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Application of fixed point theorem to solvability for fractional integral equations in Banach Space

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Abstract

This article is involved with the solvability to fractional integral equations concerning Riemann-Liouville on a Banach space C([0, b]) arising in some engraining problem. The key findings of the article are based on theoretical concepts pertaining to the fractional calculus and the measure of non-compactness (MNC). To find this purpose, we utilize the Petershyn's fixed-point theorem (PFPT) in the Banach space. In addition, we deliver numerical examples to show the applicability of our results to the theory of fractional integral equations.

Keywords: Fractional integral equation (FIE), Measure of non-compactness (MNC), Fixed point theorem.
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1 Introduction

Fractional calculus is a famous mathematical tool for the characterization of abnormal and non-local diffusion concurrently with physical exploration and has also found applications in different fields from physics and engineering to the analysis of natural phenomena and financial analysis. The domain of fractional calculus recreates a major role in mathematical analysis which analyses the derivatives and integrals of any real or complex order by utilizing the Euler gamma function[27, 37]. The theory of integral equations play an important role to study the real life problems. It has a significant contribution not only the field of mathematics but also the other branches like engineering, physics, mechanics, modeling,

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etc [15, 32, 36]. Here, we study the solvability for the following FIE

$$\begin{aligned} u(\tau) &= f\Big(\tau, u(\tau), u(\alpha(\tau)), \frac{1}{\Gamma(h)} \int_0^{\beta(\tau)} \frac{p(\tau, s, u(\mu(s))}{(\beta(\tau)-s)^{1-h}} ds\Big) \\ &\quad \times g\Big(\tau, u(\tau), u(\phi(\tau)), \int_0^{\varphi(\tau)} q(\tau, s, u(\nu(s))ds\Big), \end{aligned} \tag{1}$$

for $\tau \in I_b = [0, b],\ 0 < h \leq 1$.

We have found the following integral equations as particular type of the equation (1).

• Darwish [18] analyze the solvability for FIE

$$\begin{aligned} u(\tau) &= f\Big(\tau, u(\alpha(\tau)), \frac{1}{\Gamma(h)} \int_0^{\tau} \frac{p(\tau, s, u(s))}{(\tau-s)^{1-h}} ds\Big) \\ &\quad \times g\Big(\tau, u(\phi(\tau)), u(\tau) \int_0^{1} q(\tau, s, u(s))ds\Big),\ \tau \in [0, 1]. \end{aligned} \tag{2}$$

• Banaś and Rzepka [8] analyze the solvability for FIE

$$u(\tau) = A(\tau) + \frac{\hat{f}(\tau, u(\tau))}{\Gamma(h)} \int_0^{\tau} \frac{p(s, u(s))}{(\tau-s)^{1-h}} ds, \quad \tau \in [0, 1]. \tag{3}$$

• Banaś and Regan [10] studied the solvability for FIE

$$u(\tau) = A(\tau) + \frac{\hat{f}(\tau, u(\tau))}{\Gamma(h)} \int_0^{\tau} \frac{p(\tau, s, u(s))}{(\tau-s)^{1-h}} ds, \quad \tau \in [0, \infty). \tag{4}$$

• Darwish and Henderson [19] studied the solvability for FIE

$$u(\tau) = f(\tau, u(\tau)) + \frac{\hat{f}(\tau, u(\tau))}{\Gamma(h)} \int_0^{\tau} \frac{p(\tau, s, u(s))}{(\tau-s)^{1-h}} ds, \quad \tau \in [0, \infty). \tag{5}$$

• Balachandran et al. [2] analyze the FIE

$$u(\tau) = f(\tau, u(\alpha(\tau))) + \frac{\hat{f}(\tau, u(\hat{\alpha}(\tau)))}{\Gamma(h)} \int_0^{\tau} \frac{p(\tau, s, u(\mu(s)))}{(\tau-s)^{1-h}} ds, \quad \tau \in [0, \infty). \tag{6}$$

• Darwish [17] analyze the solvability for FIE

$$u(\tau) = A(\tau) + \frac{u(\tau)}{\Gamma(h)} \int_0^{\tau} \frac{p(s, u(s))}{(\tau-s)^{1-h}} ds,\ \tau \in I_b. \tag{7}$$

• Deepmala [26] analyzed the following FIE

$$\begin{aligned} u(\tau) &= \bigg(\hat{f}(\tau, u(\tau)) + f\Big(\tau, \int_0^{\tau} p(\tau, s, u(s))ds, u(\alpha(\tau))\Big)\bigg) \\ &\quad \times \bigg(g\Big(\tau, \int_0^{b} q(\tau, s, u(s))ds, u(\phi(\tau))\Big)\bigg),\ \tau \in I_b. \end{aligned} \tag{8}$$

• Banaś [5] as well as Maleknejad et al. [39] analyzed the following FIE

$$u(\tau) = f\Big(\tau, \int_0^{\tau} p(\tau, s, u(\mu(s))ds, u(\alpha(\tau))\Big) \times g\Big(\tau, \int_0^{b} q(\tau, s, u(\nu(s))ds, u(\phi(\tau))\Big),\ \tau \in I_b \tag{9}$$

• Caballero et al. [12] analyzed the following FIE

$$u(\tau) = f\Big(\tau, \int_0^\tau p(\tau, s, u(s))ds, u(\alpha(\tau))\Big) \times g\Big(\tau, \int_0^b u(s)q(\tau, s, u(s))ds, u(\phi(\tau))\Big), \ \tau \in I_b. \tag{10}$$

• Hu and Yan [31] examined the solvability for FIE

$$u(\tau) = f\left(\tau, u(\tau), \int_0^\tau p(\tau, s, u(s))ds\right), \quad \tau \in [0, \infty). \tag{11}$$

• Maleknejad et al. [40, 41] examined the solvability of FIEs

$$u(\tau) = \hat{f}(\tau, u(\tau)) + f\Big(\tau, \int_0^\tau p(\tau, s, u(s)ds, u(\alpha(\tau))\Big) \tag{12}$$

$$u(\tau) = \hat{f}(\tau, u(\alpha(\tau)) \int_0^\tau p(\tau, s, u(s)ds, \quad \tau \in I_b. \tag{13}$$

• Banaś [6, 11] analyzed the following FIEs

$$u(\tau) = \hat{f}(\tau, u(\tau)) \int_0^\tau p(\tau, s, u(s))ds, \tag{14}$$

$$u(\tau) = c(\tau) + \hat{f}(\tau, u(\tau)) \int_0^\tau p(\tau, s, u(s)ds, \quad \tau \in [0, \infty). \tag{15}$$

• Cakan [13], Özdemir et al. [43], Özdemir [44] analyzed the following FIE

$$u(\tau) = f(\tau, u(\alpha(\tau)) + \hat{f}(\tau, u(\hat{\alpha}(\tau))) \int_0^{\beta(\tau)} p(\tau, s, u(\mu(s)))ds, \tag{16}$$

$$u(\tau) = \hat{f}(\tau, u(\alpha(\tau)) + f\Big(\tau, \int_0^{\beta(\tau)} p(\tau, s, u(\mu(s))ds, u(\alpha(\tau))\Big), \ \tau \in I_b. \tag{17}$$

• Aghajani and Jalilian, [1] have studied the equation

$$u(\tau) = f\left(\tau, u(\alpha(\tau)), \int_0^{\beta(\tau)} p(\tau, s, u(\mu(s)))ds\right), \ \tau \in [0, \infty). \tag{18}$$

• Cichon and Metwali [16] have studied the equation

$$u(\tau) = f(\tau, u(\alpha(\tau)) + \hat{f}(\tau, u(\tau)) \int_0^1 p(\tau, s, u(\mu(s)))ds, \ \tau \in [0, 1]. \tag{19}$$

• Vetro and Vetro [48] have the studied the equation

$$u(\tau) = f(\tau, u(\alpha(\tau))) + \int_0^{\beta(\tau)} p(\tau, s, u(\mu(s)))ds, \ \tau \in [0, \infty). \tag{20}$$

• Hashem and Rwaily [28] have the studied the equation

$$u(\tau) = f(\tau, u(\alpha(\tau))) + \hat{f}(\tau, u(\hat{\alpha}(\tau))) \int_0^{\beta(\tau)} p(\tau, s, u(\mu(s)))ds, \ \tau \in [0, \infty). \tag{21}$$

• Banaś and Dhage [9] have the studied the equation

$$u(\tau) = f(\tau, u(\alpha(\tau)) + \int_0^{\beta(\tau)} p(\tau, s, u(\mu(s)))ds, \ \tau \in [0, \infty). \tag{22}$$

Further, some famous equations of first order, Volterra equation, Urysohn equation, Abel equation and Chandrasekhar type [15] have the form

$$u(\tau) = f(\tau, u(\alpha(\tau))) \tag{23}$$

$$u(\tau) = A(\tau) + \frac{1}{\Gamma(h)} \int_0^{\tau} \frac{p(\tau, s, u(s))}{(\tau - s)^{1-h}} ds \tag{24}$$

$$u(\tau) = A(\tau) + \int_0^{\tau} p(\tau, s, u(s)ds \tag{25}$$

$$u(\tau) = A(\tau) + \int_0^{b} p(\tau, s, u(s)ds \tag{26}$$

$$u(\tau) = 1 + u(\tau) \int_0^{b} \frac{\tau}{\tau + s} \hat{\beta(s)} u(s)ds. \tag{27}$$

It should notice that equation (1) is additionally standard than equations (2)-(27) and so on. Numerous authors have examined the various types of integral equations by the ideas of MNC with fixed point theorems in the Banach function spaces, we refer [4, 7, 14, 20, 21, 22, 23, 24, 25, 26, 29, 30, 33, 34, 35, 40, 41, 46, 47] and reference therein.
The paper is organized into 4 sections with the introduction. In section 2, we recognize preliminaries and establish the concept of MNC. Section 3 is applied to state and prove a theorem for equation (1) including densifying operators by Petryshyn's fixed point theorem. In the previous section, we offer some examples that test the facts of this class of FIE.

2 Preliminaries

First, we recognize the concept of the fractional integral of order h for the $u(\tau)$

Definition 2.1. [37] Let $u \in C[a, b]$ and $a < \tau < b$, then

$$I_{a^+}^h u(\tau) = \frac{1}{\Gamma(h)} \int_a^{\tau} \frac{u(s)}{(\tau - s)^{1-h}} ds, \ h > 0$$

is known as the Riemann-Liouville fractional integral of order h. Γ the gamma function is defined as

$$\Gamma(h) = \int_0^{\infty} s^{h-1} e^{-s} ds.$$

Let E be a real Banach space and B_{r_0} denote closed ball with center at 0 and radius r_0. $\partial B_{r_0} = \{u \in E : \|u\| = r_0\}$ for the sphere in E around 0 with radius $r_0 > 0$. The MNC is a helpful tool to apply fixed point theory in non-linear analysis in Banach space E.

Definition 2.2. [3] The Kuratowski MNC

$$\theta(F) = \inf\left\{\epsilon > 0 : F = \bigcup_{i=1}^{n} F_i \text{ with } \operatorname{diam} F_i \le \epsilon,\ i = 1, 2, ..., n\right\}.$$

Definition 2.3. [3] The Hausdroff MNC

$$\psi(F) = \inf\{\epsilon > 0 : \exists \text{ a finite } \epsilon\text{-net for } F \text{ in } E\},$$

where a finite ϵ-net for F in E it means, it is a set $\{u_1, u_2, ..., u_n\} \subset E$ such that $B_\epsilon(E, u_1), B_\epsilon(E, u_2)$, ..., $B_\epsilon(E, u_m)$ over F. These MNC are respectively similar in the sense that

$$\psi(F) \le \theta(F) \le 2\psi(F),$$

for any bounded set $F \subset E$.

Theorem 2.1. *Let $F, \hat{F} \subset E$ bounded and $\lambda \in \mathbb{R}$. Then*

(i) $\psi(F \cup \hat{F}) = \max\{\psi(F), \psi(\hat{F})\}$;

(ii) $F \subseteq \hat{F} \implies \psi(F) \le \psi(\hat{F})$;

(iii) $\psi(\bar{co}F) = \psi(F)$;

(iv) $\psi(F) = 0$ *if and only if F is relatively-compact;*

(v) $\psi(\lambda F) = |\lambda|\psi(F)$;

(vi) $\psi(F + \hat{F}) \le \psi(F) + \psi(\hat{F})$.

Further, Banach space $C[0, b]$ is the set of all real valued continuous function on $[0, b]$ with the sup norm

$$\|u\| = \max\{|u(\tau)| : \tau \in [0, b]\}.$$

Also, space $E = C[0, b]$ is the formation of Banach algebra.
Suppose a fix set $H \in C[0, b]$. For $\epsilon > 0$ and $u \in H$, the modulus of continuity of u defined by

$$\omega(u, \epsilon) = \sup\{|u(\tau_2) - u(\tau_1)| : \tau_2, \tau_1 \in [0, b], |\tau_2 - \tau_1| \le \epsilon\}.$$

Further,

$$\omega(F, \epsilon) = \sup\{\omega(u, \epsilon) : u \in F\},\ \omega_0(F) = \lim_{\epsilon \to 0}(F, \epsilon).$$

Theorem 2.2. *[33] The Hausdorff MNC is equivalent to*

$$\psi(F) = \lim_{\epsilon \to 0} \sup_{u \to F} \omega(u, \epsilon) \tag{28}$$

for all bounded sets $F \subset C[0, b]$.

Definition 2.4. [42] Suppose $T : E \to E$ be a continuous mapping of E. T is said a k-set contraction if for all $G \subset E$ with G bounded, $T(G)$ is bounded and $\theta(TG) \le k\theta(G), k \in (0,1)$. If $\theta(TG) < \theta(G),\ \forall\, \theta(G) > 0$, then T is called the densifying or condensing mapping.

Theorem 2.3. *[45] Assume $T : B_{r_0} \to E$ is a condensing mapping, which satisfied the boundary condition if $T(u) = ku$, for some $u \in \partial B_{r_0}$ then $k \le 1$. Then $\mathbf{F}(T)$ (the set of fixed points of T) in B_{r_0} is non-empty.*

3 Main results

We analyze the equation (1) under the following conditions:

(1) $f, g \in C(I_b \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $p \in C(I_b \times [0, C_1] \times \mathbb{R}, \mathbb{R})$, $q \in C(I_b \times [0, C_2] \times \mathbb{R}, \mathbb{R})$, and $\beta, \varphi : I_b \to \mathbb{R}^+$, $\mu : [0, C_1] \to I_b$, $\nu : [0, C_2] \to I_b$, $\alpha, \phi : I_b \to I_b$ are continuous, $\beta(\tau) \le C_1, \varphi(\tau) \le C_2$ for every $\tau \in I_b$.

(2) $\exists$ constants $h_i,\ h_1 + h_2 < 1$ and $h_4 + h_5 < 1$ for $i = 1, 2, ...6$ such that

$$|f(\tau, u_1, u_2, u_3) - f(\tau, v_1, v_2, v_3| \le h_1|u_1 - v_1| + h_2|u_2 - v_2| + h_3|u_3 - v_3|,$$
$$|g(\tau, u_1, u_2, u_3) - g(\tau, v_1, v_2, v_3| \le h_4|u_1 - v_1| + h_5|u_2 - v_2| + h_6|u_3 - v_3|.$$

(3) $\exists$ a $r_0 > 0$ such that the following bounded condition satisfied

$$\sup\Big(\{|(H_1) \times (H_2)|\}\Big) \le r_0,$$

here

$$\sup H_1 = \sup\left\{|f(\tau, u_1, u_2, u_3)| : \text{ for all } \zeta \in I_b, u_1, u_2 \in [-r_0, r_0] \text{ and } u_3 \in [-\frac{M_1 C_1^h}{\Gamma(h+1)}, \frac{M_1 C_1^h}{\Gamma(h+1)}]\right\},$$
$$\sup H_2 = \sup\{|g(\tau, u_1, u_2, u_3)| : \textit{for all } \tau \in I_b, u_1, u_2 \in [-r_0, r_0] \text{ and } u_3 \in [-M_2C_2, M_2C_2]\},$$
$$M_1 = \sup\{|p(\tau, s, u)| : \text{for all } \tau \in I_b, s \in [0, C_1] \text{ and } u \in [-r_0, r_0]\},$$
$$M_2 = \sup\{|q(\tau, s, u)| : \text{for all } \tau \in I_b, s \in [0, C_2] \text{ and } u \in [-r_0, r_0]\}.$$

Theorem 3.1. *Using the assumptions $(1) - (3)$, the equation (1) has at least one solution in $E = C(I_b)$.*

Proof. Define $f, g : B_{r_0} \to E$ in the following form

$$(fu)(\tau) = \left(f\Big(\tau, u(\tau), u(\alpha(\tau)), \frac{1}{\Gamma(h)}\int_0^{\beta(\tau)} \frac{p(\tau, s, u(\mu(s))}{(\beta(\tau) - s)^{1-h}} ds\Big)\right),$$
$$(gu)(\tau) = \left(g\Big(\tau, u(\tau), u(\phi(\tau)), \int_0^{\varphi(\tau)} q(\tau, s, u(\nu(s)) ds\Big)\right), \text{ for } \tau \in [0, b].$$

Further, setting operator T such that

$$Tu = (fu)(gu).$$

Now, we show that f is continuous on B_{r_0}. For this, $\epsilon > 0$ and any $u, z \in B_{r_0}$ such that $\|u - z\| < \epsilon$. Then,

$$
\begin{aligned}
&|(fu)(\tau) - (fz)(\tau)| \\
&= \left| f\Big(\tau, u(\tau), u(\alpha(\tau)), \frac{1}{\Gamma(h)} \int_0^{\beta(\tau)} \frac{p(\tau, s, u(\mu(s))}{(\beta(\tau) - s)^{1-h}} ds\Big) - f\Big(\tau, z(\tau), z(\alpha(\tau)), \frac{1}{\Gamma(h)} \int_0^{\beta(\tau)} \frac{p(\tau, s, z(\mu(s))}{(\beta(\tau) - s)^{1-h}} ds\Big) \right| \\
&\le h_1|u(\tau) - z(\tau)| + h_2|u(\alpha(\tau)) - z(\alpha(\tau))| + h_3 \frac{1}{\Gamma(h)} \int_0^{\beta(\tau)} \frac{|p(\tau, s, u(\mu(s)) - p(\tau, s, z(\mu(s))|}{(\beta(\tau) - s)^{1-h}} ds \\
&\le (h_1 + h_2)\|u(\tau) - z(\tau)| + \frac{h_3}{\Gamma(h+1)} C_1^h \omega(p, \epsilon),
\end{aligned}
$$

where

$$\omega(p, \epsilon) = \sup\{|p(\tau, s, u) - p(\tau, s, z)| : \tau \in I_b, s \in [0, C_1], u, z \in [-r_0, r_0], |u - z| \le \epsilon\}.$$

By uniformly continuous of $p = p(\tau, s, u)$ on $I_b \times [0, C_1] \times [-r_0, r_0]$, we refer that $\omega(p, \epsilon) \to 0$ as $\epsilon \to 0$. From the above inequality f is continuous on B_{r_0}.
Further,

$$
\begin{aligned}
&|(gu)(\tau) - (gz)(\tau)| \\
&= \left| g\Big(\tau, u(\tau), u(\phi(\tau)), \int_0^{\varphi(\tau)} q(\tau, s, u(\nu(s)) ds\Big) - g\Big(\tau, z(\tau), z(\phi(\tau)), \int_0^{\varphi(\tau)} q(\tau, s, z(\nu(s)) ds\Big) \right| \\
&\le h_4|u(\tau) - z(\tau)| + h_5|u(\phi(\tau)) - z(\phi(\tau))| + h_6 \int_0^{\varphi(\tau)} |q(\tau, s, u(\nu(s)) - q(\tau, s, z(\nu(s))| ds \\
&\le (h_4 + h_5)\|u(\tau) - z(\tau)| + h_6 C_2 \omega(q, \epsilon),
\end{aligned}
$$

where

$$\omega(q, \epsilon) = \sup\{|q(\tau, s, u) - q(\tau, s, z)| : \tau, s \in I_b, u, z \in [-r_0, r_0], |u - z| \le \epsilon\}.$$

Using uniformly continuous of $q = q(\tau, s, u)$ on $I_b \times I_b \times [-r_0, r_0]$, we refer that $\omega(q, \epsilon) \to 0$ as $\epsilon \to 0$. By above inequality g is continuous on B_{r_0}. Hence, T is a continuous operator on B_{r_0}.
Next, we prove that the f satisfy the condensing condition on B_{r_0} with respect to ψ. Take F of B_{r_0}. Select $\epsilon > 0$ and $\tau_1, \tau_2 \in I_b$ such that $\tau_1 - \tau_2 \le \epsilon$. We get

$$
\begin{aligned}
&|(fu)(\tau_2) - (fu)(\tau_1)| \\
&= \left| f\Big(\tau_2, u(\tau_2), u(\alpha(\tau_2)), \frac{1}{\Gamma(h)} \int_0^{\beta(\tau_2)} \frac{p(\tau_2, s, u(\mu(s))}{(\beta(\tau_2) - s)^{1-h}} ds\Big) \right. \\
&\left. - f\Big(\tau_1, u(\tau_1), u(\alpha(\tau_1)), \frac{1}{\Gamma(h)} \int_0^{\beta(\tau_1)} \frac{p(\tau_1, s, u(\mu(s))}{(\beta(\tau_1) - s)^{1-h}} ds\Big) \right| \\
&\le \left| f\left(\tau_2, u(\tau_2), u(\alpha(\tau_2)), \frac{1}{\Gamma(h)} \int_0^{\beta(\tau_2)} \frac{p(\tau_2, s, u(\mu(s))}{(\beta(\tau_2) - s)^{1-h}} ds\right) \right.
\end{aligned}
$$

$$
\begin{aligned}
&- f\left(\tau_2, u(\tau_2), u(\alpha(\tau_2)), \frac{1}{\Gamma(h)}\int_0^{\beta(\tau_1)} \frac{p(\tau_1, s, u(\mu(s))}{(\beta(\tau_1)-s)^{1-h}}ds\right)\Bigg| \\
&+ \Bigg| f\left(\tau_2, u(\tau_2), u(\alpha(\tau_2)), \frac{1}{\Gamma(h)}\int_0^{\beta(\tau_1)} \frac{p(\tau_1, s, u(\mu(s))}{(\beta(\tau_1)-s)^{1-h}}ds\right)| \\
&- f\left(\tau_2, u(\tau_2), u(\alpha(\tau_1)), \frac{1}{\Gamma(h)}\int_0^{\beta(\tau_1)} \frac{p(\tau_1, s, u(\mu(s))}{(\beta(\tau_1)-s)^{1-h}}ds\right)\Bigg| \\
&+ \Bigg| f\left(\tau_2, u(\tau_2), u(\alpha(\tau_1)), \frac{1}{\Gamma(h)}\int_0^{\beta(\tau_1)} \frac{p(\tau_1, s, u(\mu(s))}{(\beta(\tau_1)-s)^{1-h}}ds\right) \\
&- f\left(\tau_2, u(\tau_1), u(\alpha(\tau_1)), \frac{1}{\Gamma(h)}\int_0^{\beta(\tau_1)} \frac{p(\tau_1, s, u(\mu(s))}{(\beta(\tau_1)-s)^{1-h}}ds\right)\Bigg| \\
&+ \Bigg| f\left(\tau_2, u(\tau_1), u(\alpha(\tau_1)), \frac{1}{\Gamma(h)}\int_0^{\beta(\tau_1)} \frac{p(\tau_1, s, u(\mu(s))}{(\beta(\tau_1)-s)^{1-h}}ds\right) \\
&- f\left(\tau_1, u(\tau_1), u(\alpha(\tau_1)), \frac{1}{\Gamma(h)}\int_0^{\beta(\tau_1)} \frac{p(\tau_1, s, u(\mu(s))}{(\beta(\tau_1)-s)^{1-h}}ds\right)\Bigg| \\
&\leq \frac{h_3}{\Gamma(h)}\left|\int_0^{\beta(\tau_2)} \frac{p(\tau_2, s, u(\mu(s))}{(\beta(\tau_2)-s)^{1-h}}ds - \int_0^{\beta(\tau_1)} \frac{p(\tau_1, s, u(\mu(s))}{(\beta(\tau_1)-s)^{1-h}}ds\right| \\
&+ h_1|u(\tau_2)-u(\tau_1)| + h_2|u(\alpha(\tau_2)) - u(\alpha(\tau_1))| + \omega_f(I_b, \epsilon) \\
&\leq h_1\omega(u,\epsilon) + h_2\omega(u,\omega(\alpha,\epsilon)) + \omega_f(I_b,\epsilon)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{h_3}{\Gamma(h)}\Bigg[\left|\int_0^{\beta(\tau_2)} \frac{p(\tau_2, s, u(\mu(s))}{(\beta(\tau_2)-s)^{1-h}}ds - \int_0^{\beta(\tau_2)} \frac{p(\tau_1, s, u(\mu(s))}{(\beta(\tau_2)-s)^{1-h}}ds\right| \\
&+ \left|\int_0^{\beta(\tau_2)} \frac{p(\tau_1, s, u(\mu(s))}{(\beta(\tau_2)-s)^{1-h}}ds - \int_0^{\beta(\tau_2)} \frac{p(\tau_1, s, u(\mu(s))}{(\beta(\tau_1)-s)^{1-h}}ds\right| \\
&+ \left|\int_0^{\beta(\tau_2)} \frac{p(\tau_1, s, u(\mu(s))}{(\beta(\tau_1)-s)^{1-h}}ds - \int_0^{\beta(\tau_1)} \frac{p(\tau_1, s, u(\mu(s))}{(\beta(\tau_1)-s)^{1-h}}ds\right|\Bigg] \\
&\leq h_1\omega(u,\epsilon) + h_2\omega(u,\omega(\alpha,\epsilon)) + \omega_f(I_b,\epsilon) \\
&+ \frac{h_3}{\Gamma(h)}\Bigg[\int_0^{\beta(\tau_2)} \frac{|p(\tau_2, s, u(\mu(s)) - p(\tau_1, s, u(\mu(s))}{(\beta(\tau_2)-s)^{1-h}}ds \\
&+ \int_0^{\beta(\tau_2)} |p(\tau_1, s, u(\mu(s))|[(\beta(\tau_2)-s)^{h-1} - (\beta(\tau_1)-s)^{h-1}]ds + \int_{\beta(\tau_1)}^{\beta(\tau_2)} \frac{|p(\tau_1, s, u(\mu(s))}{(\beta(\tau_1)-s)^{1-h}}ds\Bigg] \\
&\leq h_1\omega(u,\epsilon) + h_2\omega(u,\omega(\alpha,\epsilon)) + \omega_f(I_b,\epsilon) \\
&+ \frac{h_3}{\Gamma(h+1)}\omega_p(I_b,\epsilon)(\beta(\tau_2))^h + \frac{h_3}{\Gamma(h+1)}M_1[(\beta(\tau_2))^h - (\beta(\tau_1))^h] \\
&\leq h_1\omega(u,\epsilon) + h_2\omega(u,\omega(\alpha,\epsilon)) + \omega_f(I_b,\epsilon) \\
&+ \frac{h_3}{\Gamma(h+1)}\omega_p(I_b,\epsilon)(\beta(\tau_2))^h + \frac{h_3}{\Gamma(h+1)}M_1[(\beta(\tau_2)) - (\beta(\tau_2))]^h,
\end{aligned}
$$

where

$$\begin{aligned}\omega_f(I_b,\epsilon) &= \sup\{|f(\tau_2,u_1,u_2,u_3)-f(\tau_1,u_1,u_2,u_3)|:\tau_2,\tau_1\in I_b,\\ &\qquad u_3\in[-\frac{m_1C_1^h}{\Gamma(h+1)},\frac{M_1C_1^h}{\Gamma(h+1)}],u_1,u_2\in[-r_0,r_0],|\tau_2-\tau_1|\le\epsilon\},\\ \omega_p(I_b,\epsilon) &= \sup\{|p(\tau_2,s,u)-p(\tau_1,s,u)|:|\tau_2-\tau_1|\le\epsilon,\tau_2,\tau_1\in I_b,u\in[-r_0,r_0],s\in[0,C_1]\},\\ \omega(\alpha,\epsilon) &= \sup\{|\alpha(\tau_2)-\alpha(\tau_1)|:\tau_2,\tau_1\in I_b,|\tau_2-\tau_1|\le\epsilon\},\\ \omega(\beta,\epsilon) &= \sup\{|\beta(\tau_2)-\beta(\tau_1)|:\ \tau_2,\tau_1\in I_b,|\tau_2-\tau_1|\le\epsilon\}.\end{aligned}$$

From above estimate

$$\begin{aligned}\omega(fF,\epsilon) &\le h_1\omega(u,\epsilon)+h_2\omega(F,\omega(\alpha,\epsilon))+\omega_f(I_b,\epsilon)\\ &\quad+\frac{h_3}{\Gamma(h+1)}\omega_p(I_b,\epsilon)(C_1)^h+\frac{h_3}{\Gamma(h+1)}M_1[\omega(\beta,\epsilon)]^h.\end{aligned}$$

Apply limit as $\epsilon\to 0$, we obtain

$$\omega_0(fF\le(h_1+h_2)\omega_0(F).$$

We get

$$\psi(fF)\le(h_1+h_2)\psi(F).$$

Hence f is a condensing map.
Similarly,

$$\begin{aligned}&|(gu)(\tau_2)-(gu)(\tau_1)|\\ &=\left|g\Big(\tau_2,u(\tau_2),u(\phi(\tau_2)),\int_0^{\varphi(\tau_2)}q(\tau_2,s,u(\nu(s))dsds\Big)-g\Big(\tau_1,u(\tau_1),u(\phi(\tau_1)),\int_0^{\varphi(\tau_1)}q(\tau_1,s,u(\nu(s))dsds\Big)\right|\\ &\le\left|g\Big(\tau_2,u(\tau_2),u(\phi(\tau_2)),\int_0^{\varphi(\tau_2)}q(\tau_2,s,u(\nu(s))dsds\Big)-g\Big(\tau_2,u(\tau_2),u(\phi(\tau_2)),\int_0^{\varphi(\tau_1)}q(\tau_1,s,u(\nu(s))dsds\Big)\right|\\ &+\left|g\Big(\tau_2,u(\tau_2),u(\phi(\tau_2)),\int_0^{\varphi(\tau_1)}q(\tau_1,s,u(\nu(s))dsds\Big)-g\Big(\tau_2,u(\tau_2),u(\phi(\tau_1)),\int_0^{\varphi(\tau_1)}q(\tau_1,s,u(\nu(s))dsds\Big)\right|\\ &+\left|g\Big(\tau_2,u(\tau_2),u(\phi(\tau_1)),\int_0^{\varphi(\tau_1)}q(\tau_1,s,u(\nu(s))dsds\Big)-g\Big(\tau_2,u(\tau_1),u(\phi(\tau_1)),\int_0^{\varphi(\tau_1)}q(\tau_1,s,u(\nu(s))dsds\Big)\right|\\ &+\left|g\Big(\tau_2,u(\tau_1),u(\phi(\tau_1)),\int_0^{\varphi(\tau_1)}q(\tau_1,s,u(\nu(s))dsds\Big)-g\Big(\tau_1,u(\tau_1),u(\phi(\tau_1)),\int_0^{\varphi(\tau_1)}q(\tau_1,s,u(\nu(s))dsds\Big)\right|\\ &\le h_6\left|\int_0^{\varphi(\tau_2)}q(\tau_2,s,u(\nu(s))ds-\int_0^{\varphi(\tau_1)}q(\tau_1,s,u(\nu(s))ds\right|\\ &+h_4|u(\tau_2)-u(\tau_1)|+h_5|u(\phi(\tau_2))-u(\phi(\tau_1))|+\omega_g(I_b,\epsilon)\\ &\le h_4\omega(u,\epsilon)+h_5(u,\omega(\phi,\epsilon))+\omega_g(I_b,\epsilon)\\ &+h_6\int_0^{\varphi(\tau_1)}|q(\tau_2,s,u(\nu(s))-q(\tau_1,s,u(\nu(s))|ds+h_6\int_{\varphi(\tau_1)}^{\varphi(\tau_2)}|q(\tau_2,s,u(\nu(s))|ds\\ &\le h_4\omega(u,\epsilon)+h_5(u,\omega(\phi,\epsilon))\\ &+h_6C_2\omega_q(I_b,\epsilon)+h_6M_2\omega(\varphi,\epsilon),\end{aligned}$$

where

$$\omega_g(I_b,\epsilon)=\sup\{|g(\tau_2,u_1,u_2,u_3)-g(\tau_1,u_1,u_2,u_3)|:\tau_2,\tau_1\in I_b,$$

$$
\begin{aligned}
&u_3 \in [-C_2M_2, C_2M_2], u_1, u_2 \in [-r_0, r_0], |\tau_2 - \tau_1| \le \epsilon\},\\
\omega_q(I_b, \epsilon) &= \sup\{|q(\tau_2, s, u) - q(\tau_1, s, u)| : |\tau_2 - \tau_1| \le \epsilon, \tau_2, \tau_1 \in I_b, u \in [-r_0, r_0], s \in [0, C_2]\},\\
\omega(\phi, \epsilon) &= \sup\{|\phi(\tau_2) - \phi(\tau_1)| : \tau_2, \tau_1 \in I_b, |\tau_2 - \tau_1| \le \epsilon\},\\
\omega(\varphi, \epsilon) &= \sup\{|\varphi(\tau_2) - \varphi(\tau_1)| : \tau_2, \tau_1 \in I_b, |\tau_2 - \tau_1| \le \epsilon\}.
\end{aligned}
$$

From above estimate

$$\omega(gF, \epsilon) \le h_4\omega(u, \epsilon) + h_5(F, \omega(\phi, \epsilon)) + h_6C_2\omega_q(I_b, \epsilon) + h_6M_2\omega(\varphi, \epsilon).$$

Apply limit as $\epsilon \to 0$, we get

$$\omega_0(gF) \le (h_4 + h_5)\omega_0(F).$$

This gives the following estimate

$$\psi(gF) \le (h_4 + h_5)\psi(F).$$

Hence g is a condensing map. So, T is also a condensing map. Now, let $u \in \partial B_{r_0}$ and if $Tu = ku$ then, $\|Tu\| = k\|u\| = kr_0$ and by (3),

$$
\begin{aligned}
\|Tu(\tau)\| &= \left(f\Big(\tau, u(\tau), u(\alpha(\tau)), \frac{1}{\Gamma(h)}\int_0^{\beta(\tau)} \frac{p(\tau, s, u(\mu(s))}{(\beta(\tau) - s)^{1-h}}ds\Big)\right)\\
&\quad\times \left(g\Big(\tau, u(\tau), u(\phi(\tau)), \int_0^{\varphi(\tau)} q(\tau, s, u(\nu(s))ds\Big)\right)\\
&\le r_0, \ \forall\ \tau \in I_b.
\end{aligned}
$$

Hence $\|Tu\| \le r_0$ i.e., $k \le 1$. □

4 Examples

In this section, we provide some examples of equations to illustrate the usefulness of our results.

Example 4.1. *Let the following fractional integral equation*

$$
\begin{aligned}
u(\tau) &= \left(\frac{1}{4}e^{-\tau^2} + \frac{\tau^4}{3+3\tau^4}\ln(1+|u(\tau^3)|) + \frac{1}{(5+\sin(|u(\sqrt{\tau})|))\Gamma(\frac{1}{3})}\int_0^\tau \frac{e^{-3\tau^2}(e^\tau + \tau\cos(1+s) + \frac{1}{3}(u(\sqrt{s})))}{(\tau - s)^{\frac{2}{3}}}ds\right)\\
&\quad\times\left(\frac{1}{3}\cos(u(1-\tau)) + \frac{1}{4(e^\tau + |\cos(u(\tau))|)}\int_0^{\tau^3}\left[e^{-2\tau^4}\left(e^\tau + \tau\cos(s) + \cos\left(\frac{u(s)}{1+u(s)}\right)\right)\right]ds\right),\\
&\quad\tau \in [0,1]. \qquad (29)
\end{aligned}
$$

Here $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \alpha, \beta, \mu, \varphi, \phi, v : [0,1] \to [0,1], p, q : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ and comparing (29) with equation (1), we get

$$
\begin{aligned}
&\alpha(\tau) = \varphi(\tau) = \tau^3, \mu = \sqrt{\tau}, \phi = \beta = \tau, h = \frac{1}{3}, C_1 = C_2 = 1 \ \text{ for all } \tau \in [0,1],\\
&f(\tau, u_1, u_2, z) = \frac{1}{4}e^{-\tau^2} + \frac{\tau^4}{3+3\tau^4}\ln(1+|u_2|) + \frac{z}{5+\sin(|u_2|)}, \quad z = \frac{1}{\Gamma(\frac{1}{3})}\int_0^\tau \frac{p(\tau, s, u(\mu(s)))}{(\tau - s)^{\frac{2}{3}}}ds,\\
&g(\tau, u_1, u_2, w) = \frac{1}{3}\cos(u_2) + \frac{w}{4(e^t + |\cos(u_2)|)}, \quad w = \int_0^{\tau^3} q(\tau, s, u(v(s)))ds,\\
&p(\tau, s, u(\mu(s))) = e^{-3\tau^2}(e^\tau + \tau\cos(1+s) + \frac{1}{3}(u(\sqrt{s}))), \quad |p(\tau, s, u)| \le e + 1 + \frac{1}{3}|u|\\
&q(\tau, s, u(v(s))) = e^{-2\tau^4}(e^\tau + \tau\cos(s) + \cos(\frac{u(s)}{1+u(s)})), \ |q(\tau, s, u)| \le e + 2
\end{aligned}
$$

for all $\tau \in [0,1]$. Above functions fullfil the assumptions (1) and (2). Now, we review that (3) too holds. Assume $\|u\| \le r_0, r_0 > 0$, then,

$$\begin{aligned}|u(\tau)| = &\Bigg|\left(\frac{1}{4}e^{-\tau^2} + \frac{\tau^4}{3+3\tau^4}\ln(1+|u(\tau^3)|)\right.\\ &+ \left.\frac{1}{(5+\sin(|u(\sqrt{\tau})|))\Gamma(\frac{1}{3})}\int_0^{\tau}\frac{e^{-3\tau^2}(e^{\tau}+\tau\cos(1+s)+\frac{1}{3}(u(\sqrt{s})))}{(\tau-s)^{\frac{2}{3}}}ds\right)\\ &\times\left(\frac{1}{3}\cos(u(1-\tau)) + \frac{1}{4(e^{\tau}+|\cos(u(\tau))|)}\int_0^{\tau^3}[e^{-2\tau^4}(e^{\tau}+\tau\cos(s)+\cos(\frac{u(s)}{1+u(s)})]ds\right)\Bigg|\\ &\le r_0,\ \text{for all } \tau \in [0,1].\end{aligned}$$

Hence (3) holds if,

$$\left(\frac{1}{4}+\frac{1}{3}r_0+\frac{1}{5\Gamma(\frac{4}{3})}(\frac{1}{3}r_0+e+1)\right)\left(\frac{1}{3}+\frac{1}{4}(e+2)\right) \le r_0.$$

Thus, (3) holds if $r_0 \ge 4.2796$. Hence, from Theorem 3.1, (29) has at least one solution in $C[0,1]$.

5 Conflict of interest

The authors declare no conflict of interest.

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