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Convergence of Iterates of q-Bernstein operators on square with two curved sides via contraction principle

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Abstract. In this paper, fixed points of Phillips type Bernstein operators, their products and Boolean sums have been computed on square with two curved sides. Convergence of the iterates of these operators has been established by employing the Banach Contraction Principle. Further, it is also proved that these operators are Weakly Picard Operators.

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1 Introduction

The foundations of Approximation Theory were laid down by the famous theorem by Karl Weierstrass[2]. A constructive proof of his theorem was produced by S.N. Bernstein in the year 1912 [30]. He constructed a sequence of polynomials popularly called as Bernstein polynomials utilising the probabilistic theory approach.Thereafter different approximation operators have been constructed to improve various approximation parameters. E.g. one can refer [15, 8, 4, 9, 12, 18, 31, 23]. Inverse results in approximation theory were studied in [26].
With the development of quantum calculus, rational type generalisations of Bernstein Operators were constructed by Lupaş in 1987 [3]. In 1997, Phillips [5] generalised Bernstein polynomials using the notions of q-calculus though adopting a different approach. For more details on q-calculus one can refer [6, 27] and for more details on

generalised q Bernstein polynomials one can refer [36, 37].

Approximation operators on various domains are needed in the numerical analysis for solving differential equations with known boundary conditions. As a result, several researchers constructed and generalised various operators for better approximation on diverse domains. On the square domains too approximation by operators and their bases have been used in the problems concerning CAGD (Computer Aided Geometric Design) [10]. q-Bernstein Operators (Philips type) have been studied on square shaped regions along with their approximation properties in [1].

In the paper by Rus[24] and through the Kelisky-Rivlin's result in [25] it was stated that $\forall\ t \in C[0,1], \alpha \in [0,1]$ and $\nu \in \mathbb{N}$, $lim_{m\to\infty} B_n^m(t)(\alpha) = t(0) + [t(1) - t(0)]\alpha$ (where $B_n^m(t)(\alpha)$ denotes the mth iterate of successive approximation for the classical Bernstein operator $B_n(t)(\alpha)$). The authors of the above work have beautifully used the Banach Fixed Point Theorem. Also, in [7, 11, 13, 28, 32, 34, 35] the limiting behaviour of the iterates of certain classes of operators which are positive as well as linear is studied. New characterisations e.g, Korovkin type results and the asymptotic nature of the iterates for positive linear operators were studied and described in [11, 34, 35]. Motivated by the work done in [33] we consider the q-Bernstein Approximation operators along with their product and Boolean sum on a domain bounded by a square having two curved edges and study the convergence of their iterates. We prove the existence and uniqueness of their fixed points by applying the contraction principle. One can consult the following for the relevant work in the area [14, 16, 17, 19, 20, 21, 22].

2 Weakly Picard operators

Let us now look at the following definitions:[29].
Let $(\mathfrak{X}, \mathfrak{d})$ be a metric space. Let $\mathfrak{L} : \mathfrak{X} \to \mathfrak{X}$ be an operator. Consider the following sets

1. $\mathcal{I}_{\mathfrak{L}} := \{t \in \mathfrak{X} | \mathfrak{L}(t) = t\}$, i.e. the set containing fixed points of $\mathfrak{L}$.
2. $\mathcal{J}(\mathfrak{L}) := \{\mathcal{F} \subset \mathfrak{X} |\ \mathfrak{L}(\mathcal{F}) \subset \mathcal{F}, \mathcal{F} \neq \Phi\}$ i.e. the set denoting the nonempty and invariant subsets of $\mathfrak{X}$.

Also consider the following notations:
$\mathfrak{L}^0 := \mathcal{I}_{\mathfrak{X}}, \mathfrak{L}^1 := \mathfrak{L}, \cdots, \mathfrak{L}^{n+1} := \mathfrak{L} o \mathfrak{L}^n, \quad n \in \mathbb{N}.$

Definition 2.1 *$\mathfrak{L} : \mathfrak{X} \to \mathfrak{X}$ is said to be a Picard Operator if $\exists\ \eta^* \in \mathfrak{X}$ such that*
(i) $\mathcal{I}_{\mathfrak{L}} = (\eta^*)$;
(ii) the sequence $(\mathfrak{L}^n(\eta_0))_{n\in\mathbb{N}}$ converges and it converges to η^* $\forall\ \eta_0 \in \mathfrak{X}$.

Definition 2.2 *$\mathfrak{L} : \mathfrak{X} \to \mathfrak{X}$ is said to be a Weakly Picard operator $\Leftrightarrow \exists$ a partition of $\mathfrak{X}$ i.e. $\mathfrak{X} = \cup_{\upsilon\in\Lambda}\mathfrak{X}_\upsilon$, such that*
(1) $\mathfrak{X}_\upsilon \in \mathcal{J}(\mathfrak{L})$, for all $\upsilon \in \Lambda$.
(2) $\mathfrak{L}|_{\mathfrak{X}_\upsilon} : \mathfrak{X}_\upsilon \to \mathfrak{X}_\upsilon$ is a Picard operator $\forall\ \upsilon \in \Lambda$.

Definition 2.3 *Let $\mathfrak{L}$ is a Weakly Picard operator. We define the operator $\mathfrak{L}^\infty$, in the following way*

$$\mathfrak{L}^\infty : \mathfrak{X} \to \mathfrak{X}, \text{such that}$$
$$\mathfrak{L}^\infty(\eta) := \lim_{n\to\infty} \mathfrak{L}^n(\eta), \qquad n \in \mathbb{N}$$

3 Philips q-Bernstein Operators on $\mathbb{D}_h$

Let $\mathbb{D}_h$ be a square region bounded by the sides/edges viz.Γ_1, Γ_2, Γ_3, and Γ_4 out which which the edges Γ_3 and Γ_4 are the curved sides taken to be $\Gamma_3 = g(\beta)$ and $\Gamma_4 = d(\alpha)$. Through the point $(\alpha, \beta) \in \mathbb{D}_h$, let us consider a line parallel to the edge Γ_1 intersecting the edge Γ_2 at $(0, \beta)$ and Γ_4 at $(g(\beta), \beta)$ and another line parallel to the edge Γ_2 intersecting the edge Γ_1 at $(\alpha, 0)$ and the edge Γ_3 at $(\alpha, d(\alpha))$. Let us now consider the following two uniform partitions of the intervals $[0, g(\beta)]$ and $[0, d(\alpha)]$,-
$\square^\alpha_m = \{\frac{[i_q]}{[m]_q} g(\beta), i = \overline{0, m}\}$ and $\square^\beta_n = \{\frac{[j]_q}{[n]_q} d(\alpha), j = \overline{0, n}\}$..
Let F is a real valued function on $\mathbb{D}_h$, $h \in \mathbb{R}_+$. Now on the domain $\mathbb{D}_h$, we can define the Phillips type q-Bernstein operators in the following manner:

$$(\mathcal{B}^\alpha_{m,q} f) = \sum_{i=0}^{m} \tilde{p}_{m,i}(\alpha, \beta) F\left(\frac{[i]_q}{[m]_q} g(\beta), \beta\right), \tag{3.1}$$

and

$$(\mathcal{B}^\beta_{n,q} f) = \sum_{j=0}^{n} \tilde{q}_{n,j}(\alpha, \beta) F\left(\alpha, \frac{[j]_q}{[n]_q} d(\alpha)\right), \tag{3.2}$$

where,

$$\tilde{p}_{m,i}(\alpha, \beta) = \frac{\begin{bmatrix} m \\ i \end{bmatrix}_q \alpha^i \prod_{w=0}^{m-i-1} (g(\beta) - q^w \alpha)}{(g(\beta))^m}, \qquad (\alpha, \beta) \in \mathbb{D}_h, \tag{3.3}$$

and

$$\tilde{q}_{n,j}(\alpha, \beta) = \frac{\begin{bmatrix} n \\ j \end{bmatrix}_q \beta^j \prod_{z=0}^{n-j-1} (d(\alpha) - q^z \beta)}{(d(\alpha))^n}, \qquad (\alpha, \beta) \in \mathbb{D}_h, \tag{3.4}$$

The interpolation properties for $\mathcal{B}^\alpha_{m,q}$ and $\mathcal{B}^\beta_{n,q}$ can be stated by means of the following theorems..

Theorem 3.1 *[1] For a real-valued function F on $\mathbb{D}_h$,*

$(i)\mathcal{B}_{m,q}^{\alpha}F = F$ *on the two edges* Γ_2 *and* Γ_4;

$(ii)(\mathcal{B}_{m,q}^{\alpha}e_{ij})\ (\alpha,\beta) = \alpha^i\beta^j, \quad i = 0,1; j \in \mathbb{N},$

$$(iii)\mathcal{B}_{m,q}^{\alpha}e_{i2} = \left(\alpha^2 + \frac{\alpha(g(\beta)-\alpha)}{[m]_q}\right)\beta^i, \quad i \in \mathbb{N}$$

Theorem 3.2 *For a real-valued function F on $\mathbb{D}_h$,*

$(i)\mathcal{B}_{n,q}^{\beta}F = F$ *on* $\Gamma_1 \cup \Gamma_3$;

$(ii)(\mathcal{B}_{n,q}^{\beta}e_{ij})\ (\alpha,\beta) = \alpha^i\beta^j, \quad j = 0,1; i \in \mathbb{N},$

$$(iii)\mathcal{B}_{n,q}^{\beta}e_{i2} = \left(\beta^2 + \frac{\beta(d(\alpha)-\beta)}{[n]_q}\right)\alpha^i, \quad i \in \mathbb{N}$$

We now come to the main results of this section wherein F is a real valued function on $\mathbb{D}_h$, $h \in \mathbb{R}_+$.

Theorem 3.3 *It holds that $\mathcal{B}_{m,q}^{\alpha}$ and $\mathcal{B}_{n,q}^{\beta}$ are weakly Picard operators and*
$\big(\mathcal{B}_{m,q}^{\alpha,\infty}F\big)(\alpha,\beta) = \frac{F(g(\beta),\beta)-F(0,\beta)}{g(\beta)}\alpha + F(0,\beta)$

$\big(\mathcal{B}_{n,q}^{\beta,\infty}F\big)(\alpha,\beta) = \frac{F(\alpha,d(\alpha))-F(\alpha,0)}{d(\alpha)}\beta + F(\alpha,0)$

Proof: Keeping in view the interpolation results of $\mathcal{B}_{m,q}^{\alpha}$ and $\mathcal{B}_{n,q}^{\beta}$ (from Theorem 3.1 and Theorem 3.2), let us consider the sets-

$$Y^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}} = \{F \in C(\mathcal{D}_h)|F(0,\beta) = \tau|_{\Gamma_2}, F(g(\beta),\beta) = \tau|_{\Gamma_4}\},\ for\ \beta \in [0,d(\alpha)],$$

$$Y^{(2)}_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}} = \{F \in C(\mathcal{D}_h)|F(\alpha,0) = \chi|_{\Gamma_1}, F(\alpha,d(\alpha)) = \chi|_{\Gamma_3}\},\ for\ \alpha \in [0,g(\beta)],$$

and denote

$$F^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}}(\alpha,\beta) := \frac{\tau|_{\Gamma_4} - \tau|_{\Gamma_2}}{g(\beta)}\alpha + \tau|_{\Gamma_2}$$

$$F^{(2)}_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}}(\alpha,\beta) := \frac{\chi|_{\Gamma_3}-\chi|_{\Gamma_1}}{d(\alpha)}\alpha+\chi|_{\Gamma_1}$$

with $\tau,\chi \in \mathrm{C}(\mathcal{D}_h)$.

Let us now observe the following:

(i)$Y^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma 4}}$ and $Y^{(2)}_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}}$ are closed subsets of the set $C(\mathcal{D}_h)$;
(ii)$Y^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}}$ and $Y^{(2)}_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}}$ are invariant subsets of $\mathcal{B}^{\alpha}_{m,q}$ and $\mathcal{B}^{\beta}_{n,q}$ respectively where $\tau,\chi \in \mathrm{C}(\mathcal{D}_h)$; $n,m\in \mathbb{N}$;
(iii)$\mathrm{C}(\mathcal{D}_h)=\cup_{\tau\in C(\mathcal{D}_h)}Y^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}}$ and $\mathrm{C}(\mathcal{D}_h)=\cup_{\chi\in C(\mathcal{D}_h)}Y^{(2)}_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}}$ are partitions of $\mathrm{C}(\mathcal{D}_h)$;
(iv)$F^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}}\in Y^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}}\cap F_{\mathcal{B}^{\alpha}_{m,q}}$ and $F^{(2)}_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}}\in Y^{(2)}_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}}\cap F_{\mathcal{B}^{\beta}_{n,q}}$; where$F_{\mathcal{B}^{\alpha}_{m,q}}$ and $F_{\mathcal{B}^{\beta}_{n,q}}$ are the fixed points sets of the operators $\mathcal{B}^{\alpha}_{m,q}$ and $\mathcal{B}^{\beta}_{n,q}$.
Statements (i) and (iii) are easy to prove by applying the definition of closed subset and varying $\tau,\chi\in\mathrm{C}(\mathcal{D}_h)$.
(ii) Since the operator $\mathcal{B}^{\alpha}_{m,q}$ is a linear operator and by virtue of Theorem 3.1, it follows that $\forall\ F^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}}\in Y^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}}$ and $\forall\ F^{(2)}_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}}\in Y^{(2)}_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}}$;

we have
$\mathcal{B}^{\alpha}_{m,q}F^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}}:=F^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}}$,
$\mathcal{B}^{\beta}_{n,q}F^{(2)}_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}}:=F^{(1)}_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}}$.
Therefore, $Y^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}}$ and $Y^{(2)}_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}}$ are invariant subsets of $\mathcal{B}^{\alpha}_{m,q}$ and $\mathcal{B}^{\beta}_{n,q}$ respectively for $\tau,\chi\in C(\mathcal{D}_h)$ and $n,m\in\mathbb{N}$.
We now prove that the maps, $\mathcal{B}^{\alpha}_{m,q}|_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}}:Y^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}}\to Y^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}}(\alpha,\beta)$ and $\mathcal{B}^{\beta}_{n,q}|_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}}:Y^{(2)}_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}}\to Y^{(2)}_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}}(\alpha,\beta)$ are contraction maps for $\tau,\chi\in\mathrm{C}(\mathcal{D}_h)$ and $n,m\in\mathbb{N}$.
Consider $G,\ H\in Y^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}}$.
Then for a fixed β, let

$$u_m=\underset{0\le\alpha\le g(\beta))}{min}\{\tilde{p}_{m,0}(\alpha,\beta)+\tilde{p}_{m,m}(\alpha,\beta)\}$$

$$u_m=\underset{0\le\alpha\le g(\beta))}{min}\left\{\frac{\prod_{w=0}^{m-1}(g(\beta)-q^w\alpha)}{g(\beta)^m}+\left(\frac{\alpha}{g(\beta)}\right)^m\right\}.$$

As we know that $\sum_{i=0}^{m}\tilde{p}_{m,i}(\alpha,\beta)=1$, therefore $0<u_m<1$.
Consequently,
$|\mathcal{B}^{\alpha}_{m,q}(G)(\alpha,\beta)-\mathcal{B}^{\alpha}_{m,q}(H)(\alpha,\beta)|=|\mathcal{B}^{\alpha}_{m,q}(G-H)(\alpha,\beta)|\le|1-u_m|\ \|G-H\|_\infty$.
In other terms,
$||\mathcal{B}^{\alpha}_{m,q}(G-H)(\alpha,\beta)||\le|1-u_m|\ \|G-H\|_\infty$ for all $G,\ H\in Y^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}}$

Hence $\mathcal{B}^{\alpha}_{m,q}|_{Y^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}}}$ is a contraction for $\tau \in C(\mathcal{D}_h)$.

Similarly for a fixed α, let

$$v_n = \min_{0\leq\beta\leq d(\alpha))} \{\tilde{q}_{n,0}(\alpha,\beta) + \tilde{q}_{n,n}(\alpha,\beta)\}$$

$$v_n = \min_{0\leq\beta\leq d(\alpha))} \left\{ \left(\frac{\prod_{z=0}^{n-1}(d(\alpha) - q^z\beta)}{d(\alpha)^n} \right) + \left(\frac{\beta}{d(\alpha)} \right)^n \right\}.$$

Since $\sum_{j=0}^{n} \tilde{q}_{n,j}(\alpha,\beta) = 1$ we have, $0 < v_n < 1$. Again we observe that,
$|\mathcal{B}^{\beta}_{n,q}(G)(\alpha,\beta) - \mathcal{B}^{\beta}_{n,q}(H)(\alpha,\beta)| = |\mathcal{B}^{\beta}_{n,q}(G-H)(\alpha,\beta)| \leq |1-v_n| \; \|G-H\|_\infty$

$$||\mathcal{B}^{\beta}_{n,q}(G)(\alpha,\beta) - \mathcal{B}^{\beta}_{n,q}(H)(\alpha,\beta)||_\infty \leq |1-v_n| \|G-H\|_\infty, for\ all\ G,\ H \in Y^{(2)}_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}}$$

Since $0 < v_n < 1$, $\mathcal{B}^{\beta}_{n,q}|_{Y^{(2)}_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}}}$ is a contraction for $\chi \in C(\mathcal{D}_h)$.

Moreover, $(\frac{(\tau|_{\Gamma_4} - \tau|_{\Gamma_2})}{g(\beta)})(.) + \tau|_{\Gamma_2} \in Y^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}}$,
$(\frac{(\chi|_{\Gamma_3} - \chi|_{\Gamma_1})}{(d(\alpha))})(.) + \chi|_{\Gamma_3})(d(\alpha)) \in Y^{(2)}_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}}$ are the fixed points of maps $\mathcal{B}^{\alpha}_{m,q}$ and $\mathcal{B}^{\beta}_{n,q}$ respectively, that is,

$$\mathcal{B}^{\alpha}_{m,q}\left(\frac{\tau|_{\Gamma_4} - \tau|_{\Gamma_2}}{g(\beta)}(.) + \tau|_{\Gamma_2}\tau|_{\Gamma_4} \right) = \frac{\tau|_{\Gamma_4} - \tau|_{\Gamma_2}}{g(\beta)}(.) + \tau|_{\Gamma_2}\tau|_{\Gamma_4}$$

$$\mathcal{B}^{\beta}_{n,q}\left(\frac{\chi|_{\Gamma_3} - \chi|_{\Gamma_1}}{d(\alpha)}(.) + \chi|_{\Gamma_1}\chi|_{\Gamma_3} \right) = \frac{\chi|_{\Gamma_3} - \chi|_{\Gamma_1}}{d(\alpha)}(.) + \chi|_{\Gamma_1}\chi|_{\Gamma_3}$$

Using Banach Contraction principle, we can state that $F^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}}(\alpha,\beta)$ $:= \big(\frac{(\tau|_{\Gamma_4} - \tau|_{\Gamma_2})}{(g(\beta))}\big)(\alpha) + (\tau|_{\Gamma_2})$ is the only fixed point of $\mathcal{B}^{\alpha}_{m,q}$ in $Y^{(1)}_{\tau|_{\Gamma_2},\tau|_{\Gamma_4}}$ and $\mathcal{B}^{\alpha}_{m,q}|_{Y^{(1)}_{\tau|_{\Gamma_2}},\tau|_{\Gamma_4}}$ is a Picard operator, with

$$\big(\mathcal{B}^{\alpha,\infty}_{m,q}F\big)(\alpha,\beta) = \frac{F(g(\beta),\beta) - F(0,\beta)}{g(\beta)}\alpha + F(0,\beta)$$

and, similarly $F^{(2)}_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}}(\alpha,\beta) := \big(\frac{(\chi|_{\Gamma_3} - \chi|_{\Gamma_1})}{(d(\alpha))}\big)(\beta) + \chi|_{\Gamma_1}$ is the unique fixed point of operator $Y^{(2)}_{\chi|_{\Gamma_1},\chi|_{\Gamma_3}}$ and

$$\big(\mathcal{B}^{\beta,\infty}_{n,q}F\big)(\alpha,\beta) = \frac{F(\alpha,0) - F(\alpha,0)}{d(\alpha)}\beta + F(\alpha,0).$$

Consequently, it follows that the operators $\mathcal{B}^{\alpha}_{m,q}$ and $\mathcal{B}^{\beta}_{n,q}$ are weakly Picard operators.

4 Product operators on $\mathbb{D}_h$

Let $\mathcal{P}_{mn,q}$=$\mathcal{B}^{\alpha}_{m,q}\mathcal{B}^{\beta}_{n,q}$ and $\mathcal{Q}_{nm,q}$ =$\mathcal{B}^{\beta}_{n,q}\mathcal{B}^{\alpha}_{m,q}$, i.e. the products of the two Philips type Bernstein operators on the domain $\mathbb{D}_h$.
Then these are defined by the following expressions-

$$\Big(\mathcal{P}_{mn,q}F\Big)(\alpha,\beta)=\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(\alpha,\beta)\tilde{q}_{n,j}\bigg([i]_q\frac{g(\beta)}{[m]_q},\beta\bigg)F\bigg([i]_q\frac{g(\beta)}{[m]_q},\frac{[j]_q}{[n]_q}d\bigg(\frac{[i]_qg(\beta)}{[m]_q}\bigg)\bigg)$$

$$\Big(\mathcal{Q}_{nm,q}F\Big)(\alpha,\beta)=\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(\alpha,[j]_q\frac{d(\alpha)}{[n]_q})\tilde{q}_{n,j}(\alpha,\beta)F\bigg(\frac{[i]_q}{[m]_q}g\bigg(\frac{[j]_q}{[n]_q}d(\alpha)\bigg),\frac{[j]_q}{[n]_q}d(\alpha)\bigg)$$

Theorem 4.1 *Given that, $\alpha\in[0,g(\beta)]$ and $\beta\in[0,d(\alpha)]$, product operator $\mathcal{P}_{mn,q}$ follow the relations:*
(i) $(\mathcal{P}_{mn,q}F)(\alpha,0)=(B^{\alpha}_{m,q}F)(\alpha,0)$,
(ii) $(\mathcal{P}_{mn,q}F)(0,\beta)=(B^{\beta}_{n,q}F)(0,\beta)$,
(iii) $(\mathcal{P}_{mn,q}F)(\alpha,d(\alpha))=(B^{\alpha}_{m,q}F)(\alpha,d(\alpha))$,
(iv) $(\mathcal{P}_{mn,q}F)(g(\beta),\beta)=(B^{\beta}_{n,q}F)(g(\beta),\beta)$.

Theorem 4.2 *$\mathcal{P}_{mn,q}$ is a weakly Picard operator and*
$\big(\mathcal{P}^{\infty}_{mn,q}F\big)(\alpha,\beta)=\frac{1}{[g(\beta)][d(\alpha)]}[g(\beta)d(0)F(0,0)+\frac{\alpha}{[g(\beta)][d(\alpha)]}[d(0)F(g(\beta),0)-d(0)F(0,0)]+\frac{\beta}{[g(\beta)][d(\alpha)]}[g(\beta)F(0,d(0))-g(\beta)F(0,0)]+\frac{ts}{[g(\beta)][d(\alpha)]}[F(0,0)+F(g(\beta),d(g(\beta))-F(0,d(0))-F(g(\beta),0)]$.

Proof: Consider $Y_{\mu,\nu,\xi,\zeta}=\{F\in C(\mathcal{D}_h)|F(0,0)=\mu,\ F(g(0),0)=\nu,F(g(h),d(h))=\xi,F(0,d(0))=\zeta\}$ and denote

$F_{\mu,\nu,\xi,\zeta}(\beta,\alpha):=[g(\beta)d(0)\mu-g(\beta)d(0)\nu]\times([g(\beta)][d(\alpha)])^{-1}+\frac{d(g(\beta))\zeta+d(0)\nu-d(0)\mu}{[g(\beta)][d(\alpha)]}\alpha$
$+\frac{g(\beta)\zeta-g(\beta)\mu}{[g(\beta)][d(\alpha)]}\beta+\frac{\mu+\xi-\nu-\zeta}{[g(\beta)][d(\alpha)]}\alpha\beta$

with $\mu,\nu,\xi,\zeta\in\mathbb{R}$.
Let us consider the following statements
(i) $Y_{\mu,\nu,\xi,\zeta}$ is a closed subset of the set C($\mathcal{D}_h$)
(ii) $Y_{\mu,\nu,\xi,\zeta}$ is invariant subset for $\mathcal{P}_{mn,q}$ for $\mu,\nu,\xi,\zeta\in\mathbb{R}$ and $n,m\in\mathbb{N}$;
(iii) $C(\mathcal{D}_h)$=$\cup_{\mu,\nu,\xi,\zeta}Y_{\mu,\nu,\xi,\zeta}$ i.e. $Y_{\mu,\nu,\xi,\zeta}$ forms the partition of $C(\mathcal{D}_h)$
(iv)$F_{\mu,\nu,\xi,\zeta}\in Y_{\mu,\nu,\xi,\zeta}\cap F_{\mathcal{P}_{mn,q}}$,where $F_{\mathcal{P}_{mn,q}}$ denotes the sets of fixed points for $\mathcal{P}_{mn,q}$
It is easy to prove statements (i) and (iii).
(ii) Applying same idea i.e that of theorem 3.2, we use the linearity of operators and Theorem 4.1, to note that $Y_{\mu,\nu,\xi,\zeta}$ is invariant subset of $\mathcal{P}_{nm,q}$ for $\mu,\nu,\xi,\zeta\in\mathbb{R}$ and $n,m\in\mathbb{N}$
(iv) Further we prove that

$\mathcal{P}_{mn,q}|_{Y_{\mu,\nu,\xi,\zeta}} : Y_{\mu,\nu,\xi,\zeta} \to X_{\mu,\nu,\xi,\zeta}$
is contraction map for $\mu, \nu, \xi, \zeta \in \mathbb{R}$ $n, m \in \mathbb{N}$. Let $G, H \in Y_{\mu,\nu,\xi,\zeta}$ and $j_{mn} = u_m v_n$, i.e.
$j_{mn} = \underset{0\leq\alpha\leq g(\beta)}{min}\left\{\frac{\prod_{w=0}^{m-1}(g(\beta)-q^w\alpha)}{g(\beta)^m} + \left(\frac{\alpha}{g(\beta)}\right)^m\right\} \underset{0\leq\beta\leq d(\alpha)}{min}\left\{\left(\frac{\prod_{z=0}^{n-1}(d(\alpha)-q^z\beta)}{d(\alpha)^n}\right) + \left(\frac{\beta}{d(\alpha)}\right)^n\right\}$,
we have $|\mathcal{P}_{mn,q}(G)(\alpha,\beta) - \mathcal{P}_{mn,q}(H)(\alpha,\beta)| = |\mathcal{P}_{mn,q}(G-H)(\alpha,\beta)|$

$\leq |1 - u_m v_n| \; \|G - H\|_\infty$
$\leq |1 - j_{mn}| \; \|G - H\|_\infty$.
Since $0 < u_m < 1$ and $0 < v_m < 1$, it follows that $0 < j_{mn} < 1$. Thus it is proved that, $\mathcal{P}_{mn,q}|_{Y_{\mu,\nu,\xi,\zeta}}$ is a contraction map for $\mu, \nu, \xi, \zeta \in \mathbb{R}$.

Consequently, $\|\mathcal{P}_{mn,q}(G)(\alpha,\beta) - \mathcal{P}_{mn,q}(H)(\alpha,\beta)\|_\infty \leq (1 - j_{mn})\|G - H\|_\infty$

Applying the contraction principle, it is proved that $F_{\mu,\nu,\xi,\zeta}$ is unique fixed point of $\mathcal{P}_{mn,q}$ in $Y_{\mu,\nu,\xi,\zeta}$. Moreover, $\mathcal{P}_{mn,q}|_{Y_{\mu,\nu,\xi,\zeta}}$ is a Picard operator.As a result, one can note that $\mathcal{P}_{mn,q}$ is a weakly Picard operator.

Remark 4.3 *We can prove the same result for the operator $\mathcal{Q}_{nm,q}$ analogously.*

5 Boolean sum operator on $\mathbb{D}_h$

Let us define the Boolean Sum Operators by the following expressions-

$$\mathcal{U}_{mn,q} := \mathcal{B}^\alpha_{m,q} \oplus \mathcal{B}^\beta_{n,q} = \mathcal{B}^\alpha_{m,q} + \mathcal{B}^\beta_{n,q} - \mathcal{B}^\alpha_{m,q}\mathcal{B}^\beta_{n,q},$$
$$\mathcal{V}_{nm,q} := \mathcal{B}^\beta_{n,q} \oplus \mathcal{B}^\alpha_{m,q} = \mathcal{B}^\beta_{n,q} + \mathcal{B}^\alpha_{m,q} - \mathcal{B}^\beta_{n,q}\mathcal{B}^\alpha_{m,q},$$

We have the following result for the above operators.

Theorem 5.1 *[1] For any real-valued function F defined on $\mathbb{D}_h$,*

$$\mathcal{U}_{mn,q}F\Big|_{\partial\mathbb{D}_h} = F\Big|_{\partial\mathbb{D}_h}$$

and

$$\mathcal{V}_{nm,q}F\Big|_{\partial\mathbb{D}_h} = F\Big|_{\partial\mathbb{D}_h}.$$

where $\partial\mathbb{D}_h$ denotes the boundary of the region $\mathbb{D}_h$.

Theorem 5.2 *The Boolean sum operator $\mathcal{U}_{mn,q}$ is weakly Picard and*
$\left(\mathcal{S}^\infty_{mn,q}F\right)(\alpha,\beta) = \frac{F(g(\beta),\beta)-F(0,\beta)}{g(\beta)}\alpha + \frac{g(\beta)F(g(\beta),\beta)}{g(\beta)} + \frac{F(\alpha,d(\alpha))-F(\alpha,0)}{d(\alpha)}\beta + \frac{d(\alpha)F(\alpha,0)}{d(\alpha)} - \frac{1}{[g(\beta)][d(\alpha)]}[g(\beta)d(0)F(0,0) - g(\beta)d(g(\beta))F(0,d(0))] - \frac{\alpha}{[g(\beta)][d(\alpha)]}[d(0)F(g(\beta),0) - d(0)F(0,0)] - \frac{\beta}{[g(\beta)][d(\alpha)]}[g(\beta)F(0,d(0)) - g(\beta)F(0,0)] - \frac{ts}{[g(\beta)][d(\alpha)]}[F(0,0) + F(g(\beta),d(g(\beta)) - F(0,d(0)) - F(g(\beta),0)],$

Proof: The proof can be done on the lines similar to that of the previous theorem. We only need to use the inequality,

$$\|\mathcal{U}_{mn,q}(G)(\alpha,\beta) - \mathcal{U}_{mn,q}(H)(\alpha,\beta)\|_\infty \leq [1-(u_m+v_n-j_{mn})]\|G-H\|_\infty$$

in order to show that $\mathcal{U}_{mn,q}$ is contraction map.

Remark 5.3 *Analogously one can achieve similar result for operator* $\mathcal{V}_{nm,q}$.

6 Conclusion

The Phillips type q-Bernstein operators, their products and Boolean sums have fixed points on a square domain with two curved sides. These operators happen to be Weakly Picard Operators on this domain. Moreover, the iterates of these operators converge on this doamin. The present study paves the way for any future analysis into different types of domains and it will be interesting to examine the existence of the fixed points of the analogously created operators on new domains and study the behaviour of their iterates.

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