Proceedings of the Jangjeon Mathematical Society
27 (2024), No. 3, pp. 511 – 523

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2024.27.3.511

SOLVING NON-LINEAR FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS WITH COMBINED ORTHONORMAL BERNSTEIN AND IMPROVED BLOCK-PULSE FUNCTIONS

PRAVEEN AGARWAL, MOHAMED A. RAMADAN, AND HEBA S. OSHEBA

Abstract. The purpose of this paper is to offer a computational way to solve Fredholm integro-differential equations (NFIDEs) of nonlinear type using a mixed of polynomials of Bernstein (BPs) and an block- pulse basis functions of improved form. The nonlinear part is approximated using combined BP operational matrices and an improved block pulse function, while the differential part is approximated using derivative combined BP operational matrices and an improved block pulse function. Finally, a set equations of nonlinear form is obtained by transforming the main equation that we solve for the undetermined coefficients required to obtain the approximate series solution. Using specific computational examples, we also demonstrate how the operational matrices can be used to solve (NFIDEs). 2010 Mathematics Subject Classification. 65L10, 65L20. Keywords and phrases. Bernstein and improved block-pulse functions method Fredholm integro-differential equations, convergence analysis, accuracy.

1. Introduction

Integro-differential equations are frequently used to represent physical phenomena mathematically. Fredho lmintegro-differential equations are used in fluid dynamics, elasticity, economics, biomechanics, motion, heat and mass transfer, theory of oscillation, and theory of airfoil (FIDE).

One of the earliest mathematical modeling problems is the development of numerical solutions for FIDEs [1, 3]. We discovered that there is no clear algebraic approach to solving NFIDEs, so approximation methods such as the Walsh function method [4, 5], homotopy analysis technique [6, 7], method of differential transform [8, 9], Chebyshev polynomial technique [10], sinc-collocation method [10], and the method of wavelet [11, 12] are used. BP are useful in a wide range of mathematical fields. Many researchers, for example, [13–16], solved integral equations, differential equations, and approximation theory with polynomials. In addition, the enhanced block pulse function [17] was developed and is utilized to solve linear integral equations. [18]. The purpose of this study is to create a combined function that combines BP and IBPFs to numerically solve the NFIDE.

$$\sum_{i=0}^{s} p_i(x)\, u^{(i)}(x) = f(x) + \lambda \int_0^1 k(x,t)\,[u(t)]^q dt \ , \tag{1}$$

and the imposed conditions

$$u^{(i)}(0) = \alpha_i \quad , \quad 0 = i = s - 1$$

where $u^{(i)}(x)$ is the *ith* derivative of the unknown function that will be determined, $k(x,t)$ is the kernel of the integral, $f(x)$ and $p_i(x)$ are known analytic functions, q is a positive integer and (λ, α_i) are suitable constants.

The body of this paper is as. Section 2 combines BP and IBPFs. Section 3 describes the suggested method for numerically approximating NFIDEs using the HBIBP basis. The error estimate for the proposed technique is shown in Section 4. Section 5 includes numerical test examples of the proposed solution for the second class of NFIDEs in order to demonstrate its long-term viability and accuracy. We also give some closing remarks.

2. Hybrid of Bernstein and improved block-pulse functions(HBIBPFs) [18]

Definition 2.1. *$HBIBP_{i,j}(x)$ is a set of complete orthogonal system composed BP and IBPFs, all of which are complete and orthogonal. $HBIBP_{i,j}(x)$ where $j = 0,1,\dots,M, i = 1,2,\dots,N+1$, and i and j are the order of both IBPFs and degree of BPs, in respective way. $HBIBP(x)$ is given on [0, 1) as below:*

(2)

$$HBIBP_{i,j}(x) = \begin{cases} B_{j,M}\left(\frac{2x}{h}\right) & , \quad x \in \left[0, \frac{h}{2}\right), \\ 0 & , \quad otherwise, \end{cases} \quad for\ i = 1,\ j = 0,1,\dots,M$$

(3)

$$HBIBP_{i,j}(x) = \begin{cases} B_{j,M}\left(\frac{x}{h} + \frac{3}{2} - i\right) & , x \in \left[(i-2)h + \frac{h}{2}, (i-1)h + \frac{h}{2}\right) \\ 0 & , \quad otherwise, \\ for\ i = 2,3,\dots,N, j = 0,1,\dots,M \end{cases}$$

$$HBIBP_{i,j}(x) = \begin{cases} B_{j,M}\left(\frac{2x}{h} - \frac{2}{h} + 1\right) & , \quad x \in \left[1 - \frac{h}{2}, 1\right), \\ 0 & , \quad otherwise, \end{cases}$$

(4) $$for\ i = N+1,\ j = 0,1,\dots,M$$

A a result, our new basis is$\{HBIBP_{1,0}, HBIBP_{1,1}, \dots, HBIBP_{N+1,M}\}$ and one can then approximate the functions, where N is an integer, that is positive, and $h = \frac{1}{N}$. The technique of approximating these functions is addressed in section 2 presented below.

2.1. Approximation of functions.

The function $u(x)$ can be represented using the HBIBP basis as follows:

$$u(x) = \sum_{i=1}^{N+1} \sum_{j=0}^{M} c_{i,j}.HBIBP_{i,j}(x) = C^T HBIBP(x) \ , \tag{5}$$

where

$$HBIBP(x) = [HBIBP_{1,0}, HBIBP_{1,1}, \dots, HBIBP_{N+1,M}]^T \ , \tag{6}$$

and

$$C = [c_{1,0}, c_{1,1}, \dots, c_{N+1,M}]^T, \tag{7}$$

we have

$$C^T < HBIBP\left(x\right), HBIBP\left(x\right) > = < u\left(x\right), HBIBP\left(x\right), \tag{8}$$

then

$$C = L^{-1} < u\left(x\right), HBIBP > , \tag{9}$$

with $\langle ., . \rangle$ is the dot product and L is an $((N+1)(M+1) \times (N+1)(M+1))$ matrix that is referred to as the dual matrix that is

$$\begin{aligned} L &= < HBIBP\left(x\right), HBIBP\left(x\right) > \\ &= \int_0^1 HBIBP\left(x\right) . HBIBP^T\left(x\right) dx \\ &= \begin{pmatrix} L_1 & 0 & 0 & \cdots & 0 \\ 0 & L_2 & 0 & \cdots & 0 \\ 0 & 0 & L_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & L_{n+1} \end{pmatrix} \end{aligned} \tag{10}$$

$L_i(i= 1, 2, \ldots, n+1)$ is as follows defined

$$\begin{aligned} \left(L_1\right)_{i+1,j+1} &= \int_0^{\frac{h}{2}} B_{i,M}\left(\frac{2x}{h}\right) B_{j,M}\left(\frac{2x}{h}\right) dx \\ &= \frac{h}{2} \int_0^1 B_{i,M}\left(x\right) B_{j,M}\left(x\right) dx \\ &= \frac{h \binom{M}{i} \binom{M}{j}}{2\left(2M+1\right) \binom{2M}{i+j}}, \qquad for i, j = 0, \ldots, M \quad , \end{aligned} \tag{11}$$

$$\begin{aligned} \left(L_r\right)_{i+1,j+1} &= \int_{(i-2)h+\frac{h}{2}}^{(i-1)h+\frac{h}{2}} B_{i,M}\left(\frac{x}{h} + \frac{3}{2} - i\right) B_{j,M}\left(\frac{x}{h} + \frac{3}{2} - i\right) dx, \\ & \quad for \ r = 2, \ldots, N \\ &= h \int_0^1 B_{i,M}\left(x\right) B_{j,M}\left(x\right) dx \\ &= \frac{h \binom{M}{i} \binom{M}{j}}{\left(2M+1\right) \binom{2M}{i+j}}, \quad for i, j = 0, \ldots, M \quad , \end{aligned} \tag{12}$$

(13)

$$\begin{aligned} \left(L_{N+1}\right)_{i+1,j+1} &= \int_{1-\frac{h}{2}}^1 B_{i,M}\left(\frac{2x}{h} - \frac{2}{h} + 1\right) B_{j,M}\left(\frac{2x}{h} - \frac{2}{h} + 1\right) dx \\ &= \frac{h}{2} \int_0^1 B_{i,M}\left(x\right) B_{j,M}\left(x\right) dx = \frac{h \binom{M}{i} \binom{M}{j}}{2\left(2M+1\right) \binom{2M}{i+j}}, \\ & \quad for i, j = 0, \ldots, M \quad , \end{aligned}$$

The function $k\,(x,t) \in L^2(\,[0,1] \times [0,1])$ can also be represented as

$$k\,(x,t) = HBIBP^T\,(x)\,.K.HBIBP(t),$$

where the following matrix K is an $(M+1)(N+1)$is to be obtained as follows:

$$K = L^{-1}\,\langle HBIBP\,(x)\,, \langle k\,(x,t)\,, HBIBP(t)\rangle\rangle\,L^{-1}.$$

2.2. **Operational product matrix.** Let $C^T = [C_1^T\ ,\ C_2^T\ ,\ldots\ ,\ C_{N+1}^T]$ be $1 \times (N+1)\,(M+1)$ matrix where C_i^T is $1 \times (M+1)$ for $i = 1,\ 2,\ldots\ ,\ N+1$, hence $\hat{C}$ is $(N+1)\,(M+1) \times (N+1)\,(M+1)$ whenever a product's operational matrix

$$C^T HBIBP\,(x)\,HBIBP\,(x)^T \simeq HBIBP\,(x)^T \hat{C}. \tag{14}$$

We know have

$$C^T B\,(x)\,B\,(x)^T \simeq B\,(x)^T \hat{C}_i, \qquad i = 1, 2, \ldots,\ N+1,$$

which $\hat{C}_i$ is operational matrix of multiplication of BP found in [19, 20], so

$$\begin{aligned}
&C^T HBIBP\,(x)\,HBIBP\,(x)^T \\
&= C^T \begin{pmatrix} d_1 & \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & d_1 & \cdots & \overline{0} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{0} & \overline{0} & \cdots & d_1 \end{pmatrix} \\
&= \begin{pmatrix} C_1 d_1 & \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & C_2 d_2 & \cdots & \overline{0} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{0} & \overline{0} & \cdots & C_{N+1} d_{N+1} \end{pmatrix} \\
&= \begin{bmatrix} HBIBP_{1,m}\,(x)^T \hat{C}_1 & \overline{0} & \cdots & \overline{0} \\ \overline{0} & HBIBP_{2,m}\,(x)^T \hat{C}_2 & \cdots & \overline{0} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{0} & \overline{0} & \cdots & HBIBP_{N+1,m}\,(x)^T \hat{C}_{N+1} \end{bmatrix} \\
&= HBIBP\,(x)^T \hat{C},
\end{aligned}$$

where

$$\begin{aligned}
d_1 &= HBIBP_{1,m}\,(x)\,HBIBP_{1,m}\,(x)^T, \\
d_2 &= HBIBP_{2,m}\,(x)\,HBIBP_{2,m}\,(x)^T, \\
d_1 &= HBIBP_{N+1,m}\,(x)\,HBIBP_{N+1,m}\,(x)^T.
\end{aligned}$$

for $= 0,\ \ldots, M$, where

$$\hat{C} = \begin{bmatrix} \hat{C}_1 & \overline{0} & \cdots & \overline{0} \\ \overline{0} & \hat{C}_2 & \cdots & \overline{0} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{0} & \overline{0} & \cdots & \hat{C}_{N+1} \end{bmatrix},$$

with $\overline{0}$ is $(m+1) \times (m+1)$ matrix.

2.3. **Matrix of operational integration.** It should be possible to expand the integration of HBIBP functions into HBIBP functions using the matrix of coefficients P. The operational matrix of the integration matrix $\overline{P}$'s is provided by

$$\int_0^x HBIBP\,(t)\,dt \simeq \overline{P}HBIBP\,(x)\,, \qquad 0 \leq x \leq 1 \tag{15}$$

where $\overline{P}$ is $(N+1)\,(M+1)$ and $HBIBP(x)$ is as in Eq. (2)-(4). It is simple to see:

$$\int_0^1 B_{i,m}\,(x)\,dx = \frac{1}{m+1} \quad , \quad i = 0, 1, \dots, m.$$

Then

$$\int_0^1 B_{i,m}\,(kx)\,dx = \frac{1}{k(m+1)} \quad , \quad i = 0, 1, \dots, m.$$

On the other hand, we are aware of

$$\int_0^x B\,(t)\,dt \simeq PB\,(x) \quad ,$$

which P is Bernstein function operational integration matrix $(B(x))$ and information on getting this matrix are found in [19] and [20].

$$\int_0^x HBIBP_{i,j}\,(t)\,dt = \left[\begin{array}{cccc} \frac{P}{2N}, & \frac{\overline{1}}{2N(m+1)}\,, & \cdots & \frac{\overline{1}}{2N(m+1)} \end{array}\right] HBIBP\,(x)\,,$$

$$for\ i = 1,\ j = 0, 1, \dots, M$$

$$\int_0^x HBIBP_{i,j}\,(t)\,dt =$$
$$\left[\begin{array}{cccc} \underbrace{\overline{0}, \ \cdots \ \overline{0},}_{i\ times} & \frac{P}{N}, & \frac{\overline{1}}{N(m+1)}, & \cdots & \frac{\overline{1}}{N(m+1)} \end{array}\right] HBIBP\,(x)\,,$$

$$for\ i = 2, 3, \dots, N, j = 0, 1, \dots, M.$$

$$\int_0^x HBIBP_{i,j}\,(t)\,dt = \left[\begin{array}{cc} \underbrace{\overline{0}, \ \cdots \ \overline{0},}_{i\ times} & \frac{P}{2N} \end{array}\right] HBIBP\,(x)\ ,$$

$$for\ i = N+1,\ j = 0, 1, \dots, M\ ,$$

where $\overline{1}$ is a an $(M+1) \times (M+1)$ matrix whose all of its elements are 1^s, while the zero matrix $\overline{0}$ of size $(M+1) \times (M+1)$.

As a result, the operational integration matrix p is found in this manner: Suppose $A_1 = \frac{P}{2N}$, $A_2 = \frac{P}{N}$, $B_1 = \frac{\overline{1}}{2N(m+1)}$ and $B_2 = \frac{\overline{1}}{N(m+1)}$

$$\overline{P} = \begin{pmatrix} A_1 & B_1 & B_1 & \cdots & B_1 \\ \overline{0} & A_2 & B_2 & \cdots & B_2 \\ \overline{0} & \overline{0} & A_2 & \cdots & B_2 \\ \vdots & \vdots & \vdots & \ddots & B_2 \\ \overline{0} & \overline{0} & \overline{0} & \cdots & A_1 \end{pmatrix}$$

2.4. **Matrix of operational differentiation.** The operational matrix of differentiation $\overline{D}$ is given by

$$\frac{dHBIBP(x)}{dx} = \overline{D}HBIBP(x)$$

We have

$$\frac{dB(x)}{dx} = DB(x)$$

Which D is the operational matrix of differentiation of $B(x)$ and information on obtaining this matrix are given in [19, 20].

$$\frac{dHBIBP_{i,j}(x)}{dx} = \begin{bmatrix} 2ND, & \overline{0}\ , & \cdots & \overline{0} \end{bmatrix} HBIBP(x)\,,$$
$$for\ i = 1,\ j = 0, 1, \ldots, M$$

$$\frac{dHBIBP_{i,j}(x)}{dx} = \begin{bmatrix} \underbrace{\overline{0}, \ \cdots \quad \overline{0},}_{i-1\ times} & ND, & \overline{0}, & \cdots & \overline{0} \end{bmatrix} HBIBP(x)\,,$$
$$for\ i = 2, 3, \ldots, N, j = 0, 1, \ldots, M.$$
$$\frac{dHBIBP_{i,j}(x)}{dx} = \begin{bmatrix} \underbrace{\overline{0}, \ \cdots \ \overline{0},}_{i-1\ times} & 2ND \end{bmatrix} HBIBP(x)\ ,$$
$$for\ i = N + 1,\ j = 0, 1, \ldots, M\ ,$$

where $\overline{0}$ is an $(M + 1) \times (M + 1)$ $matrix$ whose elements are all zeros.

So

$$\overline{D} = N \begin{pmatrix} 2D & \overline{0} & \overline{0} & \ldots & \overline{0} \\ \overline{0} & D & \overline{0} & \ldots & \overline{0} \\ \overline{0} & \overline{0} & D & \ldots & \overline{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{0} & \overline{0} & \overline{0} & \cdots & 2D \end{pmatrix}$$

3. Outline of Solution

Here the derivation of the method for solving the sth-order NFIDE with initial conditions (1) is to be presented.

First step: the function $u(x)$ is approximated by

$$u(x) \simeq U^T HBIBP(x) = HBIBP^T(x)\,\mathrm{U} \quad , \tag{16}$$

where U is $(N + 1)(M + 1)$-unknown matrix and basis coefficient *HBIBP* is found in (2) - (5) .

Second step: The functions $u^{(i)}(x)$, $i = 0,\ 1, \ldots, s$ are approximated by

$$u^{(i)}(x) \simeq U^T (HBIBP(x))^{(i)} = U^T \overline{D}^i HBIBP(x)\,, \quad i = 0, 1, \ldots, s \tag{17}$$

where $\overline{D}$ is $(M + 1)(N + 1) \times (M + 1)(N + 1)$ operational matrix of derivative, $\overline{D}^i$ is operational matrix of derivative with power i and $\frac{d}{dx}HBIBP(x) = \overline{D}HBIBP(x)$.

Third step: The function $k(x, t)$ is apprximated by

$$k(x, t) \simeq HBIBP^T(x)\,KHBIBP(t)\,, \tag{18}$$

where K is a $(N+1)(M+1) \times (N+1)(M+1)$-matrix.

Forth Step: In this step, we use the $[u(x)]^q$ general approximation formula.

We obtain using Eqs. (5) and (14).

$$\begin{aligned} u^2(x) &\simeq [U^T HBIBP(x)]^2 \\ &= U^T HBIBP(x).HBIBP^T(x)U \\ &= HBIBP(x)^T \hat{U} U \end{aligned}$$

$$\begin{aligned} u^3(x) &\simeq \left[U^T HBIBP(x)\right]^3 \\ &= U^T HBIBP(x).\left[U^T HBIBP(x)\right]^2 \\ &= U^T HBIBP(x).HBIBP(x)^T \hat{U} U \\ &= HBIBP(x)^T \hat{U}\hat{U} U \\ &= HBIBP^T(x)\left(\hat{U}\right)^2 U \end{aligned}$$

As a result, $u^q(x)$ will be approximated as using induction.

$$u^q(x) \simeq HBIBP^T(x)\left(\hat{U}\right)^{q-1} U, \tag{19}$$

Replacing NFIDE of second kind equations (1) with approximation (16) - (19).

$$\sum_{i=0}^{s} HBIBP^T(x)(\overline{D}^i)^T U =$$

$$f(x) + \lambda \int_0^1 HBIBP^T(x).K.HBIBP(t)HBIBP^T(t)(\hat{U})^{q-1} U dt,$$

$$\sum_{i=0}^{s} HBIBP^T(x)(\overline{D}^i)^T U =$$

$$f(x) + \lambda.HBIBP^T(x).K \int_0^1 HBIBP(t)HBIBP^T(t)(\hat{U})^{q-1} U dt,$$

where $L = \int_0^1 HBIBP(t).HBIBP^T(t)\,dt$ is defined in Eq. (10).

$$\sum_{i=0}^{s} HBIBP^T(x)(\overline{D}^i)^T U = f(x) + \lambda.HBIBP^T(x).K.L.(\hat{U})^{q-1} U \quad .$$

Therefore,

$$\sum_{i=0}^{s} HBIBP^T(x)(\overline{D}^i)^T U - \lambda.HBIBP^T(x).K.L.\left(\hat{U}\right)^{q-1} U \quad = f(x). \tag{20}$$

To get U we collocate Eq. (4.1) in $(M+1)(N+1)$ Newton-Cotes nodal points as

$$x_i = \frac{2i-1}{2(M+1)(N+1)}, \quad i = 1, \ldots, (M+1)(N+1) - 1 \tag{21}$$

From Eq. (20) using collocation point (21), we have $(M+1)(N+1)$ linear equations is obtained and $(M+1)(N+1)$ unknowns. We can obtain the unknown vector by solving the above linear system. In the expansion of the HBIBP function, U and Eq. (16) can be aplied to get the $u(x)$ solution.

4. Convergence Analysis

This section will provide a complete study of our numerical method's convergence speed.

Theorem 4.1. *The series solution*

$$u_m(x) = \sum_{i=1}^{N+1}\sum_{j=0}^{M} c_{i,j}.HBIBP_{i,j}(x) = C^T HBIBP(x),$$

defined in Eq. (16) *converges to the exact solution* $u(x)$ *then*

$$\lim_{m\to\infty} \|u(x) - u_m(x)\|^2 = 0.$$

Proof. Suppose $L^2(\mathbb{R})$ is Hilbert space and $HBIBP_{i,j}(x)$ shown in Eqs. (2-4) gives a basis of combined Bernstein functions that are orthonormal and block-pulse functions in its improved form.

Let $u(x) \cong \sum_{j=0}^{M} c_{i,j}.HBIBP_{i,j}(x)$ for a fixed i be the approximate solution of the Eq. (1) where $c_{i,j} = L^{-1}\langle u(x), HBIBP_{i,j}(x)\rangle$ defined *in Eq.* (9) and L is defined *in* Eq.(10) as the dual matrix. Let us denote $HBIBP(x) = HBIBP(x_m)$ and $\alpha_m = \langle u(x),\ HBIBP(x_m)\rangle$. Define the sequence of partial sums $\{S_i\}$ of $(\alpha_m HBIBP(x_m))$. Let $\{S_i\}$ and $\{S_j\}$ be the partial sums with $i = j$. We need to prove $\{S_i\}$ is a Cauchy sequence in the space of Hilbert.

Let $S_i = \sum_{m=1}^{i} \alpha_m HBIBP(x_m)$

Now,

$$\langle u(x),\ S_i\rangle = \left\langle u(x), \sum_{m=1}^{i} \alpha_m HBIBP(x_m)\right\rangle =$$

$$\sum_{m=1}^{i} \alpha_m \langle u(x), HBIBP(x_m)\rangle = \sum_{m=1}^{i} \alpha_m\alpha_m = \sum_{m=1}^{i} |\alpha_m|^2.$$

We assert that

$$\begin{aligned}\|S_i - S_j\|^2 &= \left\|\sum_{m=1}^{i} \alpha_m HBIBP(x_m)\right\|^2 \\ &= \left\langle \sum_{m=j+1}^{i} \alpha_m HBIBP(x_m), \sum_{m=j+1}^{i} \alpha_m HBIBP(x_m)\right\rangle \\ &= \sum_{m=j+1}^{i}\sum_{m=j+1}^{i} \alpha_m\alpha_m \langle HBIBP(x_m), HBIBP(x_m)\rangle \\ &= \sum_{m=j+1}^{i} |\alpha_m|^2.\end{aligned}$$

Therefore,

$$\left\|\sum_{m=j+1}^{i}\alpha_m HBIBP(x_m)\right\|^2=\sum_{m=j+1}^{i}|\alpha_m|^2,\qquad fori>j.$$

Based on Bessel's inequality, we have $\sum_{m=j+1}^{i}|\alpha_m|^2$ is convergent, and thus

$$\left\|\sum_{m=j+1}^{i}\alpha_m HBIBP(x_m)\right\|^2\to 0\quad ,asi,j\to\infty$$

Hence, we have

$$\left\|\sum_{m=j+1}^{i}\alpha_m HBIBP(x_m)\right\|\to 0\quad ,$$

and $\{S_i\}$ is a Cauchy sequence that leads to convergence to s (say). One can claim $u(x)=s$.

Now,

$$\begin{aligned}&\langle s-u(x),HBIBP(x_m)\rangle\\&=\langle s,HBIBP(x_m)\rangle-\langle u(x),HBIBP(x_m)\rangle\\&=\left\langle\lim_{i\to\infty}S_i\ ,HBIBP(x_m)\right\rangle-\alpha_m\\&=\lim_{i\to\infty}\langle\alpha_m HBIBP(x_m),HBIBP(x_m)\rangle\ -\alpha_m\\&=\alpha_m-\alpha_m=0.\end{aligned}$$

We reach the conclusion that

$$\langle s-u(x),HBIBP(x_z)\rangle=0.$$

Hence $u(x)=s$ and $S_i=\sum_{m=1}^{i}\alpha_m HBIBP(x_m)$ leads to converges to $u(x)$ as $i?8$ The above relationship is only possible if $u(x)=s$. So that $u(x)$ and S_i converges to the same value which guarantees the proposed $HBIBP$ method is convergent. □

5. Computational Illustrations

This part contains test problems that show the feasibility and precision of the proposed approach in this paper. MATLAB software (R2018b) was used to perform all of the calculations (R2018b).

Example 5.1. *Consider the NFIDE of first order given by*

$$u'(x)+u(x)=\frac{1}{2}\left(e^{-2}-1\right)+\int_{t=0}^{1}u^2(t)\,dt\quad ,$$

with initial condition $u(0)=1$. *This problem's exact solution is* $u(x)=e^{-x}$.

TABLE 1. shows Example 1's numerical results with $M = 1, \quad N = 2$.

x	*Absolute Error*					
	Presented technique	*Method of [22]*	*Method of [23]*	*Method of [12]*	*Method of [21] for*	
		for m = 3	*for* m = 4	*for* N = 128	$n = 3, \quad m = 35$	$n = 4, \ m15$
0.125	9.0205620751e-17	2.4509E-010	9.4E-006	3.7591E-007	5.5200E-011	1.6710E-011
0.250	8.3266726847e-17	1.0202E-010	5.1E-006	6.6413E-007	8.9982E-011	3.9705E-012
0.375	7.6327832943e-17	1.6139E-010	3.0E-005	8.6917E-007	9.4606E-011	1.2126E-011
0.500	6.9388939039e-17	3.2362E-010	4.9E-005	1.0020E-006	9.2457E-011	1.8312E-012
0.625	6.2450045135e-17	1.9197E-010	5.5E-005	1.0757E-006	7.4991E-011	8.1299E-012
0.750	5.5511151231e-17	6.6120E-011	4.5E-005	1.1029E-006	4.9442E-011	7.7237E-012
0.875	4.8572257327e-17	2.2417E-010	2.1E-005	1.0944E-006	2.6083E-011	2.5547E-012

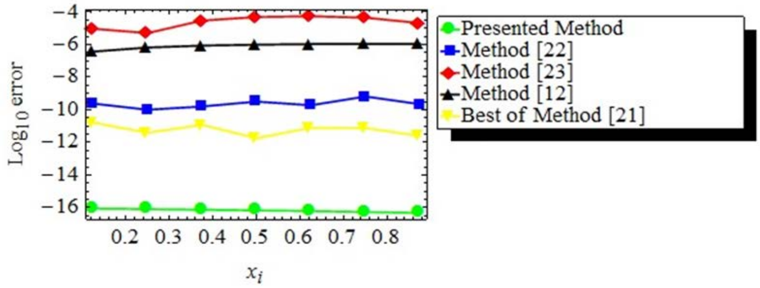

FIGURE 1. For Example 5.1, compare the absolute inaccuracy for numerical results with $\boldsymbol{M = 1, \ N = 2}$ for the presented method with other three numerical methods

We utilized our proposed HBIBPF technique to solve Example 5.1. The computed results for $M = 1, N = 2$ are tabulated in Table 1 and Fig. 1, where the AE of the calculated solution for the presented approach is compared to BPs [22] with m degree B-polynomials, wavelets and semi-orthogonal (SO) B-spline scaling functions [23], Haar wavelets technique [12], and combined block pulse functions and normalized BP (where n and m are the degrees of the orthonormal polynomials. In terms of precision, our technique is definitely superior.

Example 5.2. *Consider the NFIDE of first order given in [10, 21 and 22]*

$$u'(x) = 1 - \frac{1}{3}x + \int_{t=0}^{1} xu^2(t)\,dt \quad ,$$

where $u(0) = 0$ is the initial condition.

This problem's precise solution is $u(x) = x$.

TABLE 2. shows Example 5.2'nd numerical results with $M = 1, \quad N = 2$.

x	*Precise solution*	*Presented technique*	*Absolute Error(AE)*
0	*0.0000000000*	*0.0000000000*	*9.7144514655e-17*
0.1	*0.1000000000*	*0.1000000000*	*9.1593399532e-17*
0.2	*0.2000000000*	*0.2000000000*	*8.6042284408e-17*
0.3	*0.3000000000*	*0.3000000000*	*8.0491169285e-17*
0.4	*0.4000000000*	*0.4000000000*	*7.4940054162e-17*
0.5	*0.5000000000*	*0.5000000000*	*6.9388939039e-17*
0.6	*0.6000000000*	*0.6000000000*	*6.3837823916e-17*
0.7	*0.7000000000*	*0.7000000000*	*5.8286708793e-17*
0.8	*0.8000000000*	*0.8000000000*	*5.2735593670e-17*
0.9	*0.9000000000*	*0.9000000000*	*4.7184478547e-17*

Example 5.2 is solved using our suggested HBIBPF approach, and the obtained approximate numerical results are tabulated, as in Table 2, for $M = 1, \ N = 2$. *The exact solution can be obtained if we increase the values of M and N, as shown in methods [21] and [22], while the maximum absolute error using the sinc approach is* $1.5217E{-}03$ *[10].*

Example 5.3. *For the first order NFIDE given in [24]*

$$u'(x) = 1 - \frac{1}{4}x + \int_{t=0}^{1} xtu^2(t)\,dt \quad ,$$

whose initial condition $u(0) = 0$ *and* $u(x) = x$ *its exact solution.*

TABLE 3. The obtained computed results for example 5.3 with $M = 1, \quad N = 2$.

x	*Exact solution*	*Presented technique*	*A E*	*Method of [24]*	***A E***
0	*0.0000000000*	*0.0000000000*	*4.9004977482e-17*	*0.000000000*	*0.00000E+00*
0.1	*0.1000000000*	*0.1000000000*	*4.0505216262e-17*	*0.100000001*	*1.00000E-09*
0.2	*0.2000000000*	*0.2000000000*	*3.2005455041e-17*	*0.200000004*	*4.00000E-09*
0.3	*0.3000000000*	*0.3000000000*	*2.3505693821e-17*	*0.300000009*	*9.00000E-09*
0.4	*0.4000000000*	*0.4000000000*	*1.5005932600e-17*	*0.400000016*	*1.60000E-08*
0.5	*0.5000000000*	*0.5000000000*	*6.5061713798e-18*	*0.500000022*	*2.20000E-08*
0.6	*0.6000000000*	*0.6000000000*	*1.9935898407e-18*	*0.600000030*	*3.00000E-08*
0.7	*0.7000000000*	*0.7000000000*	*1.0493351061e-17*	*0.700000052*	*5.20000E-08*
0.8	*0.8000000000*	*0.8000000000*	*1.8993112282e-17*	*0.800000146*	*1.46000E-07*
0.9	*0.9000000000*	*0.9000000000*	*2.7492873502e-17*	*0.900000428*	*4.28000E-07*

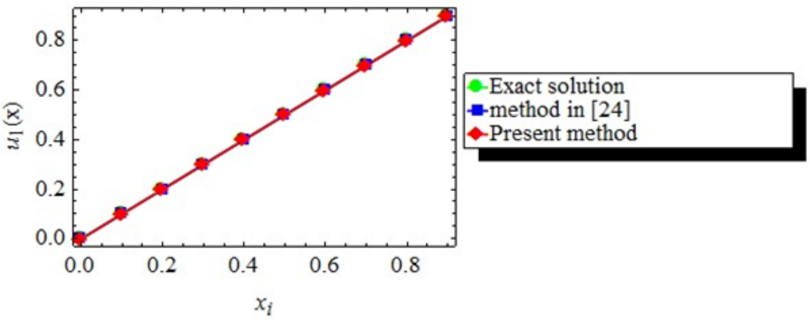

FIGURE 2. numerical results absolute error comparison for Example 5.3 with $\boldsymbol{M = 1, N = 2}$ for the presented method with BP method [24] where $\boldsymbol{n = 5}$ is degree of the Bernstein basis polynomial.

Table 3 summarizes the approximate solutions of Example 5.3 and depicts them in Fig. 2 using the provided technique (for M=1, N=2) and BP (for degree 5) [24]. The exact solution can be obtained if we increase the values of M and N. The proposed method is clearly more accurate than the method described in [24].

6. Concluding Remarks

In this study, the combined BPs and IBPFs described in [18] were used to find numerical solutions (NFIDEs). Examples of tests are provided to demonstrate the proposed method's accuracy and applicability (HBIBPFs). In terms of accuracy, the proposed combination technique outperforms BPs, Semi-orthogonal B-spline scaling functions wavelets method, Haar wavelets method, combined block pulse functions and normalized BP, and the sinc method. The numerical results show that the approach has a lot of potential for dealing with more general linear integro differential equations, which the authors are investigating.

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Department of Mathematics, Anand International College of Engineering, Jaipur-303012, India

E-mail address: goyal.praveen2011@gmail.com

Department of Mathematics, Faculty of Science, Menoufia University, Shebin El-Kom, Menoufia, Egypt

E-mail address: ramadanmohamed13@yahoo.com

Department of Mathematics, Faculty of Science, Menoufia University, Shebin El-Kom, Menoufia, Egypt

E-mail address: heba_osheba@yahoo.com