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ON CONVERGENCE PROPERTIES ASSOCIATED WITH EULER TYPE POLYNOMIALS

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Abstract. The aim of this paper is to construct a generalization of Szász-type linear positive operator by using generating function method. The operator presented in the study includes the generating function of Adjoint-Euler polynomials. Many properties of this operator are explored. Moreover, the fundamental convergence properties of this operator are given. Applying this operator to the generating function of adjoint-Euler polynomials, some new formulas and relations have been derived. We also prove Voronovskaya and Grüss-Voronovskaya type theorem for this operator. Finally, some numerical results of this operator with convergence properties associated with the rate of modulus are presented.

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1. Introduction

The generating functions have numerous uses in a variety of disciplines, including analytical number theory, practical analysis, and CAGD, among others. Generating function of a polynomial family gives some useful results such as evaluating certain integrals, using a differential recurrence relation or a pure recurrence relation.

Let $(a_n) = (a_0, a_1, ...)$ be an arbitrary sequence. Generating function of (a_n) is defined as follows [7]:

$$a(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Many researchers obtained useful and important results for polynomial families with the aid of generating functions. Recently, Simsek obtained generating function and useful results for q-Eulerian type polynomials and numbers and also constructed combinatorial sums and identities with well-known special polynomials such as Euler polynomials (*cf.* [27], [28]). In addition, he studied and gave a new family of Appell polynomials which are related to Euler and Frobenius Euler polynomials (*cf.* [29]). Kilar and Simsek obtained useful and important results for many kinds of special polynomials and numbers based on generating functions and their functional equations (*cf.* [16]). Kucukoglu *et al.* derived generating functions for new families of special numbers, and polynomials and obtained functional equations, relations and derivative formulas (*cf.* [19]). Kucukoglu and Simsek constructed a useful algorithm in order to compute some numerical values related to the k-ary Lyndon words which included the generating function of Apostol Bernoulli numbers and also found generating function for Hermite type numbers (*cf.* [20], [21]). Alkan and Simsek investigated some properties of generalized Bernoulli numbers and polynomials by using a fixed periodic group homomorphism (*cf.* [2]). Szabłowski constructed a new multivariate generating function with the aid of Chebyshev polynomials of the first or second kind (*cf.* [35]). Luo and Reina investigated generating functions of Pollaczek and other related polynomials and gave a new integral representation for these polynomials (*cf.* [22]).Costabile *et al.* obtained some important and interesting results including generating functions about polynomials sequences (*cf.* [9]). For more informations about special numbers and polynomials, we refer to works (*cf.* [17], [23], [30], [31], [32], [33], and [34]).

Approximation theory is one of the key areas in which generating functions are applied. In 1950, Szász provided the following generalization of Bernstein polynomials on an infinite interval:

$$S_n(f,x)=\sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right),$$

where $s_{n,k}=e^{-nx}\frac{(nx)^k}{k!}$ (*cf.* [36], [24]).

In 1969, with the help of Appell polynomials, Jakimovski and Leviatan presented a generalization of Szász operators at:

$$P_n(f,x)=\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right),$$

where $p_k(x)$ are called as Appell polynomials and $g(z)=\sum_{k=0}^{\infty} a_k z^k$ is an analytic function (*cf.* [15]).

In 2012, According to Varma *et al.*, the Brenke-type polynomials are included in the linear positive generalization of Szász operators as follows:

$$L_n(f,x)=\frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right),$$

where $A(y)$ and $B(y)$ are analytic functions that are a component of the Brenke type polynomials' generating function (*cf.* [37]).

In 2016, Atakut and Büyükyazıcı [5] established a generalization of the operators of the Kantorovich-type including the Brenke-type polynomials, and explored approximation properties of these operators at the following:

$$L_n^{\alpha_n,\beta_n}(f,x)=\frac{\beta_n}{A(1)B(\alpha_n x)}\sum_{k=0}^{\infty}p_k(\alpha_n x)\int_{\frac{k}{\beta_n}}^{\frac{k+1}{\beta_n}}f(t)dt,$$

where $p_k(x)$ are Brenke type polynomials and A and B are analytic functions (*cf.* [5]).

In 2022, Using the generating function of Apostol-Genocchi polynomials of degree α, Menekşe Yilmaz [39] introduced and investigated the approximation characteristics of a linear positive operator as follows:

$$A_n^{(\alpha,\beta,m)}(f,x)=e^{-(n+\mu)x}\left(\frac{2}{\beta e+1}\right)^{-\alpha}\sum_{k=0}^{\infty}\frac{\mathcal{G}_k^{\alpha}((n+\mu)x,\beta)}{k!}f\left(\frac{k+m}{n+\mu}\right),$$

where Apostol-Genocchi polynomials are referred to as $\mathcal{G}_k^{\alpha}(x,\beta)$ and their generating function is defined to be as:

$$\left(\frac{2t}{\beta e^t+1}\right)^{\alpha}=\sum_{k=0}^{\infty}\mathcal{G}_k^{\alpha}(x,\beta)\frac{t^k}{k!},(|t|<|log-\beta|),$$

(*cf.* [39]). Costabile *et al.* studied many Szász type operators involved Sheffer polynomials and investigated approximation properties by using classical techniques and compared the rate of convergence of these operators (*cf.* [10]).

Using generating functions, the following equation are used to define the Euler polynomials:

$$\sum_{k=0}^{\infty}E_k(x)\frac{t^k}{k!}=\frac{2}{e^t+1}e^{xt}, \tag{1}$$

(*cf.* [8]-[17]).

By using the generating function of Euler polynomials, many useful results are obtained about concering Euler polynomials in various fields such as mathematical analysis, analytic number theory ,and combinatorics. For example, Chen *et al.* obtained some new identities and properties by constructing second-order non-linear recursive polynomials (*cf.* [8]). Kim *et al.* constructed some new computation formulas which included Euler numbers and polynomials with the aid of its generating function (*cf.* [18]). Kilar *et al.* derived some new identities and relations related to trigonometric functions and generating functions for Euler polynomials (*cf.* [17]).

We give definitions, lemmas and theorem that enables us to obtain results in the sequel.

Natalini and Ricci [25] introduced the adjoint Appell-Euler polynomials by using Sheffer polynomial sets and investigated their important properties such as recurrence relations, generating function's ordinary and partial differential equations.

The Adjoint-Euler polynomials are defined using their generating function as follows:

$$\sum_{k=0}^{\infty} \tilde{\varepsilon}_k(x)\frac{t^k}{k!} = \frac{e^t+1}{2}e^{xt}, \tag{2}$$

(*cf.* [25]).

Remark 1. *It is clear that the Adjoint-Euler polynomials are equal to the Euler polynomials of degree* -1 *as follows:*

$$E_k^{(-1)}(x) = \tilde{\varepsilon}_k(x),$$

(cf. [16]*,* [17]*, and* [25]*).*

By applying Taylor expansion, we give some adjoint-Euler polynomials as follows:

$$\tilde{\varepsilon}_0(x) = 1.$$

$$\tilde{\varepsilon}_1(x) = x + \frac{1}{2}.$$

$$\tilde{\varepsilon}_2(x) = x^2 + x + \frac{1}{2}.$$

$$\tilde{\varepsilon}_3(x) = x^3 + \frac{3}{2}x^2 + \frac{3}{2}x + \frac{1}{2}.$$

Throughout this study, the results are given by assuming that $C[0,1)$, $C[0,\infty)$, $C_B[0,\infty)$, and $C_{B^2}[0,\infty)$ denotes respectively the space of uniformly continuous function in $[0,1)$, the space of uniformly continuous function in $[0,\infty)$, the space of uniformly continuous and bounded function in $[0,\infty)$, the space of uniformly continuous, bounded, and second-order differentiable function in $[0,\infty)$, and the space of locally bounded and second-order differentiable function.

The $C_\rho^2[0,\infty)$ is defined to be as:

$$C_\rho^2[0,\infty) = \{f \in C_p[0,\infty) : lim_{x\to\infty}\frac{f(x)}{p(x)} = k_f < \infty\},$$

where $\rho(x) = x^2+1$ is an increasing function on $[0,\infty)$, the $B_\rho[0,\infty)$ is set of real-valued functions on $[0,\infty)$ such that $|f(x)| \leq M_f p(x)$ where M is a constant. Besides, the $C_\rho[0,\infty)$ is set of $f \in B_\rho[0,\infty)$ where f is continuous.

The modulus of continuity is a mathematical tool for measuring the speed of convergence or divergence of a sequence. Now we give definitions of both moduli of continuity and related concepts.

Definition 1.1 (*cf.* [11], *P. 40*) Assume that the function f is uniformly continuous on $[0, \infty)$ and $\delta > 0$. Following is the definition of the function f's modulus of continuity by $\omega(f, \delta)$:

$$\omega(f, \delta) := sup|f(x) - f(y)|, \tag{3}$$

where $x, y \in [0, \infty)$ and $|x - y| \leq \delta$.

The following relation thus holds for any $\delta > 0$ and each $x \in [0, \infty)$:

$$|f(x) - f(y)| = \omega(f, \delta_n)\left(\frac{|x-y|}{\delta} + 1; x\right) \tag{4}$$

(*cf.* [6], [11]).

Definition 1.2 (*cf.* [26]) The continuity's second-order modulus is given as below:

$$\omega^2(f, \delta) = \sup_{0<h\leq\delta} \sup_{x\in[0,\infty)} |f(x+h) - 2f(x) + f(x-h)|, \tag{5}$$

where $f \in C_B[0, \infty)$ and $\delta > 0$.

Definition 1.3 (*cf.* [11], *P. 51*) $Lip_1(\alpha, K)$, $0 < \alpha \leq 1$, denotes the class of functions that verify the inequality $\omega_1(\phi, \sigma) \leq K\sigma^\alpha$ for all $\sigma > 0$ with positive K. Next, we have

$$|E_n^*(\phi; x) - \phi(x)| \leq K\sigma_n^\alpha(x). \tag{6}$$

Definition 1.4 (*cf.* [12]) The Peetre's K-functional is provided at the following equation:

$$K(f; \delta) = inf\{g \in C_B^2[0, \infty) : \|f - g\|_{C_B} + \delta\|g\|_{C_B^2}\} \tag{7}$$

where

$$C_B^2[0, \infty) = \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}$$

and

$$\|g\|_{C_B^2} := \|g\|_{C_B} + \|g'\|_{C_B} + \|g''\|_{C_B}.$$

The following inequality is obtained between Peetre's K-functional and the second modulus of continuity for any constant M which is independent of f and δ:

$$K(f; \delta) \leq M\{\omega_2(f; \sqrt{\delta}) + min(1, \delta)\|f\|_{C_B}\},$$

(*cf.* [4], [3], [12]).

The moment and central moment functions are mathematical tool that helps in constructing the Korovkin-Bohman theorem to examine the uniform convergence of an operator.

Definition 1.5 (*cf.* [14]) Let $L_n(f, x)$ be a linear positive operator. The r-th order moment of $L_n(f, x)$ is given at the following:

$$e_r(t) = t^r,$$

where $r = 0, 1, 2...$ and t.

With the aid of Definition 1.5, the r-th order central moment function of $L_n(f,x)$ is defined as follows:

$$L_n((e_1 - e_0x)^r, x) := L_n((t-x)^r, x).$$

Theorem 1.6 (*cf.* [11]) If $A_n : C[0,1] \longrightarrow C[0,1]$ is a sequences of positive linear maps, then

$$\lim_{n\to\infty} \|A_n f_i - f_i\| = 0,$$

where $f_i(t) = t^i$ for $i = 0, 1, 2$. Then for every $f \in C[0,1]$ we have

$$\lim_{n\to\infty} \|A_n f - f\| = 0.$$

To help of constructing Korovkin's theorem we define a set at the following:

$$E = \left\{ f \mid x \in [0,\infty), \lim_{x\to\infty} \frac{f(x)}{1+x^2} \text{ exist} \right\}.$$

Euler-type polynomials have many applications in combinatorics and applied sciences, especially in analytic number theory. Positive linear operators using special polynomials with generating functions under appropriate conditions are available in the literature. The aim of this paper is to study Adjoint-Euler polynomials in order to combine both positive linear operators and the method of generating functions.

The remainder of this study is structured as follows:

In section 2, we establish our operator with the aid of generating function of Adjoint-Euler polynomials. By using lemmas and definitions in section 1, we firstly find moment and central moment functions. Secondly we show that our operator is uniformly convergence by using Korovkin's theorem. And then, we look into a variety of convergence characteristics for our operator, such as the second-order modulus of continuity, the Lipschitz class, the Peetre's K-function, and the modulus of continuity. Finally, we construct Voronovskaya type and Voronovskaya-Grüss type theorems for our operator.

In section 3, we present some numerical examples by calculating the rate of convergence for the operator by using the modulus of continuity.

2. Main Results

In this part, we first define operator and give moment and central moment functions. Using moment functions, we present the Korovkin-type approximation theorem.

Using the adjoint-Euler polynomials' generating function for $t = 1$ and $x \to nx$, the following equation yields the operator:

$$E_n^*(f,x) = \left(\frac{2}{e+1}\right) e^{-nx} \sum_{k=0}^{\infty} \frac{\tilde{\varepsilon}_k(nx)}{k!} f\left(\frac{k}{n}\right), \tag{8}$$

in which $x \geq 0$ and $x \in N$.

By using Eq. (8), we give moment and central moment functions for $E_n^*(f,x)$.

Lemma 2.1. *For all $x \in [0,\infty)$ and $n \in N$, we get*

$$E_n^*(e_0(x),x) = 1, \tag{9}$$

$$E_n^*(e_1(x),x) = x + \frac{e}{n(e+1)}, \tag{10}$$

and

$$E_n^*(e_2(x),x) = x^2 + \left(\frac{3e+1}{e+1}\right)\frac{x}{n} + \left(\frac{2e}{e+1}\right)\frac{1}{n^2}. \tag{11}$$

Proof. Utilizing derivatives operator, we obtain the first and second derivative of adjoint Euler polynomials from the generating functions of Adjoint-Euler polynomials as follows:

$$\sum_{k=0}^{\infty} \tilde{\varepsilon}_k(x)\frac{kt^k}{k!} = \frac{1}{2}e^{xt}\left[e^t + x(e^t+1)\right].$$

Therefore,

$$\sum_{k=0}^{\infty} \tilde{\varepsilon}_k(x)\frac{k(k-1)t^k}{k!} = \frac{1}{2}e^{tx}\left(e^t(x+1)^2 + x^2\right).$$

It is simple to see from the definition of $E_n^*(f,x)$ for $f(x) = 1$,

$$E_n^*(e_0(x),x) = \left(\frac{2}{e+1}\right)e^{-nx}\sum_{k=0}^{\infty}\frac{\tilde{\varepsilon}_k(nx)}{k!} = \left(\frac{2}{e+1}\right)e^{-nx}\left(\frac{e+1}{2}\right)e^{nx} = 1.$$

Let $f(x) = x$. For $t = 1$ and $n \to nx$, the above equation reduces to the following equation:

$$\begin{aligned} E_n^*(e_1(x),x) &= \left(\frac{2}{e+1}\right)e^{-nx}\sum_{k=0}^{\infty}\frac{\tilde{\varepsilon}_k(nx)}{k!}\frac{k}{n} \\ &= \left(\frac{2}{e+1}\right)e^{-nx}\frac{1}{2n}e^{nx}[e + nx(e+1)] \\ &= x + \frac{e}{n(e+1)}. \end{aligned}$$

Let $f(x) = x^2$. For $t = 1$ and $n \to nx$, we also get

$$E_n^*(e_2(x),x) = \left(\frac{2}{e+1}\right) e^{-nx} \sum_{k=0}^{\infty} \frac{\tilde{\varepsilon}_k(nx)}{k!} \frac{k(k-1)}{n}$$
$$= \left(\frac{2}{e+1}\right) e^{-nx} \left(\sum_{k=0}^{\infty} \frac{\tilde{\varepsilon}_k(nx)}{k!} \frac{k^2}{n} + \sum_{k=0}^{\infty} \frac{\tilde{\varepsilon}_k(nx)}{k!} \frac{k}{n}\right)$$
$$= x^2 + \left(\frac{3e+1}{e+1}\right) \frac{x}{n} + \left(\frac{2e}{e+1}\right) \frac{1}{n^2}.$$

Thus the desired results are obtained. □

By using the same method as Lemma 2.1, we obtain

$$E_n^*(e_3(x),x) = x^3 + \frac{6e+1}{(e+1)n} x^2 + \frac{11e+2}{(e+1)n^2} x + \frac{8}{(e+1)n^3},$$

and

$$E_n^*(e_4(x),x) = x^4 + \frac{10e+6}{(e+1)n} x^3 + \frac{16e-11}{(e+1)n^2} x^2 + \frac{10e+26}{(e+1)n^3} x + \frac{2e+1}{(e+1)n^4}.$$

Lemma 2.2. *Let $x \in [0,\infty)$. According to Lemma 2.1, we obtain the following equalities.*

$$E_n^*((e_1 - e_0 x),x) = \frac{e}{n(e+1)}, \tag{12}$$

$$E_n^*((e_1 - e_0 x)^2,x) = \frac{5e+1}{(e+1)n} x + \frac{2e}{(e+1)n^2}. \tag{13}$$

$$E_n^*((e_1 - e_0 x)^4,x) = \frac{-16e-19}{(e+1)n^2} x^2 + \frac{10e-6}{(e+1)n^3} x + \frac{2e+1}{(e+1)n^4}. \tag{14}$$

Proof. The proof of Eq. (12) and Eq. (13) can be found in [1]. We have used the linearity property of E_n^* to find

$$E_n^*((e_1 - e_0 x)^4,x) = E_n^*(s^4,x) - 4x E_n^*(s^3,x) + 6x^2 E_n^*(s^2,x) - 4x^3 E_n^*(s,x) + x^4 E_n^*(1$$

Using the results in Lemma 2.1 and basic mathematical operations, the following result is obtained:

$$E_n^*((e_1 - e_0 x)^4,x) = \frac{-16e-19}{(e+1)n^2} x^2 + \frac{10e-6}{(e+1)n^3} x + \frac{2e+1}{(e+1)n^4}.$$

Hence, the desired results are obtained.

□

Some properties of central moment functions of $E_n^*(f,x)$ are given by the following lemmas:

Lemma 2.3. *With respect to the operators $E_n^*(f,x)$, we get*

$$E_n^*((e_1 - e_0x), x) \leq \frac{0.73}{n},$$

and

$$E_n^*((e_1 - e_0x)^2, x) \leq \frac{3.92x}{n} + \frac{1.46}{n^2}.$$

Lemma 2.4. *The following expressions hold true:*

$$\lim_{n\to\infty} nE_n^*((e_1 - e_0x), x) = \frac{e}{(e+1)}, \tag{15}$$

and

$$\lim_{n\to\infty} nE_n^*((e_1 - e_0x)^2, x) = \frac{5e+1}{(e+1)}x. \tag{16}$$

By using Lemma 2.1 and Theorem 1.6, we give Korovkin's theorem for $E_n^*(f,x)$ at the following theorem:

Theorem 2.5. *Let $f \in [0,\infty] = C[0,\infty] \cap E$.*

$$\lim_{n\to\infty} \|E_n^* f - f\| = 0. \tag{17}$$

Proof. The proof of this theorem was announced in [1]. □

The key tool used in the theory of approximation by positive linear operators is the notion of modulus of continuity. For giving quantitative estimates, this notion works well. For determining the rate of convergence in this section, we employ both the standard and second moduli of continuity.

We will give the following theorem without proof, since it will be used to obtain the numerical examples in the third chapter. See [1] for the proof of theorem.

Theorem 2.6. *Assume that f belongs to set E and is a uniformly continuous function on $[0,1]$. Then, there is*

$$|E_n(f;x) - f| \leq 2\omega\left(f; \sqrt{E_n\left((s-x)^2; x\right)}\right), \tag{18}$$

where the function's continuity modulus, ω, is used.

The following theorem satisfies a prediction for the error of the operator $E_n^*(f,x)$ to a function f belonging to the Lipschitz class of order by (5).

Theorem 2.7. *Let f be in $Lip_M(\alpha)$. For $x \geq 0$, here are*

$$|E_n^*(f;x) - f(x)| \leq M\delta^*(x), \tag{19}$$

where $\delta^*(x) := \sqrt{E_n^*((s-x)^2, x)}$.

Proof. We obtain the following from E_n^*'s monotonicity properties:

$$|E_n^*(f;x) - f(x)| \leq ME_n^*(|s-x|^{\alpha};x). \tag{20}$$

By applying the Hölder inequality, one can infer from (20)

$$|E_n^*(f;x) - f(x)| \leq M(E_n^*((s-x)^2;x))^{\frac{\alpha}{2}}. \tag{21}$$

So, the theorem's proof is finished. □

Theorem 2.8. *The following statement is true for any $f \in C_B(0,\infty)$ and $x \in (0,\infty)$:*

$$|E_n^*(f;x) - f(x)| \leq 2K(f;\lambda_n(x)), \tag{22}$$

where $\lambda_n(x) = \frac{x}{4n} + \frac{e}{e+1}\frac{n+1}{2n^2}$.

Proof. Let $h \in C_B^2(0,\infty)$. Using the linearity property of E_n^* operators and Taylor's expansion, we are able to

(23)

$$E_n^*(f;x) - f(x) = f'(x)E_n^*((s-x);x) + \frac{f'(\eta)}{2}E_n^*\left((s-x)^2;x\right), \eta \in (x,s).$$

By using Lemma 2, we have

$$|E_n^*(f;x) - f(x)| \leq \left(\frac{1}{2n} + \frac{e(n+1)}{n^2(e+1)}\right)\|h\|_{C_B^2(0,\infty)}. \tag{24}$$

On the other hand, if we apply Lemma 2.1 and expression (24), we obtain

$$\begin{aligned} |E_n^*(f;x) - f(x)| &\leq |E_n^*(f-h;x)| + |E_n^*(h;x) - h(x)| + |f(x) - h(x)| \\ &\leq 2\|f-h\|_{C_B(0,\infty)} + |E_n^*(h;x) - h(x)| \\ &\leq 2(\|f-h\|_{C_B(0,\infty)} + \lambda_n(x)\|h\|_{C_B^2(0,\infty)}). \end{aligned} \tag{25}$$

In the equation above, if we pick the infimum on the right-hand side, we get over all $h \in C_B^2(0,\infty)$; The intended result is attained as follows:

$$|E_n^*(f;x) - f(x)| \leq 2K(f;\lambda_n(x)). \tag{26}$$

□

Theorem 2.9. *For $E_n^*(f;x)$, if $f \in C_B(0,\infty)$, then we have*

$$|E_n^*(f;x) - f(x)| \leq Cw_2(f;\sqrt{v_n(x)}) + \omega\left(f;\left(\frac{e}{n(e+1)}\right)\right), \tag{27}$$

where C is constant and

$$\nu_n(x) = \frac{1}{4}\{E_n^*((s-x)^2,x)\} + \left(\frac{e}{n(e+1)}\right)^2. \tag{28}$$

Proof. Let us define an operator $\mathcal{F}_n$ by

$$\mathcal{F}_n(f,x) = E_n^*(f,x) - f\left(\frac{e}{n(e+1)} + x\right) + f(x). \tag{29}$$

We deduce from Lemma 2.2

$$\mathcal{F}_n(s-x;x) = 0. \tag{30}$$

By the Taylor formula with integral reminder term for $h \in C_B^2(0,\infty)$, we can write

$$h(s) = h(x) + (s-x)h'(x) + \int_a^b (s-u)h''(u)\,du. \tag{31}$$

By using Eq. (30) and Eq. (31), we have

$$\begin{aligned}
|\mathcal{F}(h,x) - h(x)| &= \left|\mathcal{F}_n\left(\int_x^s (s-u)h^{''}(u)\,du;x\right)\right| \\
&\leq \left|E_n^*(f,x)\left(\int_x^s (s-u)h^{''}(u)\,du;x\right)\right| \\
&+ \left[\int_x^{\frac{e}{n(e+1)}+x}\left(\frac{e}{n(e+1)} + x - u\right)h''(u)du\right] \\
&\leq \{E_n^*((e_1-e_0x))^2 + (\frac{e}{n(e+1)}+x)^2\}\|h''\|_{C_B(0,\infty)} \\
&\leq 4\nu_n(x)\|h\|_{C_B^2(0,\infty)}.
\end{aligned}$$

Considering the meaning of the E_n^* operator, using Lemma 2.1 and the above inequality, we determine that

$$\begin{aligned}
|E_n^*(f;x) - f(x)| &\leq |\mathcal{F}_n(f-h;x) - (f-h)(x)| \\
&+ |\mathcal{F}_n(h;x) - h(x)| + \left|f(\frac{e}{n(e+1)}+x) - f(x)\right| \\
&\leq 4\|f-h\|_{C_B(0,\infty)} + 4\nu_n(x)\|h\|_{C_B^2(0,\infty)} + \omega(f;\frac{e}{n(e+1)}).
\end{aligned}$$

Using the inequality stated above and accounting for Definition 1.4, We get to the conclusion that

$$\begin{aligned}
|E_n^*(f;x) - f(x)| &\leq 4K(f;\nu_n(x)) + \omega(f;\frac{e}{n(e+1)}) \\
&\leq C\omega_2(f;\sqrt{\nu_n(x)}) + \omega\left(f;\frac{e}{n(e+1)}\right).
\end{aligned}$$

The proof is finished with this. □

Now, we first present an altered version of Voronovskaya's asymptotic formula for Bernstein operators from 1932 in this section ([38]), and we then present the Grüss type Voronovskaya theorem (*cf.* [13]).

Theorem 2.10. *Let $f \in C^2_\rho$. We have;*

$$\lim_{n\to\infty} n(E_n^*(f,x) - f(x)) = \frac{e}{e+1} f'(x) + \frac{x}{2} f''(x), \tag{32}$$

for each fixed $x \in [0,\infty)$.

Proof. We use the Taylor formula for a fixed point $x_0 \in [0,\infty)$. For all $t \in [0,\infty)$, we give

$$f(t) - f(x_0) = f'(x_0)(t-x_0) + \frac{1}{2}(t-x_0)^2 f''(x_0) + g(t,x_0)(t-x_0)^2, \tag{33}$$

where $g(t,x_0)$ is a function that belongs in the space $C_E[0,\infty)$ and $\lim_{t\to x_0} g(t;x_0) = 0$. After applying the linear operator E_n^* to the Taylor series of f, we give

$$\begin{aligned} n[E_n^*(f;x_0) - f(x_0)] &= f'(x_0) n E_n^*(x - x_0; x_0) \\ &+ \frac{1}{2} f''(x_0) n E_n^*((t-x_0)^2; x_0) + n E_n^*(g(t,x_0)(t-x_0)^2; x_0). \end{aligned}$$

Cauchy-Schwarz inequality has allowed us to

$$E_n^*(g(t,x_0)(t-x_0)^2; x_0) \le \sqrt{(E_n^*(g^2(t,x_0)))(E_n^*((t-x_0)^4; x_0))}. \tag{34}$$

One has from central moment function that

$$\lim_{n\to\infty} n^2 (E_n^*(t-x_0)^4; x_0) = \frac{-16e-19}{e+1} x_0^2. \tag{35}$$

Since for the function $g(x,x_0) = h^2(x;x_0)$, $x \ge 0$ we have $g(x,x_0) \in C_E[0,\infty)$ and $\lim_{x\to x_0} g(x,x_0) = 0$. Then it follows from Theorem 1(Korovkin) that

$$\lim_{n\to\infty} (E_n^*(h^2(t,x_0),x_0);x_0) = \lim_{n\to\infty} (E_n^*(g(t,x_0),x_0);x_0) = g(x_0,x_0) = 0. \tag{36}$$

uniformly with respect to $x_0 \in [0,a]$. So, we obtain

$$\lim_{n\to\infty} n(E_n^*(g(t,x_0)(t-x_0)^2; x_0) = 0, \tag{37}$$

then taking the limit $n \to \infty$ in Eq.(35), and applying lemma 2 we have,

$$\lim_{n\to\infty} n(E_n^*(f,x) - f(x)) = \frac{e}{e+1} f'(x) + \frac{x}{2} f''(x). \tag{38}$$

The proof is completed. □

We provide a Grüss-Voronovskaya type theorem for $E_n^*(f,x)$ at the following by applying the Korovkin theorem and the Voronovskaya type theorem:

Theorem 2.11. *If f and g are bounded on I, differentiable in some neighborhood of x and has second derivative $f''(x)$, and $g''(x)$ for some $x \in I$, then*

$$\lim_{n\to\infty} n E_n^*(f,g;x) = x f'(x) g'(x). \tag{39}$$

Proof. First, let's write the first and second derivatives of $f \circ g$:

$$(f \circ g)' = f'g + fg'.$$

$$(f \circ g)''(x) = f''(x)g(x) + 2f'(x)g'(x) + g''(x).$$

Assume that

$$E_n^*(f, g; x) = E_n^*(fg; x) - E_n^*(f; x)E_n^*(g; x).$$

Using the operator E_n^*'s linearity, we are able to

$$\begin{aligned} &E_n^*(f, g; x) \\ &= E_n^*(fg; x) - f(x)g(x) - (fg)'(x)E_n^*(s - x; x) - \frac{(fg)''(x)}{2!}E_n^*((s - x)^2; x) \\ &- g(x)\left[E_n^*(f; x) - f(x) - f'(x)E_n^*(s - x; x) - \frac{f''(x)}{2!}E_n^*((s - x)^2; x)\right] \\ &- E_n^*(f; x)\left[E_n^*(g; x) - g(x) - g'(x)E_n^*(s - x; x) - \frac{g''(x)}{2!}E_n^*((s - x)^2; x)\right] \\ &+ \frac{1}{2!}E_n^*((s - x)^2; x)[f(x)g''(x) + 2f'(x)g'(x) - g''(x)E_n^*(f, x)]. \end{aligned}$$

As a result, using Lemma 2.2, we have

$$\begin{aligned} &\lim_{n\to\infty} nE_n^*(f, g; x) \\ &= \lim_{n\to\infty} n[E_n^*(fg; x) - f(x)g(x)] - \frac{e}{e+1}(fg)'(x) - \frac{(fg)''(x)}{2!}x \\ &- g(x)\left[E_n^*(f; x) - f(x) - f'(x)E_n^*(s - x; x) - \frac{f''(x)}{2!}E_n^*((s - x)^2; x)\right] \\ &- E_n^*(f; x)\left[E_n^*(g; x) - g(x) - g'(x)E_n^*(s - x; x) - \frac{g''(x)}{2!}E_n^*((s - x)^2; x)\right] \\ &+ \frac{1}{2!}E_n^*((s - x)^2; x)[f(x)g''(x) + 2f'(x)g'(x) - g''(x)E_n^*(f, x)]. \end{aligned}$$

□

When we write the results, we found in Theorem 2.5 and Theorem 2.10 in the appropriate parts in the last expression, The theorem's proof has been finished.

3. Numerical Examples

In this part, we construct error estimation tables analyzing the convergence of the operator $E_n^*(f, x)$ to a few example functions. The modulus of continuity is used to obtain the error estimates. $Maple2023^{TM}$ computing program was used to calculate the error estimates.

Example 1 In Table 1, we demonstrate the numerical results of the approximation of $E_n^*(f, x)$ to the function $f(x) = x^2e^{-2x}$.

n	**Estimation by** $\omega(f,\delta)$
10	0.2348740864
10^2	0.08950107214
10^3	0.02882962304
10^4	0.009133538684
10^5	0.002888810576
10^6	0.0009135389094
10^7	0.0002888869802

Table 1. Error of approximation of the operators $E_n^*(f,x)$ to $f(x) = x^2e^{-2x}$

Example 2 In Table 2, we show the numerical results of the approximation of $E_n^*(f,x)$ to the function $f(x) = (x - \frac{2}{3})(x - \frac{1}{2})(x - \frac{3}{5})$ for $n = 1, 2, 3, 4, 5, 6, 7$.

n	**Estimation by** $\omega(f,\delta)$
10	0.4006080608
10^2	0.2866236788
10^3	0.1161089198
10^4	0.03956970250
10^5	0.01280822557
10^6	0.004080151260
10^7	0.001293250612

Table 2. Error of approximation of the operators $E_n^*(f,x)$ to $f(x) = (x - \frac{2}{3})(x - \frac{1}{2})(x - \frac{3}{5})$ for $n = 1, 2, 3, 4, 5, 6, 7$
.

In these examples, we numerically find the approximation of $E_n^*(f,x)$ to function $f(x) = x^2e^{-2x}$ and $f(x) = (x - \frac{2}{3})(x - \frac{1}{2})(x - \frac{3}{5})$, respectively, by using the modulus of continuity. We observe that the amount of error when using ω gets smaller as n increases.

4. Discussion and Conclusion

Recently, the convergence properties of linear positive operators created with the assistance of generating functions have been studied by many researchers (*cf.* [9], [37],[5], [39], and [10]).

By the same motivation, we construct a Szász type operator involving the generating function of Adjoint-Euler polynomials were given by Natalini and Ricci in [25]. We investigated the approximation properties of our operator with the help of known convergence properties such as modulus of continuity, Peetre K functional,and Korovkin's theorem. We also proved Voronovskaya and Grüss-Voronovskaya type theorem for our operator. Finally, we illustrated the error of approximation of operators by using modulus of continuity.

The use of the Euler type polynomial family, which has important results in combinatorics and analytic number theory, to obtain positive linear operators in approximation theory has enabled this type of polynomial families to move to a new application area.

In this study, Euler polynomials of order -1 are chosen because all terms of these polynomials are positive for $x \in [0, 1]$. The examples in the numerical results section show that the use of the operator for fractional functions and polynomial functions is advantageous since the error estimation decreases with increasing n values.

For further works, the q-analog of Adjoint-Euler polynomials can be constructed and investigated their convergence properties.

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