Proceedings of the Jangjeon Mathematical Society
27 (2024), No. 3, pp. 469 - 484
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2024.27.3.469

PARTITION SEIDEL ENERGY OF GRAPHS

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Abstract. Let $G = (V, E)$ be a graph and $P = \{V_1, V_2, \ldots, V_k\}$ be a k-partition of V. In this article, we introduce the concepts of partition Seidel matrix $S_k(G)$ and partition Seidel energy $E_{S_k}(G)$, which depend on the underlying graph G and the partition of the vertex set V of G. We obtain an upper bound and a few lower bounds for partition Seidel energy, and we also obtain the partition Seidel energy of some families of graphs. We conclude the article by exploring the partition Seidel energy of some classes of double-nested graphs and nested split graphs.

2010 Mathematics Subject Classification. 05C50.

Keywords and phrases. Partition matrix, Seidel energy, divisor matrix, chain graphs, threshold graphs.

1. Introduction

Let G be a simple connected undirected graph with vertex set $V(G)$ and edge set $E(G)$, which contains the unordered pair of vertices. Two vertices $u, v \in V(G)$ are said to be adjacent (or non-adjacent) if $uv \in E(G)$ (or$uv \notin E(G)$). By the denotation $u \sim v$ (or $u \nsim v$), we mean that u is adjacent to v (or u is not adjacent to v). The open neighbourhood of a vertex u in G is denoted by $N(u) = \{v \in V(G) |\ uv \in E(G)\}$, and the closed neighbourhood of u is denoted by $N[u]$ and is defined as $N[u] = N(u) \cup \{u\}$. The degree of a vertex u is defined as the number of vertices that are adjacent to the vertex u, and we denote it by $d(u)$. For all other basic terminologies and definitions, readers can refer to [27].

The adjacency matrix $A(G) = (a_{ij})$ (or A) of a graph G, whose rows and columns correspond to the vertices $u_1, u_2, \ldots, u_n$ of G and is defined as,

$$a_{ij} = \begin{cases} 1, & \text{if } u_i \sim u_j \\ 0, & \text{otherwise.} \end{cases}$$

The concept of graph energy [11] was first introduced by Ivan Gutman in the year 1978 and is defined as the sum of the absolute values of the eigenvalues of the adjacency matrix.
The Laplacian matrix $L(G)$ of a graph G is given by $L(G) = D(G) - A(G)$, where $D(G) = Diag(d(u_1), d(u_2), \ldots, d(u_n))$.
In the year 2006, Ivan Gutman and Bo Zhou introduced the concept of Laplacian energy [12] of graphs, and they defined it as the sum of the absolute deviations of the eigenvalues of its Laplacian matrix.

In 1966, Van Lint and Seidel [26] introduced the Seidel matrix of a graph G of order n denoted by $S(G)$, which is defined as $S(G) = J - I - 2A$, where A is the adjacency matrix of the graph G, I is the identity matrix of order n, and J is the

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Submitted on 28 June 2023.

all-one square matrix of order n. Motivated by the energy of graphs, Haemers [13] defined the Seidel energy $E_s(G)$ of G which is the sum of the absolute values of all eigenvalues of its Seidel matrix.

A partial complement [9] of the graph G is a graph obtained from G by complementing all the edges in one of its induced subgraphs. Partial complement energy of a few classes of graphs has been discussed in the article [25].

The concept of L-matrix (partition matrix) [21] was introduced by E. Sampathkumar and M. A. Sriraj in 2014. Let G be a graph of order n with partition $P = \{V_1, V_2, \ldots, V_k\}$ of its vertex set. Then the L-matrix $P_k(G) = (a_{ij})$ of the graph G with respect to the partition P is an $n \times n$ matrix whose rows and columns correspond to the vertices of the graph, and the ij^{th} entry is given by,

$$a_{ij} = \begin{cases} 2, & u_i, u_j \in V_r \text{ and } u_i \sim u_j, \text{ for some } V_r \in P \\ -1, & u_i, u_j \in V_r \text{ and } u_i \nsim u_j, \text{ for some } V_r \in P \\ 1, & u_i \in V_r, u_j \in V_s \text{ and } u_i \sim u_j \text{ and } r \neq s, \text{ for some } V_r, V_s \in P \\ 0, & \text{otherwise} \end{cases}.$$

The partition energy [22] of a graph G with respect to the partition P was defined by E. Sampathkumar and others in the year 2015 as the sum of the absolute values of the eigenvalues of the L-matrix, $P_k(G)$.
Later in the year 2017, P S K Reddy and Ismail Naci Cangul defined the partition Laplacian matrix [20] of a graph as $LP_k(G) = D(G) - P_k(G)$ and also studied the partition Laplacian energy of graphs. In [23], authors obtained bounds for partition Laplacian energy and also partition Laplacian energy of some families of graphs.

Motivated by the partition energy and the partition Laplacian energy, we define the partition Seidel matrix of G as $S_k(G) = J - I - 2P_k(G)$. Similarly, the partition Seidel energy of the graph is denoted by $E_{S_k}(G)$ and is defined as the sum of the absolute values of the eigenvalues of the partition Seidel matrix $S_k(G)$.

In other words, $S_k(G) = (c_{ij})$ of the graph G with respect to the k-partition P is an $n \times n$ matrix whose rows and columns correspond to the vertices of the graph and ij^{th} entry is given by,

$$c_{ij} = \begin{cases} -3, & u_i, u_j \in V_r \text{ and } u_i \sim u_j, \text{ for some } V_r \in P \\ 3, & u_i, u_j \in V_r \text{ and } u_i \nsim u_j, \text{ for some } V_r \in P \\ -1, & u_i \in V_r, u_j \in V_s \text{ and } u_i \sim u_j \text{ and } r \neq s, \text{ for some } V_r, V_s \in P \\ 1, & u_i \in V_r, u_j \in V_s \text{ and } u_i \nsim u_j \text{ and } r \neq s, \text{ for some } V_r, V_s \in P \\ 0, & \text{if } i = j \end{cases}.$$

A chain graph, or double nested graph (DNG), is a bipartite graph in which the neighbourhoods of the vertices in each partite set form a chain with respect to set inclusion. A chain graph $G(V_1 \cup V_2, E)$ can be partitioned into h non-empty cells given by $V_1 = V_{11} \cup V_{12} \cup \ldots \cup V_{1h}$ *and* $V_2 = V_{21} \cup V_{22} \cup \ldots \cup V_{2h}$ such that $N(u) = V_{21} \cup V_{22} \cup \ldots \cup V_{2\,h-i+1}$, for any vertex $u \in V_{1i},\ 1 \leq i \leq h$. If $m_i = |V_{1i}|$ and $n_i = |V_{2i}|$, then we write $G = DNG(m_1, \ldots, m_h; n_1, \ldots, n_h)$. DNGs are characterised as being $\{2K_2, C_3, C_5\}$-free graphs.
The spectral properties of adjacency and the Laplacian matrix of chain graphs are well studied. in literature [1,2,5]. Recently, some interesting properties of the Seidel energy of a chain graph were discussed in [18].

A split graph is a graph that admits a partition of its vertex set into two parts, V_1 and V_2, such that V_1 induces a clique and V_2 induces a co-clique. Every other edge, called a cross edge, joins a vertex of V_1 with a vertex of V_2. A threshold graph, or nested split graph (NSG), is a split graph in which the adjacencies defined by the cross edges satisfy the following nesting property: Both V_1 and V_2 can be partitioned into h non-empty cells, say, $V_1 = V_{11} \cup V_{12} \cup \ldots \cup V_{1h}$ *and* $V_2 = V_{21} \cup V_{22} \cup \ldots \cup V_{2h}$ such that $N(u) = V_{11} \cup V_{12} \ldots \cup V_{1\ h-i+1}$, for any vertex $u \in V_{2i},\ 1 \leq i \leq h$. If $m_i = |V_{1i}|$ and $n_i = |V_{2i}|$, then we write $G = NSG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$. It is known that threshold graphs are characterised as being $\{2K_2, P_4, C_4\}$-free graphs. For more results related to chain graphs and threshold graphs, readers are referred to [3, 4]. The spectral properties of the adjacency matrix of threshold graphs have been well studied in [7, 8]. Seidel spectrum of threshold graphs are discussed in [17]. The authors of the article [28] showed that, with the exception of 1 and -1, the interval $(-\sqrt{2}, \sqrt{2})$ contains no Seidel eigenvalue of threshold graphs.

Section 2 discusses a few preliminary results related to the eigenvalues of the partition Seidel matrix. Bounds on partition Seidel energy are given in Section 3, and the partition Seidel energy of some families of graphs is discussed in the last section.

2. Preliminary Results

Let $G = (V, E)$ be a graph on n vertices and m edges, and $P_k = \{V_1, V_2, \ldots, V_k\}$ be a k-partition of V. Let p_i be the number of edges within $V_i, 1 \leq i \leq k$, and let q_i be the number of non-adjacent pairs of vertices within $V_i, 1 \leq i \leq k$. Let y_i be the total number of edges joining the vertices from V_i to V_j for $i \neq j, 1 \leq j \leq k$. Also, let z_i be the total number of non-adjacent pairs of vertices (u_i, u_j) such that $u_i \in V_i$ and $u_j \in V_j, i \neq j,\ 1 \leq i, j \leq k$. Let

$$n_1 = \sum_{i=1}^{k} p_i, \qquad n_2 = \sum_{i=1}^{k} q_i, \qquad n_3 = \sum_{i=1}^{k} y_i, \qquad n_4 = \sum_{i=1}^{k} z_i. \tag{1}$$

In this section we obtain few results related to eigenvalues of partition Seidel matrix. The result which gives the multiplicity of the eigenvalues $-3, -1, 3$ and 1 of $S_k(G)$ in the graph G based on its structure is given below.

Lemma 2.1. *Let G be a connected graph on n vertices and $P = \{V_1, V_2, \ldots, V_k\}$ be a k-partition of its vertex set.*

(1) *If $S \subseteq V_i$ for some $i,\ 1 \leq i \leq k$ is an independent set of size t, and for every $u_i, u_j \in S$, $N(u_i) = N(u_j)$, then*
- *-1 is an eigenvalue of $S_1(G)$ with multiplicity at least equal to $t - 1$.*
- *-3 is an eigenvalue of $S_k(G)$, $1 \leq k \leq n - 1$ with multiplicity at least $t - 1$.*
- *-1 is an eigenvalue of $S_n(G)$ with multiplicity at least $t - 1$.*

(2) *If the graph induced by $S \subseteq V_i$, for some $i,\ 1 \leq i \leq k$ is a clique of size t, and for every $u_i, u_j \in S$, $N[u_i] = N[u_j]$, then*
- *3 is an eigenvalue of $S_k(G)$, $1 \leq k \leq n - 1$ with multiplicity at least $t - 1$.*
- *1 is an eigenvalue of $S_n(G)$ with multiplicity at least $t - 1$.*

If the characteristic polynomial of $S_k(G)$ denoted by $\phi_P(G,x)$ is $c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_n$, then the coefficients c_i can be interpreted using the principal minors of $S_k(G)$. The first three coefficients of $\phi_P(G,x)$ is given in the following theorem.

Lemma 2.2. *The first three coefficients of $\phi_P(G,x)$ are,*
a) $c_0 = 1$ *b)* $c_1 = 0$ *c)* $c_2 = -(9n_1 + 9n_2 + n_3 + n_4)$.

Proof. *a)* It follows directly from the definition of the characteristic polynomial.

b) It follows from the fact that trace of the partition Seidel matrix is equal to 0.

c) We know that the coefficients c_i of $\phi_P(G,x)$ and the $i \times i$ principal minors of the partition Seidel matrix are related as $(-1)^i c_i =$ sum of all the $i \times i$ minors of $S_k(G)$.

$$\begin{aligned}(-1)^2 c_2 &= \sum_{1\le i\le j\le k} \begin{vmatrix} c_{ii} & c_{ij} \\ c_{ji} & c_{jj} \end{vmatrix} \\ &= \sum_{1\le i\le j\le k} c_{ii}c_{jj} - c_{ji}c_{ij} \\ &= \sum_{1\le i\le j\le k} -(c_{ij})^2 \\ &= -[n_1(-3)^2 + n_2(3)^2 + n_3(-1)^2 + n_4(1)^2] \\ &= -[9n_1 + 9n_2 + n_3 + n_4].\end{aligned}$$

□

Lemma 2.3. *Let $s_1, s_2, \ldots, s_n$ be the eigenvalues of the partition Seidel matrix of the graph G. Then,*

$$\sum_{i=1}^{n} s_i^2 = 2N,$$

where $N = [9n_1 + 9n_2 + n_3 + n_4]$.

Proof. We have,

$$\begin{aligned}\sum_{i=1}^{n} s_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}c_{ji} \\ &= 2\sum_{i<j}(c_{ij})^2 + \sum_{i=1}^{n}(c_{ii})^2 \\ &= 2\sum_{i<j}(c_{ij})^2 \\ &= 2[9n_1 + 9n_2 + n_3 + n_4].\end{aligned}$$

□

Few results which are used in the subsequent sections are given below.

Theorem 2.4. *[15] Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be n-tuples of real numbers satisfying*

$$0 \le m_1 \le a_i \le M_1$$

and

$$0 \leq m_2 \leq b_i \leq M_2.$$

Then,

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \frac{n^2}{3}[M_1M_2 - m_1m_2]^2.$$

Theorem 2.5. *[6] Let* $A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$ *be a* 2×2 *symmetric block matrix. Then the spectrum of* A *is the union of spectra of* $A_0 + A_1$ *and* $A_0 - A_1$.

Theorem 2.6. *[16] Let* $0 \leq a_1 \leq a_2 \leq \ldots \leq a_n$ *be a sequence of non-negative real numbers. Then*

$$\sum_{i=1}^{n} a_i + n(n-1)\left(\prod_{i=1}^{k} a_i\right)^{\frac{1}{n}} \leq \left(\sum_{i=1}^{k} \sqrt{a_i}\right)^2 \leq (n-1)\sum_{i=1}^{n} a_i + n\left(\prod_{i=1}^{k} a_i\right)^{\frac{1}{n}}.$$

For a given graph G, a partition $D : W_1 \cup W_2 \cup \ldots \cup W_k$ of $V(G)$ is called an equitable partition if every vertex in W_i has the same number of neighbours in W_j, say d_{ij}, for all $i, j \in 1, 2, \ldots, k$. Then the $k \times k$ matrix with entries $[d_{ij}]$ is called the divisor matrix of D.
In the theory of graph spectra, equitable partitions play an important role mostly because of the following result.

Theorem 2.7. *[10] Let* M *be a real symmetric matrix with a divisor matrix* D*. Then the characteristic polynomial of* D *divides the characteristic polynomial of* M.

3. Main Results

In the present section, we obtain some upper and lower bounds for $E_{S_k}(G)$. The result connecting eigenvalues of $S_{P_1}(H_1)$ and $S_{P_2}(H_2)$ is given below.

Theorem 3.1. *Let* H_1 *and* H_2 *be two graphs of order* n*. Suppose that* P_1 *and* P_2 *are* k*- partitions of vertex sets of* H_1 *and* H_2 *respectively. If* $s_1, s_2, \ldots, s_n$ *and* $s'_1, s'_2, \ldots, s'_n$ *are the eigenvalues of* $S_{P_1}(H_1)$ *and* $S_{P_2}(H_2)$ *respectively, then*

$$\sum_{i=1}^{n} s_i s'_i \leq 2\sqrt{NN'},$$

where $N = [9(n_1 + n_2) + n_3 + n_4]$ *and* $N' = [9(n'_1 + n'_2) + n'_3 + n'_4]$ *and* n_1, n_2, n_3, n_4 *are defined for* $S_{P_1}(H_1)$ *and* n'_1, n'_2, n'_3, n'_4 *are defined for* $S_{P_2}(H_2)$ *as in Equation 1.*

Proof. By Cauchy-Schwartz inequality, we have $\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$. By setting $a_i = s_i$ and $b_i = s'_i$ in the above inequality, we get

$$\begin{aligned}\left(\sum s_i s'_i\right)^2 &\leq \left(\sum s_i^2\right)\left(\sum s_i'^2\right)\\ &\leq 4[9(n_1 + n_2) + n_3 + n_4][9(n'_1 + n'_2) + n'_3 + n'_4]\end{aligned}$$

which implies $\sum_{i=1}^{n} s_i s'_i \leq 2\sqrt{[9(n_1 + n_2) + n_3 + n_4][9(n'_1 + n'_2) + n'_3 + n'_4]}$. □

In all the results discussed below, $N = [9(n_1 + n_2) + n_3 + n_4]$.

Theorem 3.2. *Let G be a graph on n vertices and $|s_1| \geq |s_2| \geq \ldots \geq |s_n|$, where s_i's are the eigenvalues of the partition Seidel matrix, $S_k(G)$. Then,*

$$E_{S_k}(G) \geq n\sqrt{\frac{2N}{n} - \frac{1}{3}(|s_1| - |s_n|)^2}.$$

Proof. By substituting $a_i = |s_i|$ and $b_i = 1$ in Theorem 2.4,

$$n\sum_{i=1}^{k}|s_i|^2 - \left(\sum_{i=1}^{k}|s_i|\right)^2 \leq \frac{n^2}{3}(|s_1| - |s_n|)^2$$

$$n\sum_{i=1}^{k}|s_i|^2 - \frac{n^2}{3}(|s_1| - |s_n|)^2 \leq (E_{S_k}(G))^2$$

$$\sqrt{2n[9(n_1+n_2)+n_3+n_4] - \frac{n^2}{3}(|s_1| - |s_n|)^2} \leq E_{S_k}(G).$$

□

Theorem 3.3. *Let G be a graph on n vertices and P be the k-partition of its vertex set. If $D = |det(S_k(G))|$, then*

$$\sqrt{2N + n(n-1)D^{\frac{2}{n}}} \leq E_{S_k}(G) \leq \sqrt{(n-1)N + nD^{\frac{2}{n}}}.$$

Proof. By substituting $a_i = s_i^2$ in the inequality of Theorem 2.6, we get

$$\sum_{i=1}^{n} s_i^2 + n(n-1)\left(\prod_{i=1}^{k} s_i^2\right)^{\frac{1}{n}} \leq \left(\sum_{i=1}^{k} s_i\right)^2 \leq (n-1)\sum_{i=1}^{n} s_i^2 + n\left(\prod_{i=1}^{k} s_i^2\right)^{\frac{1}{n}}.$$

□

Theorem 3.4. *Let G be a graph of order n and P be the k-partition of the vertex set. Then,*

$$E_{S_k}(G) \leq \sqrt{2nN}.$$

Proof. By Cauchy-Schwartz inequality, we have $\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$. By setting $a_i = |s_i|$ and $b_i = 1$ in the inequality, we get

$$(E_{S_k}(G))^2 = \left(\sum_{i=1}^{n}|s_i|\right)^2 \leq \left(\sum_{i=1}^{n}|s_i|^2\right)\left(\sum_{i=1}^{n}1^2\right)$$

$$E_{S_k}(G) \leq \sqrt{n\sum_{i=1}^{n}|s_i|^2}$$

$$\leq \sqrt{2n[9(n_1+n_2)+n_3+n_4]}.$$

□

In the following theorem, McClelland-type [19] of an improved bound for the partition Seidel energy of a graph is obtained.

Theorem 3.5. *Let G be a graph of order n and P be the k-partition of its vertex set. Then,*

$$E_{S_k}(G) \leq \sqrt{nN}.$$

Proof. We have,

$$\begin{aligned}
0 &\leq \sum_{i=1}^{n}\sum_{j=1}^{n}(|s_i|-|s_j|)^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(|s_i|^2+|s_j|^2-2|s_i||s_j|\right) \\
&= n\sum_{i=1}^{n}|s_i|^2+n\sum_{j=1}^{n}|s_j|^2-2\sum_{i=1}^{n}|s_i|\sum_{j=1}^{n}|s_j| \\
&= 2nN-2E^2_{S_k}(G).
\end{aligned}$$

□

Theorem 3.6. *Let G be a graph of order n and P be the k-partition of its vertex set. Then,*

$$E_{S_k}(G) \leq |s_1| + \sqrt{(1-n)|s_1|^2+2(n-1)N}.$$

Proof. We have,

$$\begin{aligned}
\left(\frac{\sum\limits_{i=2}^{n}|s_i|}{n-1}\right)^2 &\leq \frac{\sum\limits_{i=2}^{n}|s_i|^2}{n-1} \\
\implies \left(\sum_{i=2}^{n}|s_i|\right)^2 &\leq (n-1)\sum_{i=2}^{n}|s_i|^2 \\
\implies (n-1)|s_1|^2+(E^2_{S_k}(G)-|s_1|)^2 &\leq (n-1)\sum_{i=1}^{n}|s_i|^2 \\
\implies E^2_{S_k}(G)-2|s_1|E_{S_k}(G)+n|s_1|^2-2(n-1)N \leq 0.
\end{aligned}$$

On solving for $E_{S_k}(G)$ we get

$$\implies E_{S_k}(G) \leq \frac{2|s_1|+\sqrt{4|s_1|^2-4(n|s_1|^2-2(n-1)N)}}{2}$$

from which the, result follows. □

4. Partition Seidel energy of some family of graphs

In this section, we obtain partition Seidel energy of some classes of graphs.

Theorem 4.1. *Let G be a regular graph on n vertices with regularity r ($r > 0$). Then,*

- *$3n-6r-3$ is an eigenvalue of $S_1(G)$;*
- *$n-2r-1$ is an eigenvalue of $S_n(G)$.*

From Theorem 2.1 the following result follows.

Theorem 4.2. *If $G = K_n$, then $E_{S_1}(G) = 6n-6$ and $E_{S_n}(G) = 2n-2$.*

Proof. Note that $3(1-n)$ is an eigenvalue of $S_1(G)$ with multiplicity 1 and 3 is an eigenvalue of $S_1(G)$ with multiplicity $n-1$.
Similarly, $1-n$ is an eigenvalue of $S_n(G)$ with multiplicity 1 and 1 is an eigenvalue of $S_n(G)$ with multiplicity $n-1$. Hence the result follows. □

The crown graph S_n^0 for an integer $n \geq 3$ is the graph with the vertex set $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ and the edge set $\{u_iv_j, 1 \leq i, j \leq n, i \neq j\}$. The crown graph S_n^0 is equivalent to the complete bipartite graph $K_{n,n}$ with horizontal edges removed.
We shall need the following theorem to obtain partition Seidel energy of the crown and the cocktail party graph $K_{n\times 2}$.

Theorem 4.3. *Let G be a crown graph with the vertex set $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ and edge set $\{u_iv_j, 1 \leq i, j \leq n, i \neq j\}$. Then,*

1. $spec(S_1(G)) = \begin{pmatrix} 3 & -9 & 6n-9 \\ n & n-1 & 1 \end{pmatrix}$;
2. *with respect to $P = \{U, V\}$ where $U = \{u_1, u_2, \dots, u_n\}$, and $V = \{v_1, v_2, \dots, v_n\}$,* $spec(S_2(G)) = \begin{pmatrix} -1 & -5 & 2n-1 & 4n-5 \\ n-1 & n-1 & 1 & 1 \end{pmatrix}$,
3. $spec(S_n(G)) = \begin{pmatrix} 1 & -3 & 2n-3 \\ n & n-1 & 1 \end{pmatrix}$.

Proof.
1. The 1-partition seidel matrix of the crown graph G is given by $S_1(G) = \begin{bmatrix} 3(J-I) & 6I-3J \\ 6I-3J & 3(J-I) \end{bmatrix}$, where J is the all-1 matrix of order n and I is the unit matrix of order n.
By using Theorem 2.5, the spectrum of $S_1(G)$ is the union of the spectrum of $3I$ and $6J-9I$. The eigenvalues of the matrix $3I$ is 3 with multiplicity n. The spectrum of $6J-9I$ can be obtained in the following way.
$$|6J-9I-\lambda I| = \begin{vmatrix} -3-\lambda & 6 & 6 & \cdots & 6 \\ 6 & -3-\lambda & 6 & \cdots & 6 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 6 & 6 & 6 & \cdots & -3-\lambda \end{vmatrix}.$$
By performing $R_i \longrightarrow R_i - R_{i+1}, for\ i = 1, 2, \dots n-1$, and $R_n \longrightarrow R_n - 6R_1 - 12R_2 - 18R_3 - \dots - 6(n-1)R_{n-1}$ and on simplification we get -9 and $6n-9$ as the eigenvalues of above matrix with multiplicity $n-1$ and 1 respectively.
2. Observe that $S_2(G) = \begin{bmatrix} 3(J-I) & 2I-J \\ 2I-J & 3(J-I) \end{bmatrix}$.
By Theorem 2.5, the spectrum of $S_2(G)$ is the union of the spectrum of $2J-I$ and $4J-5I$. The eigenvalues of $2J-I$ are -1 with multiplicity $n-1$ and $2n-1$ with multiplicity 1. The spectrum of $4J-5I$ are -5 with multiplicity $n-1$ and $4n-5$ with multiplicity 1.
3. $S_n(G) = \begin{bmatrix} J-I & 2I-J \\ 2I-J & J-I \end{bmatrix}$.
By Theorem 2.5, the spectrum of $S_n(G)$ is the union of the spectrum of I and $2J-3I$. The eigenvalues of I is 1 with multiplicity n. The spectrum of $2J-3I$ are -3 with multiplicity $n-1$ and $2n-3$ with multiplicity 1.

□

Corollary 4.4. *Let G be a crown graph of order $2n$. Then,*

- $E_{S_1}(G) = 18n - 18$;
- $E_{S_2}(G) = 12n - 12$;
- $E_{S_n}(G) = 6n - 6$.

Theorem 4.5. *Let G be a cocktail party graph of order $2n$ with the vertex set $\{u_1, u_2, \dots, u_{2n}\}$ and edge set $\{u_iu_j, 1 \le i \ne j \le 2n, i \ne n+i\}$. Then,*

1. $spec(S_1(G)) = \begin{pmatrix} -3 & 9 & 9-6n \\ n & n-1 & 1 \end{pmatrix}$;
2. *with respect to the partition $P = \{U, V\}$, where $U = \{u_1, u_2, \dots, u_n\}$ and $V = \{u_{n+1}, u_{n+2}, \dots, u_{2n}\}$,*
   $spec(S_2(G)) = \begin{pmatrix} 1 & 5 & 1-2n & 5-4n \\ n-1 & n-1 & 1 & 1 \end{pmatrix}$;
3. $spec(S_n(G)) = \begin{pmatrix} -1 & 3 & 3-2n \\ n & n-1 & 1 \end{pmatrix}$.

Proof. 1. Observe that $S_1(G) = \begin{bmatrix} 3(I-J) & 6I-3J \\ 6I-3J & 3(I-J) \end{bmatrix}$, where J is the all-1 matrix of order n and I is the unit matrix of order n.
By using Theorem 2.5, the spectrum of $S_1(G)$ is the union of the spectrum of $-3I$ and $9I-6J$. The eigenvalues of the matrix $-3I$ are -3 with multiplicity n and eigenvalues of $9I-6J$ are 9 with multiplicity $n-1$ and $9-6n$ with multiplicity one.

2. $S_2(G) = \begin{bmatrix} 3(I-J) & 2I-J \\ 2I-J & 3(I-J) \end{bmatrix}$.
By using Theorem 2.5, the spectrum of $S_2(G)$ is the union of the spectrum of $I-2J$ and $5I-4J$. The eigenvalues of the matrix $I-2J$ are -1 with multiplicity $n-1$ and $1-2n$ with multiplicity one. The spectrum of $5I-4J$ is -5 with multiplicity $n-1$ and $5-4n$ with multiplicity one.

3. $S_n(G) = \begin{bmatrix} I-J & 2I-J \\ 2I-J & I-J \end{bmatrix}$.
By using Theorem 2.5, the spectrum of $S_n(G)$ is the union of the spectrum of $-I$ and $3I-2J$. The eigenvalues of the matrix $-I$ are -1 with multiplicity n. The spectrum of $3I-2J$ are 3 with multiplicity $n-1$ and $3-2n$ with multiplicity one.

□

Corollary 4.6. *Let G be a cocktail party graph of order $2n$, $n > 1$. Then,*

- $E_{S_1}(G) = 18n - 18$,
- $E_{S_2}(G) = 12n - 12$,
- $E_{S_n}(G) = 6n - 6$.

The spectrum and energy of k-partition Seidel matrix of chain graphs when $h = 1$ and 2 and for $k = 1, 2, 4, n$ are obtained below using the concept of equitable partition.

Theorem 4.7. *Let $G = DNG(p; q)$ be a chain graph on n vertices with $|V_1| = p$, and $|V_2| = q$. Then,*

1. $spec(S_1(G)) = \begin{pmatrix} 3(n-1) & -3 \\ 1 & n-1 \end{pmatrix}$;

2. *with respect to* $P = \{V_1, V_2\}$, $spec(S_2(G)) = \begin{pmatrix} \lambda_{21} & \lambda_{22} & -3 \\ 1 & 1 & n-2 \end{pmatrix}$, *where* $\lambda_{21} = \frac{1}{2}\left[3n - 6 + \sqrt{9n^2 - 14pq}\right], \lambda_{22} = \frac{1}{2}\left[3n - 6 - \sqrt{9n^2 - 14pq}\right]$;
3. $spec(S_n(G)) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$.

Proof. 1. By Theorem 2.1, we can observe that G has -3 is an eigenvalue of $S_1(G)$ with multiplicity at least $n-2$. Other two eigenvalues can be obtained through the equitable partition $D : V_1 \cup V_2$ of $V(G)$. The divisor matrix corresponding to the 1-partition Seidel matrix of D is given by $A_D = \begin{bmatrix} 3(p-1) & -3q \\ -3p & 3(q-1) \end{bmatrix}$. Eigenvalues of the divisor matrix are $3(n-1)$ and -3. From Theorem 2.7, the result follows.

2. The proof follows by Theorem 2.1 and by noting the divisor matrix corresponding to the equitable partition $D : V_1 \cup V_2$ of $V(G)$ is given by
$A_D = \begin{bmatrix} 3(p-1) & -q \\ -p & 3(q-1) \end{bmatrix}$.

3. The proof follows by Theorem 2.1 and by noting the divisor matrix corresponding to the equitable partition $D : V_1 \cup V_2$ of $V(G)$ is given by

$$A_D = \begin{bmatrix} p-1 & -q \\ -p & q-1 \end{bmatrix}.$$

Eigenvalues of the divisor matrix are $n-1$ and -1. Hence the theorem. □

Corollary 4.8. *Let* $G = DNG(p; q)$ *be a chain graph on* n *vertices with* $|V_1| = p$ *and* $|V_2| = q$. *Then,*

- $E_{S_1}(G) = 6n - 6$;
- $E_{S_2}(G) = |\lambda_{21}| + |\lambda_{22}| + 3n - 6$;
- $E_{S_n}(G) = 2n - 2$.

Note 4.9. *By putting* $p = 1$ *and* $q = n-1$ *in Corollary 4.8, we get the expression for the partition Seidel energy of the star graph* $K_{1,n-1}$.

Theorem 4.10. *Let* $G = DNG(m_1, m_2; n_1, n_2)$ *be a chain graph on* n *vertices with* $|V_{1i}| = m_i$ *and* $|V_{2i}| = n_i$. *Then,*

1. $spec(S_1(G)) = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} & -3 \\ 1 & 1 & 1 & 1 & n-4 \end{pmatrix}$,
   where $\lambda_{11}, \lambda_{12}, \lambda_{13}$ *and* λ_{14} *are the roots of the polynomial,*
   $\lambda^4 + (12 - 3n)\lambda^3 + 27(2 - n)\lambda^2 + 27(4 - 3n + 4m_1m_2n_2 + 4m_2n_1n_2)\lambda + 81(1 - n + 4m_1m_2n_2 + 4m_2n_1n_2)$;
2. *with respect to* $P = \{V_1, V_2\}$, $spec(S_2(G)) = \begin{pmatrix} \lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} & -3 \\ 1 & 1 & 1 & 1 & n-4 \end{pmatrix}$,
   where $\lambda_{21}, \lambda_{22}, \lambda_{23}$ *and* λ_{24} *are the roots of the polynomial,*
   $\lambda^4 + 3(4-n)\lambda^3 + (54 - 27n + 8(m_1 + m_2)(n_1 + n_2))\lambda^2 + (108 - 81n + 48(m_1 + m_2)(n_1 + n_2) + 12m_1m_2n_2 + 12m_2n_1n_2)\lambda + 81 - 81n + 72(m_1 + m_2)(n_1 + n_2) + 36m_1m_2n_2 + 36m_2n_1n_2 - 32m_1m_2n_1n_2$;
3. *with respect to* $P = \{V_{11} \cup V_{21}, V_{12} \cup V_{22}\}$,
   $spec(S_2'(G)) = \begin{pmatrix} \lambda'_{21} & \lambda'_{22} & \lambda'_{23} & \lambda'_{24} & -3 \\ 1 & 1 & 1 & 1 & n-4 \end{pmatrix}$,

where $\lambda'_{21}, \lambda'_{22}, \lambda'_{23}$ *and* λ'_{24} *are the roots of the polynomial,*
$\lambda^4+(12-3n)\lambda^3+(54-27n+8(m_1+n_1)(n_1+n_2))\lambda^2+(108-81n+48(m_1+n_1)(m_2+n_2)+12m_1m_2n_2+12m_2n_1n_2)\lambda+81-81n+72(m_1+n_1)(n_1+n_2)+36m_1m_2n_2+36m_2n_1n_2$;

4. *with respect to* $P=\{V_{11},V_{12},V_{21},V_{22}\}$,
$spec(S_4(G)) = \begin{pmatrix}\lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44} & -3\\ 1 & 1 & 1 & 1 & n-4\end{pmatrix}$, *where* $\lambda_{41},\lambda_{42},\lambda_{43}$ *and* λ_{44} *are the roots of the polynomial,*
$\lambda^4+3(4-n)\lambda^3+(54-27n+8m_1m_2+8n_1n_2+8(m_1+m_2)(n_1+n_2))\lambda^2+(108-81n+48(m_1m_2+n_1n_2)+48(m_1+m_2)(n_1+n_2)-20m_1m_2n_1-16m_1m_2n_2-20m_1n_1n_2-16m_2n_1n_2)\lambda+81-81n+72(m_1m_2+n_1n_2)+72(m_1+m_2)(n_1+n_2)-60m_1m_2n_1-48m_1m_2n_2-60m_1n_1n_2-48m_2n_1n_2+32m_1m_2n_1n_2$;

5. $spec(S_n(G)) = \begin{pmatrix}\lambda_{n1} & \lambda_{n2} & \lambda_{n3} & \lambda_{n4} & -1\\ 1 & 1 & 1 & 1 & n-4\end{pmatrix}$,
where $\lambda_{n1},\lambda_{n2},\lambda_{n3}$ *and* λ_{n4} *are the roots of the polynomial,*
$\lambda^4+(4-n)\lambda^3+(6-3n)\lambda^2+(4-3n+4m_1m_2n_2+4m_2n_1n_2)\lambda+1-n+4m_1m_2n_2+4m_2n_1n_2$.

Proof. 1. By Theorem 2.1 we can observe that G has -3 as an eigenvalue of $S_1(G)$ with multiplicity at least equal to $n-4$. We can obtain the other four eigenvalues through the equitable partition $D: V_{11}\cup V_{12}\cup V_{21}\cup V_{22}$ of $V(G)$. The divisor matrix corresponding to the 1-partition Seidel matrix of D is given by

$$A_D = \begin{bmatrix} 3(m_1-1) & 3m_2 & -3n_1 & -3n_2\\ 3m_1 & 3(m_2-1) & -3n_1 & 3n_2\\ -3m_1 & -3m_2 & 3(n_1-1) & 3n_2\\ -3m_1 & 3m_2 & 3n_1 & 3(n_2-1)\end{bmatrix}.$$

From Theorem 2.7, the eigenvalues of the divisor matrix along with -3 which is of multiplicity $n-4$ constitutes the spectrum.

2. The proof follows by Theorem 2.1 and by noting the divisor matrix corresponding to the equitable partition $D: V_{11}\cup V_{12}\cup V_{21}\cup V_{22}$ of $V(G)$ is given by

$$A_D = \begin{bmatrix} 3(m_1-1) & 3m_2 & -n_1 & -n_2\\ 3m_1 & 3(m_2-1) & -n_1 & n_2\\ -m_1 & -m_2 & 3(n_1-1) & 3n_2\\ -m_1 & m_2 & 3n_1 & 3(n_2-1)\end{bmatrix}.$$

3. The proof follows by Theorem 2.1 and by noting the divisor matrix corresponding to the equitable partition $D: V_{11}\cup V_{12}\cup V_{21}\cup V_{22}$ of $V(G)$ is given by

$$A_D = \begin{bmatrix} 3(m_1-1) & m_2 & -3n_1 & -n_2\\ m_1 & 3(m_2-1) & -n_1 & 3n_2\\ -3m_1 & -m_2 & 3(n_1-1) & n_2\\ -m_1 & 3m_2 & n_1 & 3(n_2-1)\end{bmatrix}.$$

4. The proof follows by Theorem 2.1 and by noting the divisor matrix corresponding to the equitable partition $D: V_{11}\cup V_{12}\cup V_{21}\cup V_{22}$ of $V(G)$ is given by

$$A_D = \begin{bmatrix} 3(m_1-1) & m_2 & -n_1 & -n_2 \\ m_1 & 3(m_2-1) & -n_1 & n_2 \\ -m_1 & -m_2 & 3(n_1-1) & n_2 \\ -m_1 & m_2 & n_1 & 3(n_2-1) \end{bmatrix}.$$

5. The proof follows by Theorem 2.1 and by noting the divisor matrix corresponding to the equitable partition $D: V_{11} \cup V_{12} \cup V_{21} \cup V_{22}$ of $V(G)$ is given by

$$A_D = \begin{bmatrix} m_1-1 & m_2 & -n_1 & -n_2 \\ m_1 & m_2-1 & -n_1 & n_2 \\ -m_1 & -m_2 & n_1-1 & n_2 \\ -m_1 & m_2 & n_1 & n_2-1 \end{bmatrix}.$$

□

Corollary 4.11. *Let $G = DNG(m_1, m_2; n_1, n_2)$ be a chain graph on n vertices with $|V_{1i}| = m_i$ and $|V_{1i}| = n_i$. Then,*

- $E_{S_1}(G) = |\lambda_{11}| + |\lambda_{12}| + |\lambda_{13}| + |\lambda_{14}| + 3n - 12;$
- $E_{S_2}(G) = |\lambda_{21}| + |\lambda_{22}| + |\lambda_{23}| + |\lambda_{24}| + 3n - 12;$
- $E'_{S_2}(G) = |\lambda'_{21}| + |\lambda'_{22}| + |\lambda'_{23}| + |\lambda'_{24}| + 3n - 12;$
- $E_{S_4}(G) = |\lambda_{41}| + |\lambda_{42}| + |\lambda_{43}| + |\lambda_{44}| + 3n - 12;$
- $E_{S_n}(G) = |\lambda_{n1}| + |\lambda_{n2}| + |\lambda_{n3}| + |\lambda_{n4}| + n - 4.$

Note 4.12. *By putting $m_1 = 1$ and $n_1 = 1$ in above result, we get the expression for the partition Seidel energy of the bi-star graph, $B(n_2, m_2)$.*

The spectrum and energy of k-partition Seidel matrix of $NSG(p;q)$ and $NSG(m_1, m_2; n_1, n_2)$ for $k = 1, 2, 4, n$ are discussed below.

Theorem 4.13. *Let $G = NSG(p;q)$ be a threshold graph on n vertices with $|V_1| = p$, which induces a clique and $|V_2| = q$, induces a co-clique. Then*

1. $spec(S_1(G)) = \begin{pmatrix} 3 & -3 & \lambda_{11} & \lambda_{12} \\ p-1 & q-1 & 1 & 1 \end{pmatrix},$
   where $\lambda_{11} = \frac{3}{2}\left[(q-p) + \sqrt{(p-2)^2 + q^2 + 6pq - 4q}\right]$
   and $\lambda_{12} = \frac{3}{2}\left[(q-p) - \sqrt{(p-2)^2 + q^2 + 6pq - 4q}\right],$
2. *with respect to $P = \{V_1, V_2\}$,* $spec(S_2(G)) = \begin{pmatrix} \lambda_{21} & \lambda_{22} & 3 & -3 \\ 1 & 1 & p-1 & q-1 \end{pmatrix},$
   where, $\lambda_{21} = \frac{1}{2}\left[3(q-p) + \sqrt{(3p-6)^2 + 9q^2 + 22pq - 36q}\right],$
   $\lambda_{22} = \frac{1}{2}\left[3(q-p) - \sqrt{(3p-6)^2 + 9q^2 + 22pq - 36q}\right],$
3. $spec(S_n(G)) = \begin{pmatrix} \lambda_{n1} & \lambda_{n2} & 1 & -1 \\ 1 & 1 & p-1 & q-1 \end{pmatrix}$
   where $\lambda_{n1} = \frac{1}{2}\left[(q-p) + \sqrt{(p-2)^2 + q^2 + 6pq - 4q}\right],$
   $\lambda_{n2} = \frac{1}{2}\left[(q-p) - \sqrt{(p-2)^2 + q^2 + 6pq - 4q}\right].$

Proof. 1. By Theorem 2.1, we can observe that G has -3 as an eigenvalue of $S_1(G)$ with multiplicity at least $q-1$ and 3 as an eigenvalue of $S_1(G)$ with multiplicity at least $p-1$. Other two eigenvalues we can obtain through the equitable partition $D: V_1 \cup V_2$ of $V(G)$. The divisor matrix corresponding to

the 1-partition Seidel matrix of D is given by $A_D = \begin{bmatrix} 3(1-p) & -3q \\ -3p & 3(q-1) \end{bmatrix}$.
Eigenvalues of the divisor matrix λ_{11} and λ_{12}, along with -3 which is of multiplicity $q-1$ and 3 which is of multiplicity $p-1$ constitutes the spectrum.

2. The proof follows by Theorem 2.1 and by noting the divisor matrix corresponding to the equitable partition $D : V_1 \cup V_2$ of $V(G)$ is given by
$$A_D = \begin{bmatrix} 3(1-p) & -q \\ -p & 3(q-1) \end{bmatrix}.$$
3. The proof follows by Theorem 2.1 and by noting the divisor matrix corresponding to the equitable partition $D : V_1 \cup V_2$ of $V(G)$ is given by
$$A_D = \begin{bmatrix} 1-p & -q \\ -p & q-1 \end{bmatrix}.$$

□

Corollary 4.14. *Let $G = NSG(p; q)$ be a threshold graph on n vertices with $|V_1| = p$ and $|V_2| = q$. Then,*

- $E_{S_1}(G) = |\lambda_{11}| + |\lambda_{12}| + 3n - 6$;
- $E_{S_2}(G) = |\lambda_{21}| + |\lambda_{22}| + 3n - 6$;
- $E_{S_n}(G) = |\lambda_{n1}| + |\lambda_{n2}| + n - 2$.

Theorem 4.15. *Let $G = NSG(m_1, m_2; n_1, n_2)$ be a threshold graph on n vertices with the graph induced by V_1 a clique and the graph induced by V_2 a co-clique. Then,*

1. $spec(S_1(G)) = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} & 3 & -3 \\ 1 & 1 & 1 & 1 & m_1+m_2-2 & n_1+n_2-2 \end{pmatrix}$,
where $\lambda_{11}, \lambda_{12}, \lambda_{13}$ and λ_{14} are the roots of the polynomial,
$\lambda^4 + 3(m_1+m_2-n_1-n_2)\lambda^3 + 9(n-2-2(m_1+m_2)(n_1+n_2))\lambda^2 + 27(n_1+n_2-m_1-m_2-4m_1m_2n_2+4m_2n_1n_2)\lambda + 81(1-n+2(m_1+m_2)(n_1+n_2)-4m_1m_2n_2-4m_2n_1n_2+8m_1m_2n_1n_2)$;
2. *with respect to $P = \{V_1, V_2\}$,*
$spec(S_2(G)) = \begin{pmatrix} \lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} & 3 & -3 \\ 1 & 1 & 1 & 1 & m_1+m_2-2 & n_1+n_2-2 \end{pmatrix}$,
where $\lambda_{21}, \lambda_{22}, \lambda_{23}$ and λ_{24} are the roots of the polynomial,
$\lambda^4+3(m_1+m_2-n_1-n_2)\lambda^3+(9n-18-10(m_1+m_2)(n_1+n_2))\lambda^2+3(9n_1+9n_2-9m_1-9m_2-4m_1m_2n_2+4m_2n_1n_2)\lambda+81-81n+90(m_1+m_2)(n_1+n_2)-36m_1m_2n_2-36m_2n_1n_2+40m_1m_2n_1n_2$;
3. *with respect to $P = \{V_{11} \cup V_{21}, V_{12} \cup V_{22}\}$,*
$spec(S_2'(G)) = \begin{pmatrix} \lambda_{21}' & \lambda_{22}' & \lambda_{23}' & \lambda_{24}' & 3 & -3 \\ 1 & 1 & 1 & 1 & m_1+m_2-2 & n_1+n_2-2 \end{pmatrix}$
where $\lambda_{21}', \lambda_{22}', \lambda_{23}'$ and λ_{24}' are the roots of the polynomial,
$\lambda^4 + 3(m_1+m_2-n_1-n_2)\lambda^3 + (-18+9n+8m_1m_2-18m_1n_1-10m_2n_1-10m_1n_2-18m_2n_2+8n_1n_2)\lambda^2+(-27m_1-27m_2+48m_1m_2+27n_1-48m_1m_2n_1+27n_2-60m_1m_2n_2-48n_1n_2+48m_1n_1n_2+60m_2n_1n_2)\lambda+81-81n+72m_1m_2+162m_1n_1+90m_2n_1-144m_1m_2n_1+90m_1n_2+162m_2n_2-180m_1m_2n_2+72n_1n_2-144m_1n_1n_2-180m_2n_1n_2+328m_1m_2n_1n_2$;
4. *with respect to $P = \{V_{11}, V_{12}, V_{21}, V_{22}\}$,*
$spec(S_4(G)) = \begin{pmatrix} \lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44} & 3 & -3 \\ 1 & 1 & 1 & 1 & m_1+m_2-2 & n_1+n_2-2 \end{pmatrix}$,
where $\lambda_{41}, \lambda_{42}, \lambda_{43}$ and λ_{44} are the roots of the polynomial,
$\lambda^4+3(m_1+m_2-n_1-n_2)\lambda^3+(9n-18-10(m_1+m_2)(n_1+n_2)+8n_1n_2+$

$8m_1m_2)\lambda^2 + (48m_1m_2 - 48n_1n_2 + 27n_1 + 27n_2 - 27m_1 - 27m_2 - 32m_1m_2n_2 + 32m_2n_1n_2 - 28m_1m_2n_1 + 28m_1n_1n_2)\lambda + 81 - 81n + 72m_1m_2 + 72n_1n_2 + 90(m_1+m_2)(n_1+n_2) - 84m_1m_2n_1 - 84m_1n_1n_2 - 96m_2n_1n_2 - 96m_1m_2n_2 + 104m_1m_2n_1n_2$;

5. $spec(S_n(G)) = \begin{pmatrix} \lambda_{n1} & \lambda_{n2} & \lambda_{n3} & \lambda_{n4} & 1 & -1 \\ 1 & 1 & 1 & 1 & m_1+m_2-2 & n_1+n_2-2 \end{pmatrix}$, *where* $\lambda_{n1}, \lambda_{n2}, \lambda_{n3}$ *and* λ_{n4} *are the roots of the polynomial,* $\lambda^4 + (m_1+m_2-n_1-n_2)\lambda^3 + (n-2-2(m_1+m_2)(n_1+n_2))\lambda^2 + (n_1+n_2-m_1-m_2-4m_1m_2n_2+4m_2n_1n_2)\lambda + 1 - n + 2(m_1+m_2)(n_1+n_2) - 4m_1m_2n_2 - 4m_2n_1n_2 + 8m_1m_2n_1n_2$.

Proof. 1. By Theorem 2.1, we can observe that G has 3 as an eigenvalue of $S_1(G)$ with multiplicity at least m_1+m_2-2 and -3 as an eigenvalue of $S_1(G)$ with multiplicity at least n_1+n_2-2. We can obtain the other four eigenvalues through the equitable partition $D : V_{11} \cup V_{12} \cup V_{21} \cup V_{22}$ of $V(G)$. The divisor matrix corresponding to the 1-partition Seidel matrix of D is given by

$$A_D = \begin{bmatrix} 3(1-m_1) & -3m_2 & -3n_1 & -3n_2 \\ -3m_1 & 3(1-m_2) & -3n_1 & 3n_2 \\ -3m_1 & -3m_2 & 3(n_1-1) & 3n_2 \\ -3m_1 & 3m_2 & 3n_1 & 3(n_2-1) \end{bmatrix}.$$

The four eigenvalues of the divisor matrix $\lambda_{11}, \lambda_{12}, \lambda_{13}$ and λ_{14} along with -3 which is of multiplicity n_1+n_2-2 and 3 which is of multiplicity m_1+m_2-2 constitutes the spectrum.

2. The proof follows by Theorem 2.1 and by noting that the divisor matrix corresponding to the equitable partition $D : V_{11} \cup V_{12} \cup V_{21} \cup V_{22}$ of $V(G)$ is given by

$$A_D = \begin{bmatrix} 3(1-m_1) & -3m_2 & -n_1 & -n_2 \\ -3m_1 & 3(1-m_2) & -n_1 & n_2 \\ -m_1 & -m_2 & 3(n_1-1) & 3n_2 \\ -m_1 & m_2 & 3n_1 & 3(n_2-1) \end{bmatrix}.$$

3. The proof follows by Theorem 2.1 and by noting that the divisor matrix corresponding to the equitable partition $D : V_{11} \cup V_{12} \cup V_{21} \cup V_{22}$ of $V(G)$ is given by

$$A_D = \begin{bmatrix} 3(1-m_1) & -m_2 & -3n_1 & -n_2 \\ -m_1 & 3(1-m_2) & -n_1 & 3n_2 \\ -3m_1 & -m_2 & 3(n_1-1) & n_2 \\ -m_1 & 3m_2 & n_1 & 3(n_2-1) \end{bmatrix}.$$

4. The proof follows by Theorem 2.1 and by noting that the divisor matrix corresponding to the equitable partition $D : V_{11} \cup V_{12} \cup V_{21} \cup V_{22}$ of $V(G)$ is given by

$$A_D = \begin{bmatrix} 3(1-m_1) & -m_2 & -n_1 & -n_2 \\ -m_1 & 3(1-m_2) & -n_1 & n_2 \\ -m_1 & -m_2 & 3(n_1-1) & n_2 \\ -m_1 & m_2 & n_1 & 3(n_2-1) \end{bmatrix}.$$

5. The proof follows by Theorem 2.1 and by noting that the divisor matrix corresponding to equitable partition $D : V_{11} \cup V_{12} \cup V_{21} \cup V_{22}$ of $V(G)$ is given by

$$A_D = \begin{bmatrix} (1-m_1) & -m_2 & -n_1 & -n_2 \\ -m_1 & (1-m_2) & -n_1 & n_2 \\ -m_1 & -m_2 & (n_1-1) & n_2 \\ -m_1 & m_2 & n_1 & (n_2-1) \end{bmatrix}.$$

□

Corollary 4.16. *Let $G = NSG(m_1, m_2; n_1, n_2)$ be a threshold graph on n vertices. Then,*

- $E_{S_1}(G) = |\lambda_{11}| + |\lambda_{12}| + |\lambda_{13}| + |\lambda_{14}| + 3n - 12$
- $E_{S_2}(G) = |\lambda_{21}| + |\lambda_{22}| + |\lambda_{23}| + |\lambda_{24}| + 3n - 12$
- $E'_{S_2}(G) = |\lambda'_{21}| + |\lambda'_{22}| + |\lambda'_{23}| + |\lambda'_{24}| + 3n - 12$
- $E_{S_4}(G) = |\lambda_{41}| + |\lambda_{42}| + |\lambda_{43}| + |\lambda_{44}| + 3n - 12$
- $E_{S_n}(G) = |\lambda_{n1}| + |\lambda_{n2}| + |\lambda_{n3}| + |\lambda_{n4}| + n - 4.$

Conclusion. In this article, we have obtained several bounds for the partition Seidel energy of graphs with k partitions. Also, the expression for partition Seidel energy of some families of graphs for a given partition is obtained. Recently, motivated by the nesting property of the extremal graphs (chain and threshold graphs), a partial chain graph (PCG) [14] is defined. Also, by extending the concept of nesting from a bipartite graph to a k-partite graph, the authors of the article [24] defined a k-nested graph (KNG). One can try to obtain partition Seidel energy for PCG and KNG.

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