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THE UNITARY CHARACTERS ON THE RADICAL OF AN ALMOST CONNECTED LIE GROUP THAT ADMIT AN EXTENSION TO A ONE-DIMENSIONAL BOUNDED PSEUDOREPRESENTATION OF THE LIE GROUP

A. I. Shtern

Abstract. We determine the unitary characters on the radical of an almost connected Lie group that admit an extension to a one-dimensional bounded pseudorepresentations of the Lie group.

§ 1. Introduction

For the definitions, notation, and generalities concerning pseudocharacters, quasicharacters, pseudorepresentations, and quasirepresentations, see [4–6]. In particular, recall that a mapping π of a given group G into the field $\mathbb{C}$ of complex numbers is said to be a *one-dimensional quasirepresentation* of G on E if $\pi(e_G) = 1 \in \mathbb{C}$, where e_G stands for the identity element of G and if

$$|\pi(g_1g_2) - \pi(g_1)\pi(g_2)| \leq \varepsilon, \qquad g_1, g_2 \in G,$$

for some $\varepsilon \geq 0$, which is usually assumed to be sufficiently small, and the least upper bound of $|\pi(g_1g_2) - \pi(g_1)\pi(g_2)|$ for a one-dimensional quasirepresentation π is referred to as the *defect* of π; a one-dimensional quasirepresentation π of G is said to be a one-dimensional *pseudorepresentation* of G if

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$\pi(g^n) = \pi(g)^n$ for any $n \in \mathbb{Z}$ and $g \in G$. For specific properties concerning one-dimensional pseudorepresentations of groups, see [7].

We describe here the unitary characters on the radical of an almost connected Lie group that admit an extension to a one-dimensional bounded pseudorepresentation of the Lie group.

§ 2. Preliminaries

Recall Dong Hoon Lee's supplement theorem (Theorem 2.13 of [8]): every almost connected locally compact group G with the connected component G_0 (i.e., a locally compact group G for which the quotient group G/G_0 is compact) admits a totally disconnected compact subgroup D such that $G = G_0 D$.

We need a well-known lemma.

Lemma 1. *Let G be a group and let π and ρ be one-dimensional pseudorepresentations of G. If $|\pi(g) - \rho(g)| \le q < \sqrt{3}$ for all $g \in G$, then $\pi = \rho$.*

For the proof, see [9].

Let us recall some preliminary results.

Theorem 1. *A one-dimensional pseudorepresentation π with a defect $\varepsilon < q_0 = \sqrt{3}/5$ of an almost connected locally compact group G with the connected component G_0 is equal to an exponential, of a pseudocharacter on G, coinciding with π on G_0 and on a Lee's supplementary subgroup D of the group G if and only if π is trivial on D.*

For the proof, see [10].

Theorem 2. *Let G be an almost connected Lie group, let G_0 be the connected component of G, and let R be the radical (the maximal connected solvable normal subgroup) of G (it coincides with the radical of G_0). Let S be a (semisimple) Levi subgroup and let D be a (finite) Lee subgroup of G. Let π be a locally bounded one-dimensional pseudorepresentation of G with a defect less than 0.05. Then there are a unitary character ϕ of D which is trivial on $D \cap G_0$, a unitary one-dimensional pseudorepresentation θ of S, and an ordinary unitary central (with respect to G) unitary character ψ of R, where θ coincides with ψ on $S \cap R$, such that the one-dimensional pseudorepresentation π of the almost connected Lie group G has the form*

$$\pi(g) = \phi(d)\theta(s)\psi(r), \qquad \text{for} \quad g = dsr, \quad \text{where} \quad d \in D, \quad s \in S, \quad r \in R. \tag{1}$$

For the proof, see [9].

Theorem 3. *Every one-dimensional pseudorepresentation of an amenable group with a defect less than* $1/4$ *is an ordinary character of the group.*

For the proof, see [11].

Theorem 4. *Let* H *be a closed subgroup of a locally compact group* G. *A continuous unitary character* ψ *of* H *admits an extension* χ *over* G *that is a continuous unitary character of* G *if and only if* ψ *is trivial on the intersection* $H \cap \overline{G'}$ *of the subgroup* H *with the closure* $\overline{G'}$ *of the commutator subgroup* G' *of* G.

For the proof, see [12].

§ 3. Main result

Theorem 5. *Let* G *be an almost connected Lie group, let* G_0 *be the connected component of* G, *and let* R *be the radical of* G (*it coincides with the radical of* G_0). *Let* S *be a* (*semisimple*) *Levi subgroup and let* D *be a* (*finite*) *Lee subgroup of* G.

(i) If the center of the universal covering group of G_0 *is finite, then every one-dimensional bounded pseudorepresentation of an almost connected locally compact group* G *is naturally isomorphic to a character* (*one-dimensional ordinary unitary complex representations*) *of the group* D *that is equal to one on the intersection* $D \cap G_0$.

(ii) If the center of the group G_0 *is infinite, then the group* $ODBP(G)$ *of one-dimensional bounded pseudorepresentations of the almost connected locally compact group* G *is naturally isomorphic to the group of triples* (ϕ, θ, ψ), *where* ϕ *is a unitary character of* D *trivial on* $D \cap G_0$, θ *is a unitary one-dimensional pseudorepresentation of* S, *and* ψ *is an ordinary unitary central* (*with respect to* G) *character of* R, *where* θ *coincides with* ψ *on* $S \cap R$, *and the one-dimensional pseudorepresentation* π *of the almost connected Lie group* G *corresponding to the triple* (ϕ, θ, ψ) *has the form* (1) *(cf.* [14]*). Thus, a unitary character on* R *admits an extension to a unitary one-dimensional pseudorepresentation of* G *if and only if it is central and its restriction to* $S \cap S$ *admits an extension to a one-dimensional pseudorepresentation of* S.

(iii) If the center of the universal covering group of G_0 *is infinite and the center of the group* G_0 *is finite, then every one-dimensional bounded pseudorepresentation of the almost connected Lie group* G *is naturally isomorphic to the group of triples* (ϕ, θ, ψ), *where* ϕ *is a unitary character of* D *trivial on* $D \cap G_0$, θ *is a unitary one-dimensional pseudorepresentation of* S, *and*

ψ is an ordinary unitary central (with respect to G) character of R, where θ coincides with psi on $S \cap R$, and the one-dimensional pseudorepresentation π of the almost connected Lie group G corresponding to the triple (ϕ, θ, ψ) has the form (1). Thus, a unitary character of the radical admits an extension to a unitary one-dimensional pseudorepresentations of S if and only if it is central and the restriction of the character to $S \cap R$ admits an extension to a one-dimensional pseudorepresentation of S.

Proof. (i) If the center of the universal covering group of G_0 is finite, then the restriction of every one-dimensional bounded pseudorepresentation of G to G_0 is automatically the identity mapping [10]. It follows from [12] that the pseudorepresentation in question is an ordinary character of $G/G_0 \sim D/D \cap G_0$. Thus, the group in question is isomorphic to the group of characters (one-dimensional ordinary unitary complex representations) of the group D that are equal to one on the intersection $D \cap G_0$.Therefore, the only unitary character of R admitting an extension to a one-dimensional pseudorepresentation is the identity character.

(ii) Let the center of the group G_0 be infinite. The restriction of every one-dimensional bounded pseudorepresentation of G to G_0 is automatically an exponential of some Guichardet–Wigner pseudocharacter of G_0 whose defect does not exceed the defect $\varepsilon < 0.05$ of the original pseudorepresentation [10]. This pseudocharacter can be extended to a pseudocharacter on G whose defect does not exceed $4 \log(1 + \varepsilon) < 0.2$ [10]. Then the defect of the product of the original pseudorepresentation and the inverse of the exponential of the extended pseudocharacter does not exceed $0.35 < 0.4$, and thus Theorem 4 can be applied to finite and commutative subgroups of the group under consideration. As in (i), this product is an ordinary character of $G/G_0 \sim D/D \cap G_0$, which completes the proof in the case of (ii).

(iii) If the center of the universal covering group of G_0 is infinite and the center of the group G_0 is finite, then the proof proceeds as in the case of (ii); however, the list of Guichardet–Wigner pseudocharacters in use is restricted by the condition that the exponential of this pseudocharacter is to take the value one on the subgroup of the center of the universal covering group such that the quotient of the center of the universal covering group by this subgroup is the center of G_0. The related modifications of the rest of the proof are obvious. This completes the proof.

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Moscow Center for Fundamental and Applied Mathematics, Moscow, 119991 Russia,
Department of Mechanics and Mathematics,
Moscow State University,
Moscow, 119991 Russia, and
Federal State Institution
"Scientific Research Institute for System Analysis
of the Russian Academy of Sciences" (FSI SRISA RAS),
Moscow, 117312 Russia
E-mail: aishtern@mtu-net.ru, rroww@mail.ru