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CONTINUITY OF LOCALLY BOUNDED HOMOMORPHISMS OF SOME EXTENSIONS OF PERFECT CONNECTED LIE GROUPS

A. I. Shtern

Abstract. We obtain sufficient conditions for the automatic continuity of locally bounded homomorphisms of some extensions of perfect connected Lie groups into connected Lie groups.

§ 1. Introduction

In this note, we obtain sufficient conditions for the continuity of every locally bounded homomorphism of some extension of a perfect connected Lie group G into a Lie group.

§ 2. Preliminaries

Let us recall some information needed below.

A (not necessarily continuous) homomorphism π of a topological group G into a topological group H is said to be *relatively compact* if there is a neighborhood $U = U_{e_G}$ of the identity element e_G in G whose image $\pi(U)$ has compact closure in H. Obviously, a homomorphism into a locally compact group is relatively compact if and only if it is *locally bounded*, i.e., there is a neighborhood U_e whose image is contained in some element of the filter $\mathfrak{V}$ of neighborhoods of e_V having compact closure.

Let us also recall the notion of discontinuity group of a homomorphism π of a topological group G into a topological group H, see [1] and [2]. Let

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$\mathfrak{U} = \mathfrak{U}_G$ be the filter of neighborhoods of e_G in G. For every (not necessarily continuous) locally relatively compact homomorphism π of G into H, the set

$$\mathrm{DG}(\pi) = \bigcap_{U \in \mathfrak{U}} \overline{\pi(U)}$$

is called the discontinuity group of π. Here and below, the bar stands for the closure in the corresponding topology (here the closure is taken in the topology of H). (See Definition 1.1.1 of [1].)

The discontinuity group of a homomorphism has some important properties. Under the above conditions, the set $\mathrm{DG}(\pi)$ is a compact subgroup of the topological group H and a compact normal subgroup of the closed subgroup $\overline{\pi(G)}$ of H. Moreover, the filter basis $\{\overline{\pi(U)} \mid U \in \mathfrak{U}\}$ converges to $\mathrm{DG}(\pi)$, and the homomorphism π is continuous if and only if $\mathrm{DG}(\pi) = \{e_H\}$. (See Theorem 1.1.2 of [1].) If G is a connected Lie group, then $\mathrm{DG}(\pi)$ is a compact connected subgroup of H. (See Lemma 1.1.6 of [1].)

Recall that a connected Lie group G is said to be *perfect* if the commutator subgroup of G coincides with G in the group-theoretic sense.

§ 3. Main Result

Theorem. *Let G be a connected Lie group containing a connected perfect Lie subgroup H and a closed supplementary central subgroup K such that $G = HK$ (i.e., every element $g \in G$ admits a representation in the form $g = hk$, where $h \in H$ and $k \in K$). Every locally bounded homomorphism of G into a Lie group is continuous if and only if its restriction to K is continuous.*

Proof. Let π be a locally bounded homomorphism of the connected Lie group G into a Lie group H. Obviously, if a homomorphism of a group is continuous, then it is continuous on each subgroup of the group, so it is sufficient to prove the "if" part.

Let H be a perfect Lie group, let K be an Abelian Lie group, and let the connected Lie group G be included in a split short exact sequence of the central extension $\{e\} \to K \xrightarrow{\iota} G \xrightarrow{\rho} B \to \{e\}$ with continuous embedding ι onto a closed subgroup of G and the canonical epimorphism ρ of the group G onto H, isomorphic to the quotient group G/K. Then the commutator subgroup G' of the group G is mapped by an epimorphism ρ onto the commutator subgroup H' of H. Moreover, G' is in a natural one-to-one correspondence with H'. Indeed, for any $k_1, k_2 \in K$ and $b, c \in G$ we have $bk_1ck_2(bk_1)^{-1}(ck_2)^{-1} = bcb^{-1}c^{-1}k_1k_1^{-1}k_2k_2^{-1} = [b,c]$ and, therefore, the commutator of bK and cK is equal to $[b,c]K$ for any $b,c \in Z$. Thus, the commutator subgroup of the group G is naturally isomorphic to the commutator of the group H. However, H is perfect, which means that $H' = H$. Hence, the subgroup G' of G coincides with the isomorphic image of the group H in G under a split mapping and is thus closed, and every element of G is the product of an element of G' and an element of K.

Now the statement of the corollary follows from Theorem 5 of [7], which was proved using the properties of the discontinuity group of a locally bounded homomorphism that are listed above in Sec. 2. This completes the proof of the theorem.

§ 4. Discussion

As a rule, a locally bounded homomorphism between Lie groups is continuous only under additional conditions (for example, see [3–6].

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Moscow Center for Fundamental and Applied Mathematics, Moscow, 119991 Russia
Department of Mechanics and Mathematics,
Moscow State University,
Moscow, 119991 Russia
Federal State Institution
"Scientific Research Institute for System Analysis of the Russian Academy of Sciences" (FSI SRISA RAS),
Moscow, 117312 Russia
E-mail: aishtern@mtu-net.ru, rroww@mail.ru