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On complete convergence for weighted sums of widely negative dependent random variables under sub-linear expectations

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Abstract. In this paper we study the complete convergence of weighted sums for widely negative dependent random variables under the sub-linear expectation spaces. Our results extend the corresponding ones of Yu and Wu(2018a) and Wu(2021) for widely negative dependent random variables under sub-linear expectations. As applications, we present some corollaries on complete convergence and almost sure convergence for END random variables under sub-linear expectations.

Keywords: Complete convergence; Almost sure convergence; Widely negative dependent; Weighted sum; Sub-linear expectation; END random variable.

Mathematics Subject Classification: 60F15; 60F05.

1 Introduction

The classical strong limit theorems such as the central limit theorem, the strong law of large number, the law of the iterated logarithm and so on, play an important role in the development of probability and statistics, and also they are established under the additivity of expectations and probability measures. However, in practice, such additivity assumption is no reasonable and cannot be well modeled in many uncertainty phenomena of applications such as statistics, finance and economics. The applications of classical limit theorem are limited to some extent; a growing number of people renounce the traditional tool of additivity of probability and instead use the new tool of nonadditive probability measure to resolve problems with uncertainty. Recently motivated by the super-hedge pricing, the risk measures and the modeling uncertainty in finance, non-additivity of probabilities and non-additivity of expectations have been used in a number of papers (see Gilboa [5], Denis and Martini [3], Maccheroni and Marinacci [11], Peng ([12]-[14]).

Peng [13] introduced the general framework of the sub-linear expectation space in a general function space by relaxing the classical linear expectation space with the linear property being replaced by the sub-additivity and positive homogeneity(see Definition 2.1 below). The sub-linear expectation provides a very flexible framework to model uncertainty problems in finance and statistics and produces many interesting properties different from those of the linear expectations. Under such frameworks, Peng ([12]-[14]) constructed the basic framework, basic properties such as central limit theorems, weak laws of large numbers the under sub-linear expectations and he give a reasonable definition of the independence through the sub-linear expectations. Zhang ([19]-[23]) studied the exponential inequalities, Rosenthal's inequalities, Donsker's invariance principle, strong law of large numbers and law of iterated logarithm under sub-linear expectations. This paper considers the general sub-linear expectations and related non-additive probabilities generated by them.

The concept of complete convergence was introduced by Hsu and Robbins [7] as follows: A sequence $\{X_n : n \in N\}$ of random variables is said to *converge completely* to a constant θ if

$$\sum_{n=1}^{\infty} P(|X_n - \theta| > \epsilon) < \infty \quad \text{for all } \epsilon > 0.$$

In view of the Borel-Cantelli lemma, the sequence of random variables $\{X_n : n \in N\}$ converging completely to a constant θ implies $X_n \to \theta$ almost sure (a.s.). Therefore the complete convergence of random variables is a very important tool in establishing a.s. convergence. There are many complete convergence theorems for sums of independent random variables (See Gut [6] for a detail survey). For complete convergence in the sub-linear expectation spaces, Lin and Feng [9] studied complete convergence and strong law of large numbers for arrays of random variables, Lu and Meng [10] established complete and complete integral convergence for arrays of row wise widely negative dependent random variables, Yu and Wu [18] obtained Marcinkiewicz type complete convergence for weighted sums, Xi, Wu and Wang [17] obtained complete convergence for arrays of rowwise END random variables and its statistical applications, Liang and Wu [8] established complete convergence of weighed sums for extended ND random variables sequence, and Feng, Wang and Wu [4] studied complete convergence for weighted sums of negatively dependent random variables under sub-linear expectation.

In this paper we study the complete convergence of weighted sums for widely negative dependent random variables under the sub-linear expectations. Note that the notion of widely negative dependence is a weaker condition than some existing concepts, such as Peng-independence, extended negatively dependence under sub-linear expectations. Our results extend the corresponding ones of Yu and Wu [18] and Wu [15] for widely negative dependent random variables under sub-linear expectation. As applications, we present some corollaries on complete convergence and almost sure convergence for END random variables under sub-linear expectation.

This paper is organized as follows: in Section 2, we summarized some basic notations and concepts, related properties under the sub-linear expectations and present the preliminary definitions and lemmas that are useful to obtain the main results. In Section 3, we give the main results including the proof.

2 Preliminaries

We use the framework and notations of Peng([12]-[14]). Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, X_2, \cdots, X_n \in \mathcal{H}$ then $\varphi(X_1, X_2, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, where $C_{l,Lip}(\mathbb{R}^n)$ denotes the linear space of local Lipschitz functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq C(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \quad \forall \mathbf{x},\ \mathbf{y} \in \mathbb{R}^n$$

for some $C > 0$ and $m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of "random variables". In this case we denote $X \in \mathcal{H}$.

Definition 2.1. A *sub-linear expectation* $\widehat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\widehat{\mathbb{E}} : \mathcal{H} \to \bar{\mathbb{R}}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$ we have

(i) Monotonicity: If $X \geq Y$ then $\widehat{\mathbb{E}}[X] \geq \widehat{\mathbb{E}}[Y]$;

(ii) Constant preserving: $\widehat{\mathbb{E}}[c] = c$;

(iii) Sub-additivity: $\widehat{\mathbb{E}}[X + Y] \leq \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$; whenever $\widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;

(iv) Positive homogeneity: $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X], \quad \lambda \geq 0$

Here $\bar{\mathbb{R}} = [-\infty, \infty]$. The triple $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called a *sub-linear expectation space.*

Given a sub-linear expectation $\widehat{\mathbb{E}}$, let us denote the conjugate expectation $\widehat{\mathcal{E}}$ of $\widehat{\mathbb{E}}$ by

$$\widehat{\mathcal{E}}[X] := -\widehat{\mathbb{E}}[-X], \quad \forall X \in \mathcal{H}.$$

From Definition 2.1, it is easily shown that

$$\widehat{\mathcal{E}}[X] \le \widehat{\mathbb{E}}[X], \ \ \widehat{\mathbb{E}}[X+c] = \widehat{\mathbb{E}}[X] + c \ \text{ and } \ \widehat{\mathbb{E}}[X-Y] \ge \widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y]$$

for all $X, Y \in \mathcal{H}$ with $\widehat{\mathbb{E}}[Y]$ being finite. Further, if $\widehat{\mathbb{E}}[|X|]$ is finite, then $\widehat{\mathbb{E}}[X]$ and $\widehat{\mathcal{E}}[X]$ are both finite, and if $\widehat{\mathbb{E}}[X] = \widehat{\mathcal{E}}[X]$, then $\widehat{\mathbb{E}}[X + aY] = \widehat{\mathbb{E}}[X] + a\widehat{\mathbb{E}}[Y]$ for any $a \in \mathbb{R}$.

Next, we consider the capacities corresponding to the sub-linear expectations.

Definition 2.2. Let $\mathcal{G} \subset \mathcal{F}$. A function $\mathbb{V} : \mathcal{G} \to [0,1]$ is called a *capacity* if

$$\mathbb{V}(\emptyset) = 0, \ \ \mathbb{V}(\Omega) = 1 \quad \text{and} \quad \mathbb{V}(A) \le \mathbb{V}(B) \quad \text{whenever } A \subset B \quad \text{and } A, B \in \mathcal{G}.$$

It is called to be *sub-additive* if $\mathbb{V}(A \cup B) \le \mathbb{V}(A) + \mathbb{V}(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$. Especially, a capacity $\mathbb{V}$ is 2-*alternating* if for all $A, B \in \mathcal{F}$,

$$\mathbb{V}(A \cup B) \le \mathbb{V}(A) + \mathbb{V}(B) - \mathbb{V}(A \cap B).$$

Let $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ be a sub-linear space. We denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\{\widehat{\mathbb{E}}[\xi] : I_A \le \xi, \xi \in \mathcal{H}\}, \ \ \mathcal{V}(A) = 1 - \mathbb{V}(A^c), \ \ \forall A \in \mathcal{F},$$

where A^c is the complement set of A. Then

$$\widehat{\mathbb{E}}[f] \le \mathbb{V}(A) \le \widehat{\mathbb{E}}[g] \quad \text{and} \quad \widehat{\mathcal{E}}[f] \le \mathcal{V}(A) \le \widehat{\mathcal{E}}[g], \tag{2.1}$$

if $f \le I_A \le g, \ f, g \in \mathcal{H}$. It is obvious that $\mathbb{V}$ is sub-additive, i.e., $\mathbb{V}(A \cup B) \le \mathbb{V}(A) + \mathbb{V}(B)$. But $\mathcal{V}$ and $\widehat{\mathcal{E}}$ are not. However, we have

$$\mathcal{V}(A \cup B) \le \mathcal{V}(A) + \mathbb{V}(B) \quad \text{and} \quad \widehat{\mathcal{E}}[X+Y] \le \widehat{\mathcal{E}}[X] + \widehat{\mathbb{E}}[Y]$$

due to the fact that

$$\mathbb{V}(A^c \cap B^c) = \mathbb{V}(A^c \setminus B) \ge \mathbb{V}(A^c) - \mathbb{V}(B) \ \text{ and } \ \widehat{\mathbb{E}}[-X-Y] \ge \widehat{\mathbb{E}}[-X] - \widehat{\mathbb{E}}[Y].$$

Further, if X is not in $\mathcal{H}$, we define $\widehat{\mathbb{E}}$ by $\widehat{\mathbb{E}}[X] = \inf\{\widehat{\mathbb{E}}[Y] : X \le Y, Y \in \mathcal{H}\}$. Then $\mathbb{V}(A) = \widehat{\mathbb{E}}[I_A]$.

In this paper we only consider the capacity generated by a sub-linear expectation. Given a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, we define a capacity:

$$\mathbb{V}(A) := \widehat{\mathbb{E}}[I_A], \quad \forall A \in \mathcal{F}$$

and also define the conjugate capacity:

$$\mathcal{V}(A) := 1 - \mathbb{V}(A^c), \quad \forall A \in \mathcal{F}.$$

It is clear that $\mathbb{V}$ is a sub-additive capacity and $\mathcal{V}(A) = \widehat{\mathcal{E}}[I_A]$.

Definition 2.3 ([21]) (1) A sub-linear expectation $\widehat{\mathbb{E}} : \mathcal{H} \to R$ is called to be *countably sub-additive* if it satisfies

$$\widehat{\mathbb{E}}[X] \le \sum_{n=1}^{\infty} \widehat{\mathbb{E}}[X_n], \text{ whenever } X \le \sum_{n=1}^{\infty} X_n, X, X_n \in \mathcal{H},$$

where $X \ge 0, X_n \ge 0$ and $n \ge 1$.

(2) A function $\mathbb{V} : \mathcal{F} \to [0,1]$ is called to be *countably sub-additive* if

$$\mathbb{V}(\cup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mathbb{V}(A_n), \ \forall A_n \in \mathcal{F}.$$

Definition 2.4. Let X be a random variable on $(\Omega, \mathcal{F})$. The *upper Choquet integral/expectation of X induced by* a capacity $\mathbb{V}$ on $\mathcal{F}$ is defined by $(C_{\mathbb{V}}, C_{\mathcal{V}})$ by

$$C_V(X) = \int_{\Omega} X dV(x) = \int_0^{\infty} V\left(X > x\right) dx + \int_{-\infty}^{0} \left(V\left(X > x\right) - 1\right) dx,$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$, respectively.

The *lower Choquet expectation of X induced by* $\mathbb{V}$ is given by $C_{\mathcal{V}}[X] := -C_{\mathbb{V}}[-X]$, which is conjugate to the upper expectation and satisfies $C_{\mathcal{V}}[X] \le C_{\mathbb{V}}[X]$.

Definition 2.5. ([2]) In a sub-linear expectation $(\Omega, \mathcal{H}, \tilde{\mathbb{E}})$, a sequence of random variable $\{X_n, n \ge 1\}$ is said to be *stochastically dominated* by a random variable X if there exist a r.v. X and a constant C satisfying

$$\widehat{\mathbb{E}}[h(X_n)] \le C\widehat{\mathbb{E}}[h(X)] \quad \textit{for all } n \ge 1, \ 0 \le h \in C_{l,Lip}(\mathbb{R}). \tag{2.1}$$

For simplicity, we only consider the upper Choquet expectation in the sequel, since the lower (conjugate) version can be considered similarly.

Lemma 2.1. ([1]) Let X, Y be two random variables on $(\Omega, \mathcal{F})$ and let $C_{\mathbb{V}}$ be the upper Choquet expectation induced by a capacity $\mathbb{V}$, then, we have

(1) Monotonicity: $C_{\mathbb{V}}[X] \le C_{\mathbb{V}}[Y]$ for $X \le Y$;

(2) Positive homogeneity: $C_{\mathbb{V}}[\lambda X] \le \lambda C_{\mathbb{V}}[X]$ for $\lambda \ge 0$;

(3) translation invariance: $C_{\mathbb{V}}[X + a] \le C_{\mathbb{V}}[X] + a$ for $\forall a \in \mathbb{R}$.

The following lemmas show that some important inequalities in classical probability theory still hold in sub-linear expectation spaces (See [10]).

Lemma 2.2. (Markov's inequality) *For any $X \in \mathcal{H}$, we have*

$$\mathbb{V}(|X| \geq x) \leq \frac{\widehat{\mathbb{E}}[|x|^p]}{x^p}$$

for any $x > 0$ and $p > 0$.

Now we give the definition of widely negative dependence on the sublinear expectation space. The concept of widely negative dependence is introduced by Lin and Feng [9] as follows.

Definition 2.6. Let $X_1, X_2, \cdots, X_{n+1}$ be real measurable random variables of $(\Omega, \mathcal{F})$.
(1) X_{n+1} is called *widely negative dependence* of $(X_1, \cdots, X_n)$ under $\widehat{\mathbb{E}}$ if for every non-negative measure function φ_i with the same monotonicity on $\mathbb{R}$ and $\widehat{\mathbb{E}}[\varphi_i(X_i)] < \infty, i = 1, 2, \cdots, n+1$, there exists a positive finite real function $g(n+1)$ such that

$$\widehat{\mathbb{E}}\left[\prod_{i=1}^{n+1}\varphi_i(X_i)\right] \leq g(n+1)\widehat{\mathbb{E}}\left[\prod_{i=1}^{n}\varphi_i(X_i)\right]\widehat{\mathbb{E}}\left[\varphi_{n+1}(X_{n+1})\right].$$

(2) $\{X_i\}_{i=1}^{\infty}$ is said to be *a sequence of widely negative dependent random variables* under $\widehat{\mathbb{E}}$ if for any $n \geq 1$, X_{n+1} is widely negative dependence of $(X_1, X_2, \cdots, X_n)$.
(3) $\{X_{ni} : 1 \leq i \leq n, n \geq 1\}$ is said to be *an array of rowwise widely negative dependent random variables* under $\widehat{\mathbb{E}}$ if for any $n \geq 1$, $\{X_{ni} : 1 \leq i \leq k_n\}$ is a sequence of widely negative dependent random variables

Remark 2.1. For a sequence of widely negative dependent random variables $\{X_i : i \geq 1\}$, we have

$$\widehat{\mathbb{E}}\left[\prod_{i=1}^{n}\varphi_i(X_i)\right] \leq \tilde{g}(n)\prod_{i=1}^{n}\widehat{\mathbb{E}}\left[\varphi_i(X_i)\right], \text{ where } \tilde{g}(n) = \prod_{i=1}^{n} g(i)$$

for any $n \geq 1$ and every nonnegative measurable function $\varphi_i(\cdot)$ with the same monotonicity on $\mathbb{R}$ and $\widehat{\mathbb{E}}[\varphi_i(X_i)] < \infty, i = 1, 2, \cdots, n$, where $g(\cdot)$ is in Definition 2.6(1).

Remark 2.2. Without loss of generality, we will assume that $g(n) \geq 1$ for any $n \geq 1$ in the sequal.

The following lemma is introduced by Lin and Feng [9].

Lemma 2.3. *Suppose that $\{X_i\}_{i=1}^{\infty}$ is a sequence of widely negative dependent random variable under $\widehat{\mathbb{E}}$, and $\{\psi_i(x)\}_{i=1}^{\infty}$ is a sequence of measurable function with the same monotonicity. Then $\{\psi_i(X_i)\}_{i=1}^{\infty}$ is also a sequence of widely negative dependent random variables.*

It is necessary to note that widely negative dependence under sub-linear expectations is defined through continuous functions in $C_{l,Lip}$ and the indicator function $I(|x| \le a)$ is not necessarily continuous. Therefore, the expression $\tilde{\mathbb{E}}[I(|X| \le a)]$ does not necessarily exist in the sub-linear expectation space. So we should modify the indicator function by functions in $I(|x| \le a)$ to ensure that the sequence of truncated random variables is also widely negative dependence (See Lu and Meng[10]).

We define the function $h(x) \in C_{l,Lip}(\mathbb{R})$ as follows. For $0 < \mu < 1$, let $h(x) \in C_{l,Lip}(\mathbb{R})$ be a nonincreasing function such that $0 \le h(x) \le 1$ for all x and $h(x) = 1$ if $|x| \le \mu$, $h(x) = 0$ if $|x| > 1$, then

$$I(|x| \le \mu) \le h(x) \le I(|x| \le 1), \qquad I(|x| > 1) \le 1 - h(x) \le I(|x| > \mu). \tag{2.2}$$

Throughout this paper, let $\{X_n : n \ge 1\}$ be a sequence of widely negative dependent random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. C will signify a positive constant that may have different values in different places. $a_n = O(b_n)$ denotes that for a sufficiently large n, there exists $C > 0$ such that $a_n \le Cb_n$ and $I(\cdot)$ denotes an indicator function.

3 Main Results and Proofs

Theorem 3.1. *Suppose that $\{X, X_n : n \ge 1\}$ is a sequence of widely negative dependent random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ which is stochastically dominated by a random variable X, and let $\mathbb{V}$ be a countably sub-additive capacity. Let $\tilde{g}(x)$ be a nondecreasing positive function on $[0, \infty)$ such that*

$$\tilde{g}(x) = \tilde{g}(n) \text{ when } x = n, \quad \tilde{g}(0) = 1 \quad \text{and} \quad \frac{\tilde{g}(x)}{x^\tau} \downarrow \quad \text{for some } 0 < \tau < 1. \tag{3.1}$$

Assume that $\{a_{nk} : 1 \le k \le n, n \ge 1\}$ is an array of real numbers satisfying

$$\max_{1 \le k \le n} |a_{nk}| = O(n^{-\alpha}) \tag{3.2}$$

and

$$c_n = \sum_{k=1}^{n} a_{nk}^p = o(\log^{-1} n), \tag{3.3}$$

where $0 < \alpha < 1$. Further assume that and

$$\widehat{\mathbb{E}}[|X|^{2p}] \le C_{\mathbb{V}}(|X|^{2p}) < \infty \tag{3.4}$$

where $\beta > -\alpha$, $p = (1+\alpha+\beta)/\alpha > 1$. *Then we have*

$$\sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{k=1}^{n} a_{nk}\left(X_k - \widehat{\mathbb{E}}[X_k]\right) > \epsilon\right) < \infty, \quad \forall \epsilon > 0. \tag{3.5}$$

and

$$\sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{k=1}^{n} a_{nk}\left(X_k - \widehat{\mathcal{E}}[X_k]\right) < -\epsilon\right) < \infty, \quad \forall \epsilon > 0. \tag{3.6}$$

In particular, if $\{X_n : n \geq 1\}$ *is a sequence of widely negative dependent random variables in* $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ *and* $\widehat{\mathbb{E}}[X_k] = \widehat{\mathcal{E}}[X_k]$ *, then*

$$\sum_{n=1}^{\infty} \mathbb{V}\left(\left|\sum_{k=1}^{n} a_{nk}\left(X_k - \widehat{\mathbb{E}}[X_k]\right)\right| > \epsilon\right) < \infty, \quad \forall \epsilon > 0. \tag{3.7}$$

Proof. Note that $a_{nk} = a_{nk}^{+} - a_{nk}^{-}$, where $a_{nk}^{+} = \max\{0, a_{nk}\}$ and $a_{nk}^{-} = \max\{0, -a_{nk}\}$. Without loss of generality, we may assume that $\widehat{\mathbb{E}}[X_n] = 0, \widehat{\mathbb{E}}[X_n^2] = 1, a_{nk} \geq 0$ for all $1 \leq k \leq n, n \geq 1$. For $\alpha > 0$, set $\rho = \alpha^2/(\alpha+1)$, and then $\rho > 0$. Let N be a positive integer such that $N > (\alpha+1)/\beta$ for some $\beta > -\alpha$. Since $n^{-\rho} \to 0$ as $n \to \infty$, for any $\epsilon > 0$ and for some $N > 0$, there exists a positive integer N_0 such that $n \geq N_0 \Rightarrow n^{\rho} < \epsilon/N$. Let

$$\begin{aligned}
X_{nk}^{(1)} &= X_k I(a_{nk}X_k \leq n^{-\rho}) + a_{nk}^{-1} n^{-\rho} I(a_{nk}X_k > n^{-\rho}), \\
X_{nk}^{(2)} &= \left(X_k - a_{nk}^{-1} n^{-\rho}\right) I\left(|X_k| > \epsilon/N\right), \\
X_{nk}^{(3)} &= \left(X_k - a_{nk}^{-1} n^{-\rho}\right) I\left(n^{-\rho} < a_{nk}X_k \leq \epsilon/N\right), \\
T_n &= \sum_{k=1}^{n} a_{nk}X_k = T_n^{(1)} + T_n^{(2)} + T_n^{(3)},
\end{aligned} \tag{3.8}$$

where

$$X_k = X_{nk}^{(1)} + X_{nk}^{(1)} + X_{nk}^{(1)} \quad \text{and} \quad T_n^{(i)} = \sum_{k=1}^{n} a_{nk} X_{nk}^{(i)}, \quad i = 1, 2, 3$$

Since

$$\{T_n > 3\epsilon\} \subset \left\{T_n^{(1)} > \epsilon\right\} \cup \left\{T_n^{(2)} > \epsilon\right\} \cup \left\{T_n^{(3)} > \epsilon\right\},$$

to prove that $\sum_{n=1}^{\infty} \mathbb{V}(T_n > 3\epsilon) < \infty$, we only need to show that

$$\sum_{n=1}^{\infty} \mathbb{V}\left(T_n^{(1)} > \epsilon\right) < \infty, \tag{3.9}$$

$$\sum_{n=1}^{\infty} \mathbb{V}\left(T_n^{(2)} > \epsilon\right) < \infty, \tag{3.10}$$

$$\sum_{n=1}^{\infty} \mathbb{V}\left(T_n^{(3)} > \epsilon\right) < \infty. \tag{3.11}$$

For $n \geq 1$, let $u_n = \min\{\epsilon/(2Cc_n, n^\rho\}$, where $c_n = \sum_{k=1}^n a_{nk}^2$. It is easily checked from Lemma 2.10 that for any fixed $n \geq 1$, $\left\{a_{nk}X_{nk}^{(1)}, 1 \leq k \leq n\right\}$ are still widely negative dependent random variables, and $\left\{\exp\left(u_n a_{nk}X_{nk}^{(1)}\right), 1 \leq k \leq k\right\}$ are also widely negative dependent random variables. Hence, it follows from Definition 2.7 that

$$\begin{aligned}\widehat{\mathbb{E}}\left[\exp\left(u_n T_n^{(1)}\right)\right] &= \widehat{\mathbb{E}}\left[\prod_{k=1}^n \exp\left(u_n a_{nk} X_{nk}^{(1)}\right)\right] \\ &\leq \tilde{g}(n)\prod_{k=1}^n \widehat{\mathbb{E}}\left[\exp\left(u_n a_{nk}X_{nk}^{(1)}\right)\right].\end{aligned} \tag{3.12}$$

Note that $u_n a_{nk} X_{nk}^{(1)} \leq 1$ for $1 \leq k \leq n, n \geq 1$, we have by Lemma 3.1

$$\widehat{\mathbb{E}}\left[\exp\left(u_n a_{nk}X_{nk}^{(1)}\right)\right] \leq \exp\left(u_n a_{nk}\widehat{\mathbb{E}}\left[X_{nk}^{(1)}\right] + u_n^2 a_{nk}^2 \widehat{\mathbb{E}}\left[X_{nk}^{(1)}\right]^2\right), \tag{3.13}$$

which together with $\widehat{\mathbb{E}}\left[X_{nk}^{(1)}\right] \leq \widehat{\mathbb{E}}\left[X_{nk}\right] = 0$ and $\widehat{\mathbb{E}}\left[X_k^{(1)}\right]^2 \leq \widehat{\mathbb{E}}\left[X_{nk}\right]^2 = 1$ yields that

$$\widehat{\mathbb{E}}\left[\exp\left(u_n a_{nk}X_{nk}^{(1)}\right)\right] \leq \exp\left(u_n^2 a_{nk}^2\right), \tag{3.14}$$

and thus by (3.4)

$$\widehat{\mathbb{E}}\left[\exp\left(u_n T_n^{(1)}\right)\right] \leq \tilde{g}(n)\exp\left(u_n^2 c_n\right). \tag{3.15}$$

Since X is a random variable with $X \leq 1$ a.s., then

$$\widehat{\mathbb{E}}[\exp(X)] \leq \exp\left(\widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[X^2]\right)$$

(See [8]). By Markov's inequality and $(3, 10)$, we get that

$$\begin{aligned}\mathbb{V}\left(T_n^{(1)} > \epsilon\right) &\leq \exp\left(-\epsilon u_n\right)\widehat{\mathbb{E}}\left[\exp\left(u_n T_n^{(1)}\right)\right] \\ &\leq \tilde{g}(n)\exp\left\{-\epsilon u_n + u_n^2 c_n\right\}.\end{aligned} \tag{3.16}$$

Let $\mathbb{N}_1 = \{n : \epsilon/(2c_n) \geq n^\rho\}$ and $\mathbb{N}_2 = \{n : \epsilon/(2c_n) < n^\rho\}$. If $n \in \mathbb{N}_1 \Leftrightarrow \epsilon/(2c_n) \geq n^\rho$, then by definition of u_n, we have $u_n = n^\rho$ and $-\epsilon u_n + u_n^2 c_n \leq -\frac{\epsilon}{2}n^\rho$. For any $\epsilon > 0$, we have $\epsilon n^\rho/2 \geq \log n^2$ for sufficiently large n, it follows that

$$\sum_{n\in\mathbb{N}_1} \mathbb{V}\left(T_n^{(1)} > \epsilon\right) \leq \sum_{n=1}^\infty \tilde{g}(n)\exp\left(-\epsilon n^\rho/2\right) < \infty. \tag{3.17}$$

If $n \in \mathbb{N}_2 \Leftrightarrow \epsilon/(2c_n) < n^\rho$, then by definition of u_n, we have $u_n = \epsilon/(2c_n)$ and $-\epsilon u_n + u_n^2 c_n = -\frac{\epsilon^2}{4c_n}$. Since $c_n < \epsilon^2/(8\log n)$ for sufficiently large n from $c_n = o(\log^{-1} n)$, we have

$$\sum_{n\in\mathbb{N}_2} \mathbb{V}\left(T_n^{(1)} > \epsilon\right) \leq \sum_{n=1}^\infty \tilde{g}(n)\exp\left(-\epsilon^2/(4c_n)\right) < \infty. \tag{3.18}$$

Therefore, we have from (3.11) and (3.12)

$$\sum_{n=1}^{\infty} \mathbb{V}\left(T_n^{(n)} > \epsilon\right) = \sum_{n\in\mathbb{N}_1} \mathbb{V}\left(T_n^{(1)} > \epsilon\right) + \sum_{n\in\mathbb{N}_2} \mathbb{V}\left(T_n^{(1)} > \epsilon\right) < \infty. \tag{3.19}$$

Next, note that for any $c > 0$

$$\begin{aligned} C_{\mathbb{V}}\left(|X|^{2/\alpha}\right) &= \int_0^{\infty} \mathbb{V}\left(|X|^{2/\alpha} > x\right) dx = \int_0^{\infty} \mathbb{V}\left(|X| > x^{\alpha/2}\right) dx \\ &= c^{2/\alpha} \int_0^{\infty} \mathbb{V}\left(|X| > cy^{\alpha/2}\right) dy \\ &= 2c^{2/\alpha} \int_0^{\infty} z\mathbb{V}\left(|X| > cz^{\alpha}\right) dz, \end{aligned}$$

then

$$\begin{aligned} C_{\mathbb{V}}\left(|X|^{2/\alpha}\right) < \infty \quad &\Leftrightarrow \quad \sum_{n=1}^{\infty} \mathbb{V}\left(|X| > cn^{\alpha/2}\right) < \infty \\ &\Leftrightarrow \quad \sum_{n=1}^{\infty} n\mathbb{V}\left(|X| > cn^{\alpha}\right) < \infty. \end{aligned}$$

Since $p = (1 + \alpha + \beta)/\alpha > 1/\alpha$, we have from Proposition 2.5(1)

$$C_{\mathbb{V}}\left(|X|^{2/\alpha}\right) \le C_{\mathbb{V}}\left(|X|^{2p}\right) < \infty. \tag{3.20}$$

By (3.3), we have

$$\left(T_n^{(2)} > \epsilon\right) \subset \cup_{k=1}^{n} \left(X_k > \frac{\epsilon a_{nk}^{-1}}{N}\right) \subset \cup_{k=1}^{n} \left(X_k > \epsilon n^{\alpha}(cN)^{-1}\right), \tag{3.21}$$

it follows from (2.1), (2.2) and (3.1) that

$$\begin{aligned} \mathbb{V}\left\{T_n^{(2)} \ge \epsilon\right\} &\le \sum_{k=1}^{n} \mathbb{V}\left(X_k > \epsilon n^{\alpha}(cN)^{-1}\right) \\ &\le \sum_{k=1}^{n} \widehat{\mathbb{E}}\left(1 - h\left(\frac{X_k}{\epsilon n^{\alpha}(cN)^{-1}}\right)\right) \\ &\le C\sum_{k=1}^{n} \widehat{\mathbb{E}}\left(1 - h\left(\frac{X}{\epsilon n^{\alpha}(cN)^{-1}}\right)\right) \\ &\le C\sum_{k=1}^{n} \mathbb{V}\left(X > \mu\epsilon n^{\alpha}(cN)^{-1}\right) \\ &\le Cn\mathbb{V}\left(|X| > \mu\epsilon n^{\alpha}(cN)^{-1}\right). \end{aligned} \tag{3.22}$$

Therefore, we have by (2.1), (3.15), (3.16) and (3.17)

$$\sum_{n=1}^{\infty} \mathbb{V}\left\{T_n^{(2)} \ge \epsilon\right\} \le C\sum_{k=1}^{\infty} \mathbb{V}\left(|X| > \mu\epsilon n^{\alpha}(cN)^{-1}\right) < \infty. \tag{3.23}$$

Next, we prove that $\sum_{n=1}^{\infty}\mathbb{V}(T_n^{(3)}>\epsilon)$ for any $\epsilon>0$. It follows by the definition of $X_{nk}^{(3)}$ that if $a_{nk}X_k\notin(n^{-\rho},\epsilon/N]$, then $a_{nk}X_{nk}^{(3)}=0$; if $a_{nk}X_k\in(n^{-\rho},\epsilon/N]$, then $a_{nk}X_{nk}^{(3)}\le\epsilon/N$. So in order to ensure $T_n^{(3)}=\sum_{k=1}^{n}a_{nk}X_{nk}^{(3)}>\epsilon$, there must exist at least a positive integer N indices k such that $n^{-\rho}<a_{nk}X_k\le\epsilon/N$. Hence by (2.1), (2.2), (3.1), Definition 2.7 and Markov's inequality, we have that

$$
\begin{aligned}
&\left\{T_n^{(3)}\ge\epsilon\right\}\\
&\subset\left\{\text{there exists at least } N \text{ indices } k \text{ such that } n^{-\rho}<a_{nk}X_k\le\epsilon/N\right\}\\
&\subset\left\{\text{there exists at least } N \text{ indices } k \text{ such that } a_{nk}X_k>n^{-\rho}\right\}\\
&\le\sum_{1\le k_1<\cdots<k_N\le n}\mathbb{V}\left(X_{k_1}>a_{n_{k_1}}^{-1}n^{-\rho},\cdots,X_{k_N}>a_{n_{k_N}}^{-1}n^{-\rho}\right)\\
&\le\sum_{1\le k_1<\cdots<k_N\le n}\widehat{\mathbb{E}}\left[\prod_{i=1}^{N}\left(1-h\left(a_{n_{k_i}}n^{\rho}X_{k_i}\right)\right)\right]\\
&\le\sum_{1\le k_1<\cdots<k_N\le n}\tilde{g}(N)\prod_{i=1}^{N}\widehat{\mathbb{E}}\left[\left(1-h\left(a_{n_{k_i}}n^{\rho}X\right)\right)\right]\\
&\le\tilde{g}(N)\sum_{1\le k_1<\cdots<k_N\le n}\prod_{i=1}^{N}\mathbb{V}\left(X>\mu a_{nk_i}^{-1}n^{-\rho}\right)\\
&\le\tilde{g}(N)\sum_{k=1}^{n}\left[\mathbb{V}\left(|X|>\mu a_{nk}^{-1}n^{-\rho}\right)\right]^N\\
&\le\tilde{g}(n)\left[\sum_{k=1}^{n}\mathbb{V}\left(|X|>\mu a_{nk}^{-1}n^{-\rho}\right)\right]^N\\
&\le C\tilde{g}(n)\left[\sum_{k=1}^{n}\frac{\widehat{\mathbb{E}}\left[|X|^{2p}\right]}{(\mu a_{nk}^{-1}n^{-\rho})^{2p}}\right]^N\\
&=C\tilde{g}(n)\left[\frac{\widehat{\mathbb{E}}\left[|X|^{2p}\right]}{\mu^{2p}n^{-2p\rho}}\sum_{k=1}^{n}a_{nk}^{2p}\right]^N\\
&\le C\frac{n^{\tau}}{n^{2N(\alpha(p-1)-p\rho)}}.
\end{aligned}
\tag{3.24}
$$

Note that

$$
\begin{aligned}
(\alpha(p-1)-p\rho)N&>\left[\alpha\left(\frac{1+\alpha+\beta}{\alpha}-1\right)-\frac{1+\alpha+\beta}{\alpha}\cdot\frac{\alpha^2}{1+\alpha}\right]\frac{1+\alpha}{\beta}\\
&=\left(1+\beta-\frac{(1+\alpha+\beta)\alpha}{1+\alpha}\right)\frac{1+\alpha}{\beta}\\
&=1+\frac{1-\alpha^2}{\beta}>1
\end{aligned}
$$

for $0<\alpha<1$, it follows from (3.1) and (3.18) that

$$
\sum_{n=1}^{\infty}\mathbb{V}\left(T_n^{(3)}>\epsilon\right)\le C\sum_{n=1}^{\infty}n^{-(2-\tau)}<\infty. \tag{3.25}
$$

Together with (3.13), (3.19), (3.20) and $\mathbb{V}$ being the countably sub-additivity, we obtain (3.8). This completes the proof of (3.6).

Furthermore , if $\{X_n, n \geq 1\}$ a sequence of widely negative dependent random variables in $(\Omega, \mathcal{F}, \widehat{\mathbb{E}})$, then $\{-X_n, n \geq 1\}$ is also satisfying the conditions of Theorem 3.1 and considering $\{-X_n, n \geq 1\}$ instead of $\{X_n, n \geq 1\}$ in (3.6), we can obtain by $\widehat{\mathbb{E}}\left(-X_k\right) = -\widehat{\mathcal{E}}\left(X_k\right)$

$$\begin{aligned}
&\sum_{k=1}^{\infty}\mathbb{V}\left(\sum_{k=1}^{n} a_{nk}\left(-X_k - \widehat{\mathbb{E}}[-X_k]\right) > \epsilon\right) \\
&\quad = \sum_{k=1}^{\infty}\mathbb{V}\left(\sum_{k=1}^{n} a_{nk}\left(X_k - \widehat{\mathcal{E}}[X_k]\right) < -\epsilon\right) < \infty, \quad \forall \epsilon > 0.
\end{aligned}$$

That is, (3.7) is established.

In particular, if $\widehat{\mathbb{E}}\left(X_k\right) = \widehat{\mathcal{E}}\left(X_k\right)$, then for any $\epsilon > 0$

$$\begin{aligned}
&\sum_{k=1}^{\infty}\mathbb{V}\left(\left|\sum_{k=1}^{n} a_{nk}\left(-X_k - \widehat{\mathbb{E}}[-X_k]\right|\right) > \epsilon\right) \\
&\quad \leq \sum_{k=1}^{\infty}\mathbb{V}\left(\sum_{k=1}^{n} a_{nk}\left(-X_k - \widehat{\mathbb{E}}[-X_k]\right) > \epsilon\right) \\
&\qquad + \sum_{k=1}^{\infty}\mathbb{V}\left(\sum_{k=1}^{n} a_{nk}\left(X_k - \widehat{\mathbb{E}}[X_k]\right) < -\epsilon\right) < \infty,
\end{aligned}$$

which completes the proof.

We can get the following corollary immediately.

Corollary 3.1. *Let $\{X, X_n, n \geq 1\}$ be a sequence of END random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ which is stochastically dominated by a random variable X satisfying* (3.4)*, and let $\mathbb{V}$ be a countably sub-additive capacity. Assume that $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ is an array of real numbers satisfying* (3.2) *and* (3.3)*. Then* (3.5) *and* (3.6) *hold.*

Corollary 3.2. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed END random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ satisfying (3.4) and $\widehat{\mathbb{E}}[X_1] = \widehat{\mathcal{E}}[X_1] = 0$, and let $\mathbb{V}$ be a countably sub-additive capacity. Assume that $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ is an array of real numbers satisfying* (3.2) *and* (3.3)*, then we have*

$$\sum_{k=1}^{n} a_{nk} X_k \longrightarrow 0 \quad \text{a.s.} \quad \mathbb{V} \quad \text{as } n \to \infty.$$

If we take $a_{nk} = n^{-\alpha}$ in Corollary 3.2, we can get the Marcinkiewicz's strong law of large numbers under sub-linear expectations ([23]).

Corollary 3.3. *Let $\{X_n, n \geq 1\}$ be a sequence of widely negative dependent and identically distributed random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ satisfying (3.4) and $\widehat{\mathbb{E}}[X_1] = \widehat{\mathcal{E}}[X_1] = 0$, and let $\mathbb{V}$ be a countably sub-additive capacity, then we have*

$$\sum_{k=1}^{n} \frac{X_k}{n^\alpha} \longrightarrow 0 \quad \text{a.s.} \quad \mathbb{V} \quad \text{as } n \to \infty,$$

where $0 < \alpha < 1$.

References

[1] J. Chen and Z. Chen, A new proof of the generalized law of large numbers under Choquet expectation, *J. Ineq. & Appli.*, (2020), 2020:158.

[2] X. Chen and Q. Wu, Complete convergence and complete integral convergence of partial sums for moving average process under sub-linear expectations, *AIMS Mathematics*, **7**(6) (2022), 9694-9715.

[3] L. Denis and C. Martini, A theoretical framework for the pricing of contingent claims in the presence of model uncertainly, *Ann. Applied Probab.*, **16**(2) (2006), 827-852.

[4] F. Feng, D. Wang and Q. Wu, Complete convergence for weigted sums of negatively dependent random variables under the sub-linear expectations, *Comm. Stati.-Theory & Methods*, **48**(6) (2019), 1351-1366.

[5] I. Gilboa, Expected utility theory with purely subjective non-additive probabilities, *J. Math. Economics*, **16**(1) (1987), 65-88.

[6] A. Gut, Complete convergence for arrays, *Periodica Math. Hungarica*, **25** (1992), 51-75.

[7] P.L. Hsu and H. Robbins, Complete convergence and the law of large numbers, *Proc. Natl. Acad. Sci.*, **33** (1947), 25-31.

[8] H. Liang and Q. Wu, Complete convergence of weighted sums for extended ND random variables sequence under sub-linear expectation, *J. Univ. Sci. & Tech. China*, **48**(8) (2018), 637-642.

[9] Y. Lin and X. Feng, Complete convergence and strong law of large numbers for arrays of random variables under sublinear expectations, *Comm. Stati.-Theory & Methods*, **49**(23) (2020), 5866-5882.

[10] D. Lu and Y. Meng, Complete and complete integral convergence for arrays of row wise widely negative dependent random variables under the sub-linear expectations, *Comm. Stati.-Theory & Methods*, **51**(9) (2020), 1-14.

[11] F. Maccheroni and M. Marinacci, A strong law of large number for capacities, *Ann. Probab.*, **33** (2005), 1171-1178.

[12] S.G. Peng, Monotonic limit theorem of BSDE and nonlinear decoposition theorem of Doob-Meyer type, *Probab. Th. & Rel. Fields*, **113** (1999), 473-499.

[13] S.G. Peng, G-expectation, G-Brownian motion and related stochastic calculus of Ito type, *Stoch. Anal. Appli.*, **2**(4) (2006), 541-567.

[14] S.G. Peng, A new central limit theorem under sublinear expectations, *J. Math.*, **53**(8) (2008), 1989-1994.

[15] Q. Wu, Complete convergence for END random variables under sublinear expectations, *Discrete Dynamics in Nature and Society*, Vol. 2021, Article ID 5529109, 10 pages.

[16] M. Xi, Y. Wu and X. Wang, Complete convergence for arrays of rowwise END random variables and its statistical applications under sub-linear expectations, *J. Korean Statist. Sco.* , **48** (2019), 412-425.

[17] D. Yu and Q. Wu, Complete convergence for weighted sums of extended negatively depedent random variables under sub-linear expectation, *Comm. Stati.-Theory* & *Metheds*, **47**(19) (2018a), 4741-4750.

[18] D. Yu and Q. Wu, Marcinkiewicz type complete convergence for weighted sums under sub-linear expectations, *J. Univ. Sci.* & *Tech. China*, **48**(2) (2018b), 89-96.

[19] L.X. Zhang, Donsker's invariance principle under the sub-linear expectation with an application to Chung's law of the iterated logarithm, *Commun. Math. Stat.*, **3** (2015), 287–214.

[20] L.X. Zhang, Exponential inequalities under the sub-linear expectations with applications to laws the iterated logarithm, *Sci. China Math.*, **59** (2016a), 2503–2526.

[21] L.X. Zhang, Rosenthal's inequalities for independent and negatively dependent random variables under sub-linear expectations with applications, *Sci. China Math.*, **59**(4) (2016b), 751–768.

[22] L.X. Zhang, Strong limit theorems for extended independent and extended negatively dependent random variables under non-linear expectations, *Acta Math. Scientia*, **42** (2022), 467-490.

[23] L.X. Zhang and J. Lin, Marcinkiewicz's strong law of large numbers for non-additive expectation, *Stat.* & *Probab. Lett.*, **137** (2018), 267–276.