(p-1)\left((2p-1)\sqrt{p-1}+\sqrt{2p-1}\right)}{p(p-1)(2p-1)}, & \text{if } i = j. \\ \frac{(p-1)\sqrt{2p-1}-(2p-1)\sqrt{p-1}}{p(p-1)(2p-1)}, & \text{if } i \neq j. \end{cases}$$

Let $\phi_P = \{S^{\frac{-1}{2}}(b_i)\}_{i=1}^{p+q}$ be the Parseval frame constructed from ϕ. Without loss of generality, let $S^{\frac{-1}{2}}(b_i)$ $(1 \le i \le p)$ be the Parseval frame vectors corresponding to p-loops, where $S^{\frac{-1}{2}}(b_i)$ is the i^{th} column of $S^{\frac{-1}{2}}$ and $S^{\frac{-1}{2}}(b_{j+p})$, for $1 \le j \le q$ denote the frame vectors corresponding to the remaining q non-loop edges in G'. Since $S^{\frac{-1}{2}}(b_{j+p})$ is the Parseval frame vectors correspond to the non-loop edges in G', that column can be written as a linear combination of two $(S^{\frac{-1}{2}}(b_i))$'s, where $S^{\frac{-1}{2}}(b_i)$ is the Parseval frame vectors correspond to the loops. Next, we prove $G(\phi_P) = K_{p+1}$.

(1) If both $S^{\frac{-1}{2}}(b_i)$ and $S^{\frac{-1}{2}}(b_j)$ are frame vectors corresponding to the loops, then

$$\langle S^{\frac{-1}{2}}(b_i), S^{\frac{-1}{2}}(b_j)\rangle = \frac{\begin{gathered}\left((p-1)\sqrt{2p-1}-(2p-1)\sqrt{p-1}\right) \\ \Big(2(p-1)\left((2p-1)\sqrt{p-1}+\sqrt{2p-1}\right)+ \\ (p-2)\left((p-1)\sqrt{2p-1}-(2p-1)\sqrt{p-1}\right)\Big)\end{gathered}}{p^2(p-1)^2(2p-1)^2}.$$

(2) If $S^{\frac{-1}{2}}(b_i)$ and $S^{\frac{-1}{2}}(b_j)$ are frame vectors corresponding to the loop and non-loop edges, then

$$\langle S^{\frac{-1}{2}}(b_i), S^{\frac{-1}{2}}(b_j)\rangle = \begin{cases} \frac{\begin{gathered}\left((p-2)(2p-1)\sqrt{p-1}+2(p-1)\sqrt{2p-1}\right)^2+ \\ 2(p-2)\left((p-1)\sqrt{2p-1}-(2p-1)\sqrt{p-1}\right)^2\end{gathered}}{p^2(p-1)^2(2p-1)^2}, & \text{if } \langle b_i, b_j\rangle \neq 0. \\[2ex] \frac{\begin{gathered}\left(2p(2p-1)\sqrt{p-1}+2p(p-1)\sqrt{2p-1}\right) \\ \left((p-1)\sqrt{2p-1}-(2p-1)\sqrt{p-1}\right)\end{gathered}}{p^2(p-1)^2(2p-1)^2}, & \text{if } \langle b_i, b_j\rangle = 0. \end{cases}$$

(3) If both $S^{\frac{-1}{2}}(b_i)$ and $S^{\frac{-1}{2}}(b_j)$ are frame vectors corresponding to the non-loop edges with $i \neq j$, then

$$\langle S^{\frac{-1}{2}}(b_i), S^{\frac{-1}{2}}(b_j)\rangle = \begin{cases} \dfrac{\begin{array}{c}\big((p-2)(2p-1)\sqrt{p-1}+2(p-1)\sqrt{2p-1}\big)\\ \big((p-6)(2p-1)\sqrt{p-1}+6(p-1)\sqrt{2p-1}\big)+\\ 4(p-3)\big((p-1)\sqrt{2p-1}\big)-(2p-1)\sqrt{p-1}\big)^2\end{array}}{P^2(p-1)^2(2p-1)^2}, & \text{if } \langle b_i, b_j\rangle \neq 0.\\ \\ \dfrac{\begin{array}{c}\big((p-1)\sqrt{2p-1}-(2p-1)\sqrt{p-1}\big)\\ \big(4p(p-1)\sqrt{2p-1}+(4p)(2p-1)\sqrt{p-1}\big)\end{array}}{p^2(p-1)^2(p-1)^2}, & \text{if } \langle b_i, b_j\rangle = 0.\end{cases}$$

The above inner product values are non zero in all three cases, which implies that all the loop and non-loop edges are mutually adjacent. Hence, the result follows. □

6. Frame Bounds

Remark 6.1. *Let Q be the signless Laplacian matrix of a graph G. Suppose $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_K$ are the eigenvalues of Q. Then $1+\lambda_1, 1+\lambda_2 \ldots 1+\lambda_k$ are the eigenvalues of S, where S is the frame operator of ϕ. Moreover, $1+\lambda_1$ is the optimal upper frame bound of S and $1+\lambda_k$ is the optimal lower frame bound of S.*

Remark 6.2. *Since $S = I + Q$, if $\varepsilon > 0$ we choose $I\sqrt{\varepsilon}$ instead of I in S, then we obtain a better optimal upper bound and optimal lower bound.*

Lemma 6.3. [5] *The multiplicity of the eigenvalue 0 of the signless Laplacian of a graph G is equal to the number of bipartite components in G.*

Theorem 6.4. *For any odd positive integer n, there exists a frame from the graph whose optimal upper frame bound is n.*

Proof. Let n be an odd positive integer and let $n = 2p+1$. Let $G \cong K_{p,p}$. Then the eigenvalues of the signless Laplacian matrix of G are $0, p, 2p$. Therefore, it follows from Remark 6.1 that $1+2p$ is the optimal upper frame bound of the frame ϕ, where ϕ is constructed from $B(G')$. □

Theorem 6.5. *For any $a, b \in 2\mathbb{Z}^+$ such that $0 < a < b < \infty$, there exists a frame whose optimal lower and upper frame bounds are a and b, respectively.*

Proof. Let $G \cong K_{p,p}$. Then, the eigenvalues of the signless Laplacian matrix of $K_{p,p}$ are $0, p, 2p$. Based on Remark 6.1 and Remark 6.2, instead of I in $B(G')$, choose $I\sqrt{a}$, where $a > 0$. Then, the corresponding frame operator is, $S = aI + Q$ which implies that the maximum and minimum eigenvalues of S are $2p + a$ $(= b)$ and a, respectively. Therefore, for any $0 < a < b < \infty$ and $a, b \in 2Z^+$, we can construct a frame from the graph $K_{p,p}$ whose optimal upper frame bound is b and optimal lower frame bound is a. □

Lemma 6.6. [2] *Let G be a regular graph of degree k. Then the following conditions hold:*

(1) *k is an eigenvalue of G*
(2) *if G is connected, then the multiplicity of k is 1*
(3) *if λ is an eigenvalue of G, then $\lambda \in [-k, k]$.*

Theorem 6.7. *If G is a k-regular graph of order p, then the frame constructed from the matrix $[I|B]$ has an optimal upper frame bound and optimal lower frame bound in the interval* $[1, 2k+1]$.

Proof. Let G be a k-regular graph of order p. Since we know that $S = I + Q$, $S = (k+1)I + A(G)$. It follows from Lemma 6.6 that the eigenvalues of S lie in $[1, 2k+1]$ and hence, the optimal upper bound and optimal lower bound lie in the interval $[1, 2k+1]$. □

Corollary 6.8. *If G is a complete graph of order p, then the frame constructed from the matrix $[I|B]$ has $(p-1)$ as an optimal frame lower bound and $(2p-1)$ as an optimal frame upper bound.*

Corollary 6.9. *There exists a k-regular graph having both upper and lower frame bounds in the interval* $(1, 2k+1)$.

The existence of such a k-regular graph given in Corollary 6.9 is shown in Example 6.10.

Example 6.10. *Consider the graph C_5, the cycle on 5 vertices. Clearly, $B(C_5')$ will be given as below:*

$$B(C_5') = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Let ϕ be the frame that is constructed from the column of $B(C_5')$. Then by Lemma 3.1, the corresponding frame operator S is written as follows:

$$S = \begin{bmatrix} 3 & 1 & 0 & 0 & 1 \\ 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 1 & 0 & 0 & 3 & 1 \end{bmatrix}$$

The maximum eigenvalue of S is $\frac{5+\sqrt{5}}{2}$ and the minimum eigenvalue of S is $\frac{5-\sqrt{5}}{2}$. Therefore the optimal bound lies between the interval $(1, 5)$.

Lemma 6.11. [5] 0 *is the eigenvalue of the signless Laplacian matrix of G if and only if G contains at least one connected bipartite component.*

Theorem 6.12. *If G is a graph of order p, then the following conditions are equivalent:*

(1) 1 *is the optimal lower frame bound constructed from the columns of* $[I|B]$.
(2) *G contains at least one connected bipartite component.*

Proof. Assume that 1 is the lower frame bound of the frame constructed from the column of $[I|B]$. We need to prove that G contains at least one connected bipartite component. Suppose G does not contain any connected bipartite component, then by Lemma 6.11, the eigenvalues corresponding to the signless Laplacian matrix is strictly greater than 0. This implies that all the eigenvalues of S are strictly greater than 1. Then the optimal lower frame bound is strictly greater than 1, which is a contradiction to our assumption. Therefore G must contain at least one connected bipartite component.

Conversely, assume that G contains at least one connected bipartite component. Now it is required to prove that 1 is the lower frame bound of the frame constructed from $[I|B]$.

It follows from Remark 6.1 and Remark 6.2 that $1+\lambda_{min}$ is the optimal lower frame bound and $1+\lambda_{max}$ is the optimal upper frame bound. Therefore, 0 is the eigenvalue of the signless Laplacian matrix of G. This implies that 1 is the minimum eigenvalue of S and hence, 1 is the lower frame bound of the frame constructed from $[I|B]$. □

Corollary 6.13. *If G is a connected graph and is not bipartite, then the frame constructed from the columns of $[I|B]$ has an optimal lower frame bound in the interval $(1,\infty)$.*

Proof. Let G be connected and not a bipartite graph. Then, the signless Laplacian matrix of G is non-singular. Hence, all the eigenvalues in Q are strictly greater than 0. Therefore, the frame constructed from the columns of $[I|B]$ has an optimal lower frame bound in the interval $(1,\infty)$. □

Lemma 6.14. [10] *Let G be a connected graph on p-vertices with signless Laplacian matrix Q and normalized Laplacian L. Let the smallest eigenvalue of Q be μ and the largest eigenvalue of L be λ. Then, $2-\frac{\mu}{\delta(G)}\leq\lambda\leq 2-\frac{\mu}{\Delta(G)}$.*

Theorem 6.15. *Let G be a connected graph of order p. Then the frame operator S has the optimal lower frame bound in $[\delta(2-\lambda)+1,\ \Delta(2-\lambda)+1]$, where λ is the largest eigenvalue of L.*

Proof. Let Q be the signless Laplacian matrix of G and $S=I+Q$. Suppose μ is the smallest eigenvalue of Q, then $1+\mu$ is the smallest eigenvalue of S. Therefore, it follows from Lemma 6.14 that $\delta(2-\lambda)\leq\mu\leq\Delta(2-\lambda)$, which implies that $1+\delta(2-\lambda)\leq 1+\mu\leq 1+\Delta(2-\lambda)$. Hence, the lower frame bound lies in $[1+\delta(2-\lambda),1+\Delta(2-\lambda)]$. □

A graph G of order n is a *dual multiplicity graph* (DM graph) [4] provided one of the members of $H(G)$ has exactly two distinct eigenvalues.

Lemma 6.16. [4] *Let $G=(V,E)$ be a dual multiplicity graph and $I\subset V$ be an independent set of G. If for each $v\in I$, there exists $w\in I$ $(w\neq v)$ such that $N(v,w)\neq\emptyset$, then $|I|\leq|CN(I)|$.*

Lemma 6.17. [1] *Let G be a nontrivial graph. Then G is associated with a tight frame in a Hilbert space of dimension k if and only if it is a DM graph.*

Theorem 6.18. *Let G be a simple undirected (p,q)-graph with $q\geq 1$ and G' be the graph obtained from G by adding a loop at each vertex in G. Then the graph $G(\phi)$ is not a tight frame graph and $A(G(\phi))$ has at least three distinct eigenvalues.*

Proof. Given that G is a simple undirected (p,q) graph with $q\geq 1$. Since $q\geq 1$, there exist vertices $u_i,u_j\in V$ and $e=(u_i,u_j)\in E(G)$. Corresponding to those vertices we add e_i,e_j in $[I|B]$. Now we construct a graph from the columns of $[I|B]$. Since e_i

and e_j are adjacent in $G(\phi)$ if and only if $\langle e_i, e_j \rangle \neq 0$, e_i and e_j are not adjacent in $G(\phi)$ and e is the common neighbor of e_i and e_j. Now let $I = \{e_i, e_j\} \subset V(G(\phi))$. Clearly I is an independent set. Therefore by Lemma 6.16 and Lemma 6.17, $G(\phi)$ is not a DM graph and $A(G(\phi))$ has at least three distinct eigenvalues. □

Remark 6.19. [8] *If the frame* $\phi = \{f_i\}_{i=1}^m$ *is Parseval in* $\mathcal{H}$*, then the norm value of each vector in* ϕ *is at most* 1.

It follows from Remark 6.19 that the frame ϕ constructed using Construction 3.3 is Parseval if and only if G is either trivial (or) totally disconnected.

7. Conclusion

In this paper, we have introduced a new construction of frame from a given graph G. Using this construction, the properties of frames have been explored in terms of the properties of the associated graph. Further, we have obtained the canonical dual frame and Parseval frame corresponding to the frame constructed from the graphs K_n and $K_{m,n}$ respectively. Moreover, we have also derived the necessary and sufficient condition for which a graph attains its optimal lower frame bound. Also, the exact range for the optimal lower frame bounds of a graph have been determined in terms of the maximum and minimum degree of a graph.

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References

[1] F. Abdollahi and H. Najafi, *Frame graph*, Linear and Multilinear Algebra 66 (2018), 1229-1243.
[2] N. L. Biggs and B. Norman, *Algebraic graph theory*, Cambridge University Press, 1993.
[3] B. G. Bodmann and V. I. Paulsen, *Frames, graphs and erasures*, Linear algebra and its Applications 404 (2005), 118-146.
[4] Z. Chen, M. Grimm, P. McMichael and C. R. Johnson, *Undirected graphs of Hermitian matrices that admit only two distinct eigenvalues*, Linear Algebra and its Applications 458 (2014), 403-428.
[5] D. Cvetkovic, P. Rowlinson, and S. Simi, *Signless Laplacians of finite graphs*, Linear Algebra and its Applications 423 (2007), 155-171.
[6] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Transactions of the American Mathematical Society 72 (1952), 341-366.
[7] V. Furst and H. Grotts, *Tight frame graphs arising as line graphs*, arXiv preprint arXiv:2007.08673 (2020).
[8] D. Han, K. Kornelson and D. Larson, *Frames for undergraduates*, American Mathematical Soc. vol. 40, 2007.
[9] R. B. Holmes and V. I. Paulsen, *Optimal frames for erasures*, Linear Algebra and its Applications 377 (2004), 31-51.
[10] S. Kirkland and D. Paul, *Bipartite subgraphs and the signless Laplacian matrix*, Applicable Analysis and Discrete Mathematics 5 (2011), 1-13.
[11] T. Strohmer and R. W. Heath Jr, *Grassmannian frames with applications to coding and communication*, Applied and computational harmonic analysis 14 (2003), 257-275.
[12] M. A. Sustik, J. A. Tropp, I. S. Dhillon and R. W. Heath Jr, *On the existence of equiangular tight frames*, Linear Algebra and its Applications 426 (2007), 619-635.
[13] S. Waldron, *On the construction of equiangular frames from graphs*, Linear Algebra and its Applications 431 (2009), 2228-2242.

[14] D. B. West, *Introduction to graph theory*, Upper Saddle River: Prentice Hall, 2001.

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