Proceedings of the Jangjeon Mathematical Society
27 (2024), No. 3, pp. 411 - 422

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2024.27.3.411

RANDIĆ TYPE LODEG ENERGY OF A GRAPH

K. N. PRAKASHA AND PUSHPALATHA MAHALANK

Abstract. The purpose of this paper is to introduce and investigate properties of the Randić type lodeg energy $RLE(G)$ of a graph G. We establish upper and lower bounds for $RLE(G)$. Also the Randić type lodeg energy for certain graphs with one edge deleted are calculated.

2010 Mathematics Subject Classification. 05C50

Keywords and phrases. Randić type lodeg index, Randić type lodeg eigenvalues, Randić type lodeg energy, k-complement, k(i)-complement, edge deletion.

1. Introduction

Consider a simple graph G with vertices $\{v_1, v_2, v_3, \cdots, v_n\}$. Let $i, j \in \{1, 2, \cdots, n\}$. If two vertices v_i and v_j of G are adjacent, then we use the notation $v_i \sim v_j$. For a vertex $v_i \in V(G)$, the degree of v_i will be denoted by $d(v_i)$ or briefly by d_i.

Basically energy of a graph was introduced by a chemist Ivan Gutman to estimate the total π-electron energy of a molecule, [3]. We can represent the conjugated hydrocarbons by means of a graph which is also called as a molecular graph. We can represent the carbon atoms by vertices and a chemical bond between two carbon atoms can be represented by an edge. In mathematical chemistry, topological indices play an important role. There are plenty of applications of such indices. Many physical properties and chemical reactivities can be predicted by these molecular descriptors. There are many topological indices such as Randić index, sum-connectivity index, atom bond connectivity index, Zagreb indices, etc. One of those numerical descriptors, the Randić type lodeg index, is the best predictor of heat capacity at constant T for octane isomers.

The Randić type lodeg index of a graph G is defined by

$$R = R(G) = \sum_{i \sim j} (\ln d_i)(\ln d_j).$$

The concept of the Randić type lodeg index motivates one to associate a symmetric square matrix $RL(G)$ to a graph G. The Randić type lodeg matrix $RL(G) = (S_{ij})_{n \times n}$ is, by this reason, defined as

$$S_{ij} = \begin{cases} \ln d_i \ln d_j & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

2. The Randić type lodeg energy of a graph

Let G be a simple, finite, undirected graph. The classical energy $E(G)$ is defined as the sum of absolute values of the eigenvalues of its adjacency matrix. For more details on energy of a graph, see [3, 4].

Let $RL(G)$ be the Randić type lodeg matrix. The characteristic polynomial of $RL(G)$ will be denoted by $\phi_{RL}(G,\lambda)$ and defined as

$$\phi_{RL}(G,\lambda) = det(\lambda I - RL(G)).$$

Since the Randić type lodeg matrix is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The Randić type lodeg energy of G is similarly defined by

$$RLE(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{1}$$

3. Some basic properties of Randić type lodeg energy of a graph

In this section, we discuss the properties of Randić type lodeg matrix and Randić type lodeg energy.

Proposition 3.1. *The first three coefficients of the polynomial $\phi_{RL}(G,\lambda)$ are as follows:*
(i) $a_0 = 1$,
(ii) $a_1 = 0$,
(iii) $a_2 = -\sum_{i<j}(\ln d_i \ln d_j)^2$.

Proof. (i) By the definition of $\phi_{RL}(G,\lambda) = det[\lambda I - RL(G)]$, we get $a_0 = 1$.
(ii) The sum of determinants of all 1×1 principal submatrices of $RL(G)$ is equal to the trace of $RL(G)$ implying that

$$a_1 = (-1)^1 \times \textit{the trace of } RL(G) = 0.$$

(iii) By the definition, we have

$$\begin{aligned}(-1)^2 a_2 = \sum_{1\leq i<j\leq n}\begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj}\end{vmatrix} &= \sum_{1\leq i<j\leq n} a_{ii}a_{jj} - a_{ji}a_{ij} \\ &= \sum_{1\leq i<j\leq n} a_{ii}a_{jj} - \sum_{1\leq i<j\leq n} a_{ji}a_{ij} \\ &= -\sum_{i<j}(\ln d_i \ln d_j)^2.\end{aligned}$$

□

Proposition 3.2. *If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the Randić type lodeg eigenvalues of $RL(G)$, then*

$$\sum_{i=1}^{n}\lambda_i^2 = 2\sum_{i<j}(\ln d_i \ln d_j)^2.$$

Proof. It follows as

$$\begin{aligned}\sum_{i=1}^{n}\lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}a_{ji}\\ &= 2\sum_{i<j}(a_{ij})^2+\sum_{i=1}^{n}(a_{ii})^2\\ &= 2\sum_{i<j}(a_{ij})^2\\ &= 2\sum_{i<j}(\ln d_i \ln d_j)^2.\end{aligned}$$

□

Using this result, we now obtain lower and upper bounds for the Randić type lodeg energy of a graph:

Theorem 3.3. *Let G be a graph with n vertices. Then*

$$RLE(G) \leq \sqrt{2n\sum_{i<j}(\ln d_i \ln d_j)^2}.$$

Proof. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of $RL(G)$. By the Cauchy-Schwartz inequality, we have

$$\left(\sum_{i=1}^{n}a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right).$$

Let $a_i = 1, b_i = |\lambda_i|$. Then

$$\left(\sum_{i=1}^{n}|\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{n}1\right)\left(\sum_{i=1}^{n}|\lambda_i|^2\right)$$

implying that

$$[RLE(G)]^2 \leq n\cdot 2\sum_{i<j}(\ln d_i \ln d_j)^2$$

and hence we get

$$RLE(G) \leq \sqrt{2n\sum_{i<j}(\ln d_i \ln d_j)^2}$$

as an upper bound. □

Theorem 3.4. *Let G be a graph with n vertices. If $R = det\ RL(G)$, then*

$$RLE(G) \geq \sqrt{2\sum_{i<j}(\ln d_i \ln d_j)^2 + n(n-1)R^{\frac{2}{n}}}.$$

Proof. By definition, we have

$$
\begin{aligned}
(RLE(G))^2 &= \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \\
&= \sum_{i=1}^{n} |\lambda_i| \sum_{j=1}^{n} |\lambda_j| \\
&= \left(\sum_{i=1}^{n} |\lambda_i|^2\right) + \sum_{i\neq j} |\lambda_i||\lambda_j|.
\end{aligned}
$$

Using arithmetic-geometric mean inequality, we have

$$
\frac{1}{n(n-1)} \sum_{i\neq j} |\lambda_i||\lambda_j| \geq \left(\prod_{i\neq j} |\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}}
$$

Therefore,

$$
\begin{aligned}
[RLE(G)]^2 &\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i\neq j} |\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}} \\
&\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i=1}^{n} |\lambda_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\
&= \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)R^{\frac{2}{n}} \\
&= 2\sum_{i<j} (\ln d_i \ln d_j)^2 + n(n-1)R^{\frac{2}{n}}.
\end{aligned}
$$

Thus,

$$
RLE(G) \geq \sqrt{2\sum_{i<j} (\ln d_i \ln d_j)^2 + n(n-1)R^{\frac{2}{n}}}.
$$

□

Let λ_n and λ_1 are the minimum and maximum values of all $\lambda_i' s$. Then the following results can easily be proven by means of the above results:

Theorem 3.5. *For a graph G of order n,*

$$
RLE(G) \geq \sqrt{2n\sum_{i<j} (\ln d_i \ln d_j)^2 - \frac{n^2}{4}(\lambda_1 - \lambda_n)^2}.
$$

Theorem 3.6. *For a graph G of order n with non-zero eigenvalues, we have*

$$
RLE(G) \geq \frac{2\sqrt{\lambda_1 \lambda_n}\sqrt{2n\sum_{i<j} (\ln d_i \ln d_j)^2}}{(\lambda_1 + \lambda_n)^2}.
$$

Theorem 3.7. *Let G be a graph of order n. Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$ be the eigenvalues in increasing order. Then*

$$RLE(G) \geq \frac{|\lambda_1||\lambda_n|n + 2\sum_{i<j}(\ln d_i \ln d_j)^2}{|\lambda_1| + |\lambda_n|}.$$

4. Randić type lodeg energy of some standard graphs

Here we obtain the Randić type lodeg energy for some standard graphs such as complete graph, star graph, crown graph, cocktail party graph, friendship graph, double star graph and complete bipartite graph etc.,

Definition 4.1. [5] *Let G and H be two graphs. The join $G \vee H$ of G and H is a graph obtained from G and H by joining each vertex of G to every vertex in H.*

Lemma 4.2. [1] *For $i = 1, 2$, let M_i be a normal matrix of order n_i having all its row sums equal to r_i. Suppose $r_i, \theta_{i2}, \theta_{i3}, ..., \theta_{in_i}$ are the eigenvalues of M_i, then for any two constants a and b, the eigenvalues of*

$$M := \begin{bmatrix} M_1 & aJ_{n_1\times n_2} \\ bJ_{n_2\times n_1} & M_2 \end{bmatrix}$$

are $theta_{ij}$ for $i = 1, 2,\ j = 2, 3, ..., n_i$ and the two roots of the quadratic equation $(x - r_1)(x - r_2) - abn_1n_2 = 0$.

Theorem 4.3. *Let G_1 be a r_1-regular graph of order n_1 and let G_2 be a r_2-regular graph of order n_2. Then the spectrum of $RRR(G_1 \vee G_2)$ consists of $[\ln(r_1 + n_2)]^2\lambda_i(G_1)$ and $[\ln(r_2 + n_1)]^2\lambda_j(G_2)$ and the two roots of the quadratic equation $(x - [\ln(r_1 + n_2)]^2r_1)(x - [\ln(r_2 + n_1)]^2r_2) - [\ln(r_1 + n_2)\ln(r_2 + n_1)n_1n_2]^2$*

Proof. Since G_1 and G_2 are regular graphs, the RRR matrix of $G_1 \vee G_2$can be obtained as follows:

$$Z1(G_1 \vee G_2) = \begin{bmatrix} [\ln(r_1 + n_2)]^2A(G_1) & \ln(r_1 + n_2)\ln(r_2 + n_1)J_{n_1\times n_2} \\ \ln(r_1 + n_2)\ln(r_2 + n_1)J_{n_2\times n_1} & [\ln(r_2 + n_1)]^2A(G_2) \end{bmatrix}$$

Setting $a = b = (r_1 + n_2 - 1)(r_2 + n_1 - 1)$ in 4.2, we arrive at the desired result. □

Theorem 4.4. *The Randić type lodeg energy of a complete graph K_n is*

$$RLE(K_n) = 2(n - 1)[\ln(n - 1)]^2.$$

Proof. Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. For this graph, the Randić type lodeg matrix is

$$\begin{bmatrix} 0 & [\ln(n-1)]^2 & [\ln(n-1)]^2 & \ldots & [\ln(n-1)]^2 & [\ln(n-1)]^2 \\ [\ln(n-1)]^2 & 0 & [\ln(n-1)]^2 & \ldots & [\ln(n-1)]^2 & [\ln(n-1)]^2 \\ [\ln(n-1)]^2 & [\ln(n-1)]^2 & 0 & \ldots & [\ln(n-1)]^2 & [\ln(n-1)]^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [\ln(n-1)]^2 & [\ln(n-1)]^2 & [\ln(n-1)]^2 & \ldots & 0 & [\ln(n-1)]^2 \\ [\ln(n-1)]^2 & [\ln(n-1)]^2 & [\ln(n-1)]^2 & \ldots & [\ln(n-1)]^2 & 0 \end{bmatrix}.$$

The characteristic equation then becomes

$$(\lambda + [\ln(n - 1)]^2)^{n-1}(\lambda - (n - 1)[\ln(n - 1)]^2) = 0$$

and the spectrum would be

$$Spec_{RL}(K_n) = \begin{pmatrix} -[\ln(n-1)]^2 & (n-1)[\ln(n-1)]^2 \\ n-1 & 1 \end{pmatrix}.$$

Therefore, $RLE(K_n) = 2(n-1)[\ln(n-1)]^2$. □

Theorem 4.5. *The Randić type lodeg energy of the star graph $K_{1,n-1}$ is*

$$RLE(K_{1,n-1}) = 0$$

Proof. Let $K_{1,n-1}$ be the star graph with vertex set $V = \{v_0, v_1, \cdots, v_{n-1}\}$ with v_0 denotes the central vertex. In this case, all the entries of Randić type lodeg matrix will be zero. Thus, the charcteristic equation is $\lambda^n = 0$. Therefore, the spectrum would be

$$Spec_{RL}(K_{1,n-1}) = \begin{pmatrix} 0 \\ n \end{pmatrix}.$$

Therefore,

$$RLE(K_{1,n-1}) = 0.$$

□

Theorem 4.6. *The Randić type lodeg energy of the friendship graph F_n^3 is*

$$RLE(F_n^3) = (2n-1)(\ln 2)^2 + \sqrt{(\ln 2)^2 + 8n[\ln 2 \ln 2n]^2}.$$

Proof. Let F_n^3 be the friendship graph with $2n+1$ vertices and let v_0 be the common vertex. The Randić type lodeg matrix is

$$\begin{bmatrix} 0 & A & A & A & A & \dots & A & A \\ A & 0 & [(\ln 2)]^2 & 0 & 0 & \dots & 0 & 0 \\ A & [(\ln 2)]^2 & 0 & 0 & 0 & \dots & 0 & 0 \\ A & 0 & 0 & 0 & [(\ln 2)]^2 & \dots & 0 & 0 \\ A & 0 & 0 & [(\ln 2)]^2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A & 0 & 0 & 0 & 0 & \dots & 0 & [(\ln 2)]^2 \\ A & 0 & 0 & 0 & 0 & \dots & [(\ln 2)]^2 & 0 \end{bmatrix}$$

where $A = (\ln 2n)(\ln 2)$. The characteristic equation becomes

$$(\lambda - [(\ln 2)])^{n-1}(\lambda + [(\ln 2)])^n(\lambda^2 - [(\ln 2)]^2\lambda - 2n[(\ln 2)(\ln 2n)]^2 = 0$$

implying that the spectrum is

$$\begin{pmatrix} -\ln 2 & \ln 2 & \frac{(\ln 2)^2 + \sqrt{(\ln 2)^2 + 8n[\ln 2 \ln 2n]^2}}{2} & \frac{(\ln 2)^2 - \sqrt{(\ln 2)^2 + 8n[\ln 2 (\ln 2n)]^2}}{2} \\ n & n-1 & 1 & 1 \end{pmatrix}.$$

Therefore, we get

$$RLE(F_n^3) = (2n-1)(\ln 2)^2 + \sqrt{(\ln 2)^2 + 8n[\ln 2 \ln 2n]^2}.$$

□

Theorem 4.7. *The Randić type lodeg energy of the cocktail party graph $K_{n\times 2}$ is*

$$RLE(K_{n\times 2}) = (4n-4)[\ln(2n-2)]^2.$$

Proof. Let $K_{n\times 2}$ be the cocktail party graph of order $2n$ having vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. The Randić type lodeg matrix is

$$RL(K_{n\times 2}) = \begin{bmatrix} 0 & 0 & B & B & \dots & B & B & B & B \\ 0 & 0 & B & B & \dots & B & B & B & B \\ B & B & 0 & 0 & \dots & B & B & B & B \\ B & B & 0 & 0 & \dots & B & B & B & B \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ B & B & B & B & \dots & 0 & 0 & B & B \\ B & B & B & B & \dots & 0 & 0 & B & B \\ B & B & B & B & \dots & B & B & 0 & 0 \\ B & B & B & B & \dots & B & B & 0 & 0 \end{bmatrix}$$

where $B = [\ln(2n-2)]^2$. In that case, the characteristic equation is

$$\lambda^n(\lambda + 2[\ln(2n-2)]^2)^{n-1}(\lambda - (2n-2)[\ln(2n-2)]^2) = 0$$

and hence the spectrum becomes

$$Spec_{RL}(K_{n\times 2}) = \begin{pmatrix} (2n-2)[\ln(2n-2)]^2 & 0 & -2[\ln(2n-2)]^2 \\ 1 & n & n-1 \end{pmatrix}.$$

Therefore we arrive at the required result. □

Theorem 4.8. *The Randić type lodeg energy of the double star graph $S_{n,n}$ is*

$$RLE(S_{n,n}) = 2[\ln n]^2.$$

Proof. The Randić type lodeg matrix is

$$RL(S_{n,n}) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & [\ln n]^2 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ [\ln n]^2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Then the characteristic equation becomes

$$\lambda^{2n-2}(\lambda + [\ln n]^2)(\lambda - [\ln n]^2) = 0.$$

Hence, the spectrum would be

$$Spec_{RL}(S_{n,n}) = \begin{pmatrix} 0 & [\ln n]^2 & -[\ln n]^2 \\ 2n-2 & 1 & 1 \end{pmatrix}$$

and therefore, we get

$$RLE(S_{n,n}) = 2[\ln n]^2.$$

□

Theorem 4.9. *The Randić type lodeg energy of a crown graph S_n^0 is*

$$RLE(S_n^0) = (4n-4)[\ln(n-1)]^2.$$

Proof. Let S_n^0 be the crown graph of order $2n$ and let the vertex set of this graph be $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. The Randić type lodeg matrix of S_n^0 is

$$RL(S_n^0) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & A & \dots & A & A \\ 0 & 0 & 0 & \dots & 0 & A & 0 & \dots & A & A \\ 0 & 0 & 0 & \dots & 0 & A & A & \dots & 0 & A \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & A & A & \dots & A & 0 \\ 0 & A & A & \dots & A & 0 & 0 & \dots & 0 & 0 \\ A & 0 & A & \dots & A & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A & A & 0 & \dots & A & 0 & 0 & \dots & 0 & 0 \\ A & A & A & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

where $A = [\ln(n-1)]^2$. Therefore the characteristic equation is

$$(\lambda - A)^{n-1}(\lambda + A)^{n-1}(\lambda + (n-1)A)(\lambda - (n-1)A) = 0$$

implying that the spectrum is

$$\begin{pmatrix} (n-1)[\ln(n-1)]^2 & -(n-1)[\ln(n-1)]^2 & [\ln(n-1)]^2 & -[\ln(n-1)]^2 \\ 1 & 1 & n-1 & n-1 \end{pmatrix}.$$

Therefore, we obtain

$$RLE(S_n^0) = (4n-4)[\ln(n-1)]^2.$$

□

Theorem 4.10. *The Randić type lodeg energy of the complete bipartite graph $K_{m,n}$ of order $m \times n$ is*

$$RLE(K_{m,n}) = 2\sqrt{mn}(\ln m)(\ln n).$$

Proof. Let $K_{m,n}$ be the complete bipartite graph of order $m \times n$ with vertex set $\{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}$. The matrix is

$$\begin{bmatrix} 0 & 0 & 0 & \dots & (\ln m)(\ln n) & (\ln m)(\ln n) & (\ln m)(\ln n) \\ 0 & 0 & 0 & \dots & (\ln m)(\ln n) & (\ln m)(\ln n) & (\ln m)(\ln n) \\ 0 & 0 & 0 & \dots & (\ln m)(\ln n) & (\ln m)(\ln n) & (\ln m)(\ln n) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (\ln m)(\ln n) & (\ln m)(\ln n) & (\ln m)(\ln n) & \dots & 0 & 0 & 0 \\ (\ln m)(\ln n) & (\ln m)(\ln n) & (\ln m)(\ln n) & \dots & 0 & 0 & 0 \\ (\ln m)(\ln n) & (\ln m)(\ln n) & (\ln m)(\ln n) & \dots & 0 & 0 & 0 \end{bmatrix}.$$

So the characteristic equation is

$$\lambda^{m+n-2}(\lambda - \sqrt{mn}(\ln m)(\ln n))(\lambda + \sqrt{mn}(\ln m)(\ln n)) = 0$$

and hence, the spectrum will be

$$Spec_{RL}(K_{m,n}) = \begin{pmatrix} \sqrt{mn}(\ln m)(\ln n) & 0 & -\sqrt{mn}(\ln m)(\ln n) \\ 1 & m+n-2 & 1 \end{pmatrix}.$$

Therefore,

$$RLE(K_{m,n}) = 2\sqrt{mn}(\ln m)(\ln n).$$

□

5. Randić type lodeg energy of complements

In this section, we calculated the Randić type lodeg energy for complements, k-complements and $k(i)$-complements, which are defined in [7]. Let G be a graph and $P_k = \{V_1, V_2, \cdots, V_k\}$ be a partition of its vertex set V. Then the k-complement of G is obtained as follows: For all V_i and V_j in P_k, $i \neq j$ remove the edges between V_i and V_j and add the edges between the vertices of V_i and V_j which are not in G and is denoted by $\overline{(G)_k}$. Similarly, the $k(i)$-complement of G is obtained as follows: For each set V_r in P_k, remove the edges of G joining the vertices within V_r and add the edges of $\overline{G}$ (complement of G) joining the vertices of V_r, and is denoted by $\overline{(G)_{k(i)}}$.

Theorem 5.1. *The Randić type lodeg energy of the complement $\overline{K_n}$ of the complete graph K_n is*

$$RLE(\overline{K_n}) = 0.$$

Proof. Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. The Randić type lodeg connectivity matrix of the complement of the complete graph K_n is

$$RL(\overline{K_n}) = \left[0\right]_{n\times n}.$$

Clearly, the characteristic equation is $\lambda^n = 0$ implying

$$RLE(\overline{K_n}) = 0.$$

□

Theorem 5.2. *The Randić type lodeg energy of the complement $\overline{K_{1,n-1}}$ of the star graph $K_{1,n-1}$ is*

$$RLE(\overline{K_{1,n-1}}) = (2n-4)[\ln(n-2)]^2.$$

Proof. Let $\overline{K_{1,n-1}}$ be the complement of the star graph with vertex set $V = \{v_0, v_1, \cdots, v_{n-1}\}$ where v_0 is the central. The Randić type lodeg matrix is

$$RL(\overline{K_{1,n-1}}) = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & [\ln(n-2)]^2 & \ldots & [\ln(n-2)]^2 & [\ln(n-2)]^2 \\ 0 & [\ln(n-2)]^2 & 0 & \ldots & [\ln(n-2)]^2 & [\ln(n-2)]^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & [\ln(n-2)]^2 & [\ln(n-2)]^2 & \ldots & 0 & [\ln(n-2)]^2 \\ 0 & [\ln(n-2)]^2 & [\ln(n-2)]^2 & \ldots & [\ln(n-2)]^2 & 0 \end{bmatrix}.$$

The corresponding characteristic equation is

$$\lambda(\lambda + [\ln(n-2)]^2)^{n-2}(\lambda - (n-2)[\ln(n-2)]^2) = 0$$

and therefore the spectrum is

$$Spec_{RL}(\overline{K_{1,n-1}}) = \begin{pmatrix} -[\ln(n-2)]^2 & 0 & (n-2)[\ln(n-2)]^2 \\ n-2 & 1 & 1 \end{pmatrix}.$$

Therefore the result follows. □

Theorem 5.3. *The Randić type lodeg energy of the complement $\overline{K_{n\times 2}}$ of the cocktail party graph $K_{n\times 2}$ of order $2n$ is $RLE(\overline{K_{n\times 2}}) = 0$.*

Proof. Let $K_{n\times 2}$ be the cocktail party graph of order $2n$ having the vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. The corresponding Randić type lodeg matrix is

$$RL(\overline{K_{n\times 2}}) = \big[0\big]_{n\times n}$$

and the characteristic equation becomes

$$\lambda^n = 0.$$

implying that the spectrum would be

$$Spec_{RL}(\overline{K_{n\times 2}}) = \binom{0}{n}$$

Therefore,

$$RLE(\overline{K_{n\times 2}}) = 0.$$

□

Theorem 5.4. *The Randić type lodeg energy of* 2*-complement of cocktail party graph* $K_{n\times 2}$ *is*

$$RLE(\overline{(K_{n\times 2})_{(2)}}) = (4n-4)[\ln n]^2.$$

Proof. Consider the 2-complement $\overline{K_{n\times 2_{(2)}}}$ of the cocktail party graph $K_{n\times 2_{(2)}}$. The Randić type lodeg matrix is

$$RL(\overline{(K_{n\times 2})_{(2)}}) = [\ln n]^2 \left(\begin{array}{cc} (J-I)_{n\times n} & (J-I)_{n\times n} \\ (J-I)_{n\times n} & (J-I)_{n\times n} \end{array} \right).$$

The characteristic polynomial is

$$\lambda^{n-1}(\lambda + 2[\ln n]^2)^{n-1}(\lambda - (n-2)[\ln n]^2)(\lambda - n[\ln n]^2) = 0$$

and therefore, the Randić type lodeg spectra is

$$Spec(\overline{(K_{n\times 2})_{(2)}}) = \begin{pmatrix} -2[\ln n]^2 & 0 & (n-2)[\ln n]^2 & n[\ln n]^2 \\ n-1 & n-1 & 1 & 1 \end{pmatrix}$$

implying the acquired result. □

6. Randić type lodeg energy of graphs with one edge deleted

In this section, we obtain the Randić type lodeg energy for certain graphs with one edge deleted.

Theorem 6.1. *Let* e *be an edge of the complete graph* K_n*. Then*
$RLE(K_n - e) = \big((n-3)[\ln(n-1)]^2 + [\ln(n-1)]C\big)$
where $C = \sqrt{(n-3)^2[\ln(n-1)]^2 + (2n-4)4[\ln(n-2)]^2}$

Proof. The Randić type lodeg matrix for $K_n - e$ is

$$\begin{bmatrix} 0 & 0 & A & \dots & A & A \\ 0 & 0 & \ln(n-1)\ln(n-2) & \dots & \ln(n-1)\ln(n-2) & \ln(n-1)\ln(n-2) \\ A & A & 0 & \dots & [\ln(n-1)]^2 & [\ln(n-1)]^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A & A & [\ln(n-1)]^2 & \dots & 0 & [\ln(n-1)]^2 \\ A & A & [\ln(n-1)]^2 & \dots & [\ln(n-1)]^2 & 0 \end{bmatrix}$$

where $A = \ln(n-1)\ln(n-2)$. Therefore the characteristic equation is
$\lambda(\lambda + [\ln(n-1)]^2)^{n-3}\left(\lambda^2 - (n-3)[\ln(n-1)]^2\lambda - (2n-4)(\ln(n-1)\ln(n-2))^2\right) = 0$
implying that the spectrum would be

$$\begin{pmatrix} [\ln(n-1)]^2 & 0 & \ln(n-1)\frac{(n-3)[\ln(n-1)]+C}{2} & \ln(n-1)\frac{(n-3)[\ln(n-1)]-C}{2} \\ n-3 & 1 & 1 & 1 \end{pmatrix}.$$

Therefore, $RLE(K_n - e) = \left((n-3)[\ln(n-1)]^2 + [\ln(n-1)]C\right)$
where $C = \sqrt{(n-3)^2[\ln(n-1)]^2 + (2n-4)4[\ln(n-2)]^2}$. □

Theorem 6.2. *Let e be an edge of the complete bipartite graph $K_{n,n}$. The Randić type lodeg energy of $K_{n,n} - e$ is*

$$RLE(K_{n,n} - e) = 2(\ln n)B$$

where $A = (n-1)\ln n$ and $B = \sqrt{(n-1)^2[\ln n]^2 + 4(n-1)[\ln(n-1)]^2}$.

Proof. The Randić type lodeg matrix for $K_{n,n} - e$ is

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & C & C \\ 0 & 0 & 0 & \dots & C & [\ln n]^2 & [\ln n]^2 \\ 0 & 0 & 0 & \dots & C & [\ln n]^2 & [\ln n]^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & C & C & \dots & 0 & 0 & 0 \\ C & [\ln n]^2 & [\ln n]^2 & \dots & 0 & 0 & 0 \\ C & [\ln n]^2 & [\ln n]^2 & \dots & 0 & 0 & 0 \end{bmatrix}$$

where $C = \ln(n)\ln(n-1)$. Then the characteristic equation is
$\lambda^{2n-4}(\lambda^2 + (n-1)[\ln n]^2\lambda - (n-1)[C]^2)(\lambda^2 - (n-1)[\ln n]^2\lambda - (n-1)[C]^2) = 0$
and hence, the spectrum would be

$$\begin{pmatrix} \frac{\ln n}{2}(A+B) & \frac{\ln n}{2}(-A+B) & \frac{\ln n}{2}(A-B) & \frac{\ln n}{2}(-A-B) & 0 \\ 1 & 1 & 1 & 1 & 2n-4 \end{pmatrix}$$

where $A = (n-1)\ln n$ and $B = \sqrt{(n-1)^2[\ln n]^2 + 4(n-1)[\ln(n-1)]^2}$.
Therefore,

$$RLE(K_{n,n} - e) = 2(\ln n)B.$$

□

The following result can easily be proven as above:

Lemma 6.3. *Let $K_{1,n-1}$ be the star graph with n vertices and let e be an edge of it. Then $RLE(K_{1,n-1} - e) = RLE(K_{1,n-2})$ for $n \geq 3$.*

References

[1] C. Adiga and B.R. Rakshith, Upper bounds for the extended energy of graphs and some extended equienergetic graphs, *Opuscula Math.* 38 (2018), no. 1, 5–13.

[2] V. Alexander, Upper and lower bounds of symmetric division deg index, *Iranian Journal of Mathematical Chemistry*, 5(2) (2014), 91-98.

[3] I. Gutman, The energy of a graph, *Ber. Math. Stat. Sekt. Forschungsz. Graz*, 103(1978), 1-22.

[4] I. Gutman, The energy of a graph: old and new results, *Combinatorics and applications, A. Betten, A. Khoner, R. Laue and A. Wassermann, (Eds.), Springer, Berlin*, 2001, 196-211.

[5] H.S. Ramane, H.B. Walikar, Construction of equienergetic graphs, *MATCH Commun. Math. Comput. Chem.* 57 (2007), 203–210.
[6] M. Randić, On characterization of molecular branching, *J. Am. Chem. Soc.*, 97(1975), 6609-6615.
[7] E. Sampathkumar, L. Pushpalatha, C. V. Venkatachalam and Pradeep Bhat, Generalized complements of a graph, *Indian J. pure appl. Math.*, 29(6)(1998), 625-639.
[8] R. Todeschini, V. Consonni, Handbook of Molecular Descriptors, *Wiley-VCH, Weinheim*, (2000), 84-90.
[9] R. Todeschini, V. Consonni, Molecular Descriptors for Chemoinformatics, *Wiley-VCH, Weinheim*, (2009), 161-172.
[10] B. Zhou, N. Trinajstic, On Sum-Connectivity Matrix and Sum-Connectivity Energy of (Molecular) Graphs, *Acta Chim. Slov*, 57 (2010), 518-523.
[11] B. Zhou, N. Trinajstic, On a novel connectivity index, *J. Math. Chem.*, 46 (2009), 1252-1270.

[1]Department of Mathematics, Vidyavardhaka College of Engineering, Mysuru, INDIA
Email address: prakashamaths@gmail.com

[2]Department of Mathematics, SDM College of Engineering and Technology, Dharwad- 580001, INDIA
Email address: mahalankpushpalatha@gmail.com