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GENERAL SUM CONNECTIVITY ENERGY OF GRAPH

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Abstract. Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G)$. The general sum connectivity matrix of a graph G whose vertex v_i has degree d_i is defined by the $n \times n$ matrix whose $(i, j)-$entry is equal to $(d_i + d_j)^\alpha$ if the vertices v_i and v_j are adjacent and 0 otherwise. The general sum connectivity energy is the sum of absolute values of the eigenvalues of general sum connectivity matrix. We provide lower and upper bounds for $GSCE$.

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1. Introduction

All graphs considered in this paper are finite, simple, undirected and without multiple edges. For notations and terminology, see [6]. For a graph G, we represent the degree of the vertex by d_i. For the connectedness of two vertices we use the notation $v_i \sim v_j$.

In 1978 Ivan Gutman introduced a novel graph spectral quantity which he called it as graph energy [4]. Let G be a simple graph of order n and $A(G)$ be it's adjacency matrix. The energy of a graph G is defined to be the sum of the absolute values of its eigen values.

A topological index of graph is a numerical quantity which characterize its topology. These are the molecular structure descriptors calculated from a molecular graph of a chemical compound.

In 2009, Bo Zhou et al., [11] defined the sum connectivity index as

$$SC = \sum_{i \sim j} \left(\frac{1}{\sqrt{d_i + d_j}} \right).$$

They also introduced sum connectivity matrix and sum connectivity energy in [10]. In 2010, Bo Zhou et al., [12] defined the general sum connectivity index as

$$GSC = \sum_{i \sim j} (d_i + d_j)^\alpha.$$

In [3], Hanyuan Deng et al., defined the general sum connectivity matrix as

$$GSC_{ij} = \begin{cases} \sum_{i<j} (d_i + d_j)^\alpha & \text{if } v_i \sim v_j, \\ 0 & otherwise. \end{cases}$$

In this paper, we discuss few more properties of general sum connectivity matrix and we introduce the general sum connectivity energy as the sum of the absolute values of the eigenvalues of the general sum connectivity matrix.

$$GSCE(G) = \sum_{i=1}^{n} |\lambda_i|$$

(where λ_i are the general sum connectivity eigenvalues.)

2. Some basic properties of general sum connectivity energy of a graph

In this section, we discuss the basic properties of general sum connectivity matrix and it's energy.

Theorem 2.1. *Let G be a graph of order n, then $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_n|$ if and only if $G = K_n$ or $G = [\frac{n}{2}]K_2$ (n is even).*

Theorem 2.2. *For a regular graph G with regularity r, $(2r)^{\alpha}E(G) = GSCE(G)$.*

Proof. The general sum connectivity matrix of a regular graph G with regularity r will be having the entries 0 and $(2r)^{\alpha}$. Let λ_i be the eigen values of the general sum connectivity matrix and β_i are the eigenvalues of the adjacency matrix, then $(2r)^{\alpha}\lambda_i = \beta_i$.
Thus the proof follows. □

Theorem 2.3. *Let G be a $r-$regular graph ($r \geq 3$) of order n and let $\lambda_1 \geq \lambda_2 \geq \lambda_3 ... \geq \lambda_n$ be the general sum connectivity eigen values of the graph G. Then the general sum connectivity energy of complement $\bar{G}$ is*

$$[2(n-r-1)]^{\alpha}\left(|n-r-1| + |\sum_{i=2}^{n} -\lambda_i(2r)^{\alpha} - 1|\right)$$

Proof. The general sum connectivity matrix of the complement $\bar{G}$ of a regular graph G with regularity r will be having the entries 0 and $(2(n-r-1))^{\alpha}$. Let λ_i be the eigen values of general sum connectivity matrix, then the eigenvalues of complement $\bar{G}$ of a regular graph G are
$(2(n-r-1))^{\alpha}$ and $(2(n-r-1))^{\alpha}(-\lambda_i(2r)^{\alpha} - 1)$ for $i = 2, 3...n$.
Thus the proof follows by the definition of general sum connectivity energy. □

Theorem 2.4. *Let G be a graph with n vertices. Then*

$$GSCE(G) \leq (d_i + d_j)^{\frac{\alpha}{2}}\sqrt{2n}$$

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $GSC(G)$. Now by the Cauchy-Schwartz inequality we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right).$$

For $a_i = 1$ and $b_i = \lambda_i$,

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$

which implies that
$$[GSCE(G)]^2 \leq 2n(d_i + d_j)^{\alpha}$$
and finally
$$GSCE(G) \leq (d_i + d_j)^{\frac{\alpha}{2}} \sqrt{2n}$$
which is an upper bound. □

Theorem 2.5. *Let G be a graph with n vertices and if R= det $GSC(G)$, then*
$$GSCE(G) \geq \sqrt{2(d_i + d_j)^{2\alpha} + n(n-1)R^{\frac{2}{n}}}.$$

Proof. By the definition of general sum connectivity energy,
$$\begin{aligned}(GSCE(G))^2 &= \left(\sum_{i=1}^{n} \mid \lambda_i \mid\right)^2 \\ &= \sum_{i=1}^{n} \mid \lambda_i \mid \sum_{j=1}^{n} \mid \lambda_j \mid \\ &= \left(\sum_{i=1}^{n} \mid \lambda_i \mid^2\right) + \sum_{i \neq j} \mid \lambda_i \mid\mid \lambda_j \mid .\end{aligned}$$
Using arithmetic mean and geometric mean inequality, we have
$$\frac{1}{n(n-1)} \sum_{i \neq j} \mid \lambda_i \mid\mid \lambda_j \mid \;\; \geq \;\; \left(\prod_{i \neq j} \mid \lambda_i \mid\mid \lambda_j \mid\right)^{\frac{1}{n(n-1)}}.$$
Therefore,
$$\begin{aligned}(GSCE(G))^2 &\geq \sum_{i=1}^{n} \mid \lambda_i \mid^2 + n(n-1) \left(\prod_{i \neq j} \mid \lambda_i \mid\mid \lambda_j \mid\right)^{\frac{1}{n(n-1)}} \\ &\geq \sum_{i=1}^{n} \mid \lambda_i \mid^2 + n(n-1) \left(\prod_{i=1}^{n} \mid \lambda_i \mid^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\ &= \sum_{i=1}^{n} \mid \lambda_i \mid^2 + n(n-1)R^{\frac{2}{n}} \\ &= 2(d_i + d_j)^{2\alpha} + n(n-1)R^{\frac{2}{n}}.\end{aligned}$$
Thus,
$$GSCE(G) \geq \sqrt{2(d_i + d_j)^{2\alpha} + n(n-1)R^{\frac{2}{n}}}.$$
□

Proposition 2.6. *The first three coefficients of the polynomial $\phi_{GSC}(G, \lambda)$ are given as follows:*

(i) $a_0 = 1$,

(ii) $a_1 = 0$,
(iii) $a_2 = -\sum_{i<j}(d_i + d_j)^{2\alpha}$.

Proof. **(i)** From the definition, $\Phi_{GSC}(G, \lambda) = det[\lambda I - GSC(G)]$ and then we get $a_0 = 1$ after easy calculations.

(ii) The sum of the determinants of all 1×1 principal submatrices of $GSC(G)$ is equal to the trace of $GSC(G)$. Therefore

$$a_1 = (-1)^1 \cdot \text{trace of } [GSC(G)] = 0.$$

(iii) Similarly we have

$$\begin{aligned}
(-1)^2 a_2 &= \sum_{1 \le i < j \le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\
&= \sum_{1 \le i < j \le n} a_{ii}a_{jj} - a_{ji}a_{ij} \\
&= \sum_{1 \le i < j \le n} a_{ii}a_{jj} - \sum_{1 \le i < j \le n} a_{ji}a_{ij} \\
&= -\sum_{i<j}(d_i + d_j)^{2\alpha}.
\end{aligned}$$

□

Proposition 2.7. *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the general sum connectivity eigenvalues of $GSC(G)$, then*

$$\sum_{i=1}^{n} {\lambda_i}^2 = 2\sum_{i<j}(d_i + d_j)^{2\alpha}.$$

Proof. We know that

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\
&= 2\sum_{i<j} a_{ij}^2 + \sum_{i=1}^{n} a_{ii}^2 \\
&= 2\sum_{i<j} a_{ij}^2 \\
&= 2\sum_{i<j}(d_i + d_j)^{2\alpha}.
\end{aligned}$$

□

Theorem 2.8. *Let G be a graph with n vertices. Then*

$$GSCE(G) \le \sqrt{2n\sum_{i<j}(d_i + d_j)^{2\alpha}}$$

.

Proof. Let $\lambda_1,\ \lambda_2,\ \dots,\ \lambda_n$ be the eigenvalues of $GSC(G)$. Now by the Cauchy-Schwartz inequality we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right).$$

We let $a_i = 1$ and $b_i = \alpha_i$. Then

$$\left(\sum_{i=1}^{n} |\alpha_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} |\alpha_i|^2\right)$$

which implies that

$$[GSCE(G)]^2 \leq n(2)(d_i + d_j)^{2\alpha}$$

and finally

$$GSCE(G) \leq \sqrt{2n \sum_{i<j} (d_i + d_j)^{2\alpha}}$$

which is an upper bound. □

3. General sum connectivity energy of some graph types

In this section, we calculate the general sum connectivity energy of some well-known and frequently used graphs.

Theorem 3.1. *The general sum connectivity energy of a complete graph K_n is*

$$GSCE(K_n) = 2(n-1)(2n-2)^{\alpha}.$$

Proof. Let K_n be the complete graph with vertex set $V = \{v_1,\ v_2,\ ...,\ v_n\}$, the general sum connectivity matrix is

$$\begin{bmatrix} 0 & (2n-2)^{\alpha} & (2n-2)^{\alpha} & \dots & (2n-2)^{\alpha} & (2n-2)^{\alpha} \\ (2n-2)^{\alpha} & 0 & (2n-2)^{\alpha} & \dots & (2n-2)^{\alpha} & (2n-2)^{\alpha} \\ (2n-2)^{\alpha} & (2n-2)^{\alpha} & 0 & \dots & (2n-2)^{\alpha} & (2n-2)^{\alpha} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (2n-2)^{\alpha} & (2n-2)^{\alpha} & \dots & (2n-2)^{\alpha} & 0 & (2n-2)^{\alpha} \\ (2n-2)^{\alpha} & (2n-2)^{\alpha} & \dots & (2n-2)^{\alpha} & (2n-2)^{\alpha} & 0 \end{bmatrix}.$$

Hence the characteristic equation will be

$$(\lambda - (2n-2)^{\alpha})^{n-1}\,(\lambda - (n-1)(2n-2)^{\alpha}) = 0$$

and therefore the spectrum becomes

$$Spec_{GSC}(K_n) = \begin{pmatrix} (2n-2)^{\alpha} & (n-1)(2n-2)^{\alpha} \\ n-1 & 1 \end{pmatrix}.$$

Therefore,

$$GSCE(K_n) = 2(n-1)(2n-2)^{\alpha}.$$

□

Definition 3.2. [1] *The cocktail party graph, denoted by $K_{n\times 2}$, is a graph with vertex set $V = \cup_{i=1}^{n}\{u_i, v_i\}$ and edge set $E = \{u_iu_j, v_iv_j, u_iv_j, v_iu_j : 1 \leq i < j \leq n\}$.*

Theorem 3.3. *The general sum connectivity energy of the cocktail party graph $K_{n\times 2}$ is*

$$GSCE(K_{n\times 2}) = 2[4^{\alpha}(n-1)^{\alpha+1}].$$

Proof. Let $K_{n\times 2}$ be the cocktail party graph of order $2n$ with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. The general sum connectivity matrix is

$$\begin{bmatrix} 0 & A & A & A & \dots & 0 & A & A & A \\ A & 0 & A & A & \dots & A & 0 & A & A \\ A & A & 0 & A & \dots & A & A & 0 & A \\ A & A & A & 0 & \dots & A & A & A & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & A & A & A & \dots & 0 & A & A & A \\ A & 0 & A & A & \dots & A & 0 & A & A \\ A & A & 0 & A & \dots & A & A & 0 & A \\ A & A & A & 0 & \dots & A & A & A & 0 \end{bmatrix}.$$

Where $A = (4n-4)^{\alpha}$.
This implies that the characteristic equation becomes

$$\lambda^{n}\left(\lambda + (4n-4)^{\alpha}\right)^{n-1}\left(\lambda - 4^{\alpha}(n-1)^{\alpha+1}\right) = 0.$$

Hence, the spectrum is

$$Spec_{GSC}(K_{n\times 2}) = \begin{pmatrix} -(4n-4)^{\alpha} & 0 & 4^{\alpha}(n-1)^{\alpha+1} \\ n-1 & n & 1 \end{pmatrix}.$$

Therefore,

$$GSCE(K_{n\times 2}) = 2[4^{\alpha}(n-1)^{\alpha+1}].$$

□

Theorem 3.4. *The general sum connectivity energy of the complete bipartite graph $K_{m,n}$ of order $m \times n$ is*

$$GSCE(K_{m,n}) = 2\sqrt{mn}(m+n)^{\alpha}.$$

Proof. Let $K_{m,n}$ be the complete bipartite graph of order $m \times n$ with vertex set $\{u_1,\ u_2,\ \cdots,\ u_n,\ v_1,\ v_2,\ \cdots,\ v_n\}$. The general sum connectivity matrix is

$$\begin{bmatrix} 0 & 0 & 0 & \dots & (m+n)^{\alpha} & (m+n)^{\alpha} & (m+n)^{\alpha} \\ 0 & 0 & 0 & \dots & (m+n)^{\alpha} & (m+n)^{\alpha} & (m+n)^{\alpha} \\ 0 & 0 & 0 & \dots & (m+n)^{\alpha} & (m+n)^{\alpha} & (m+n)^{\alpha} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (m+n)^{\alpha} & (m+n)^{\alpha} & (m+n)^{\alpha} & \dots & 0 & 0 & 0 \\ (m+n)^{\alpha} & (m+n)^{\alpha} & (m+n)^{\alpha} & \dots & 0 & 0 & 0 \\ (m+n)^{\alpha} & (m+n)^{\alpha} & (m+n)^{\alpha} & \dots & 0 & 0 & 0 \end{bmatrix}.$$

So the characteristic equation is

$$\lambda^{m+n-2}(\lambda - \sqrt{mn}(m+n)^{\alpha})(\lambda + \sqrt{mn}(m+n)^{\alpha}) = 0$$

and hence, the spectrum will be

$$Spec_{GSC}(K_{m,n}) = \begin{pmatrix} \sqrt{mn}(m+n)^{\alpha} & 0 & -\sqrt{mn}(m+n)^{\alpha} \\ 1 & m+n-2 & 1 \end{pmatrix}.$$

Therefore,

$$GSCE(K_{m,n}) = 2\sqrt{mn}(m+n)^{\alpha}.$$

□

References

[1] C. Adiga, A. Bayad, I. Gutman and S. A. Srinivas, *The minimum covering energy of graph*, Kragujevac Journal of Science, 34 (2012), 39–56.

[2] V. Alexander, *Upper and lower bounds of general sum connectivity index*, Iranian Journal of Mathematical Chemistry, 5(2) (2014), 91-98.

[3] Hanyuan Deng, He Huang, Jie Zhang, *On the Eigenvalues of General Sum-Connectivity Laplacian Matrix*, Journal of the Operations Research Society of China, 1 (2013), 347–358

[4] I. Gutman, *The energy of a graph*, Ber. Math. Stat. Sekt. Forschungsz. Graz, 103 (1978), 1-22.

[5] I. Gutman, *The energy of a graph: old and new results*, Combinatorics and applications, A. Betten, A. Khoner, R. Laue and A. Wassermann, (Eds.), Springer, Berlin, (2001), 196-211.

[6] F. Harary, (1969) Graph Theory. Addison-Wesley Publishing Company, New York.

[7] M. Randić, *On characterization of molecular branching*, J. Am. Chem. Soc., 97 (1975), 6609-6615.

[8] R. Todeschini, V. Consonni, *Handbook of Molecular Descriptors*, Wiley-VCH, Weinheim, (2000), 84-90.

[9] R. Todeschini, V. Consonni, *Molecular Descriptors for Chemoinformatics*, Wiley-VCH, Weinheim, (2009), 161-172.

[10] B. Zhou, N. Trinajstic, *On Sum-Connectivity Matrix and Sum-Connectivity Energy of (Molecular) Graphs*, Acta Chim. Slov, 57 (2010), 518-523.

[11] B. Zhou, N. Trinajstic, *On a novel connectivity index*, J. Math. Chem., 46 (2009), 1252-1270.

[12] B. Zhou, N. Trinajstić, *On general sum-connectivity index.* J Math Chem. 47 (2010), 210–218.

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