Advanced Studies in Contemporary Mathematics
34 (2024), No. 3, pp. 209 - 221

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2024.34.3.209

ON NEARLY ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN COMPLETE HYPERBOLIC SPACES

KIRAN DEWANGAN[1], LAXMI RATHOUR[2*], LAKSHMI NARAYAN MISHRA[3], AND VISHNU NARAYAN MISHRA[4]

Abstract. The aim of this paper is to establish some fixed point results in complete hyperbolic spaces and convergence of M iteration scheme for nearly asymptotically nonexpansive mappings. Our results generalize the results given by Sharma from $CAT(k)$ spaces to complete hyperbolic spaces.

2010 Mathematics Subject Classification. 47H09, 47H10.

Keywords and phrases. Hyperbolic space, nearly asymptotically nonexpansive mapping, fixed point.

date of submission 12.02.2024. date of acceptance 03.06.2024.

1. Introduction

There are many applications of the fixed point theory in solution of different mathematical problems like integral equations, differential equations, VIP, convex minimization problem, image recovery, etc. Also several fixed point results in different spaces for nonlinear mappings are available in the literature (refer to [27, 28, 29, 30, 31, 32, 33, 34, 35, 36]) etc.

The concept of asymptotically nonexpansive mappings and the asymptotically nonexpansive type mappings were introduced by Goebel and Kirk [2] as a generalization of the class of nonexpansive mappings. Recall that a mapping $T : K \to K$, where K is non-empty subset of a uniformly convex Banach space X, is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that

$$||T^n x - T^n y|| \leq k_n ||x - y||,$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

Goebel and Kirk [2] proved that "Every asymptotically nonexpansive self-mapping of a non-empty closed bounded and convex subset of a uniformly convex Banach space has a fixed point". After this, several authors have being concerned with the iterative construction of a fixed point of asymptotically nonexpansive mappings.

In 2003, Chidume et al. [37] introduced the concept of asymptotically nonexpansive nonself-mappings as a generalization of asymptotically nonexpansive self-mappings. Non-self asymptotically nonexpansive mappings have been studied by many authors (see [38, 39, 40]). In 2016, Alber et al. [41] introduced the concept of total asymptotically nonexpansive mappings that generalizes the family of mapping that is the extension of asymptotically nonexpansive mappings. There are many papers dealing with the approximation of fixed points of asymptotically nonexpansive mappings in uniformly convex Banach spaces. (refer to [18, 24, 25, 26]).

Several authors have studied approximate fixed points of asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces by using different iteration schemes (refer to [14, 16, 17, 18]) etc. In 1953, Mann [20] introduced new iteration scheme, which is powerful method for solving nonlinear operator equations. This scheme is:
Let K be a non-empty closed convex and bounded subset of a Banach space X and $T : K \to K$ be any nonlinear mapping. For fix $x_1 \in K$ the sequence $\{x_{k+1}\}$ is defined by

$$x_{k+1} = \alpha_k T(x_k) + (1 - \alpha_k)x_k, \ k \in \mathbb{N},$$

where $\{\alpha_k\}$ is a sequence in $(0, 1)$.

Using demiclosedness property that was introduced by J. Gornicki [12], Schu [18] introduced modified Mann iteration process and proved that the modified Mann iteration process converges weakly for asymptotically nonexpansive mapping in Hilbert space under certain conditions.

In 1974, Ishikawa [22] introduced the following iterative scheme:
For fix $x_1 \in K$ the sequence $\{x_{k+1}\}$ is defined by

$$\begin{cases} y_k = (1 - \beta_k)x_k + \beta_k T x_k, \\ x_{k+1} = (1 - \alpha_k)x_k + \alpha_k T y_k, \ k \in \mathbb{N}, \end{cases}$$

where $\{\alpha_k\}$ and $\{\beta_k\}$ are sequences in $(0, 1)$. If $\beta_k = 0$, then it reduces to Mann iteration.

In 2000, Noor [21] introduced three-step iteration scheme as follows:

$$\begin{cases} x_1 \in X, \\ y_k = \gamma_k x_k + (1 - \gamma_k)T x_k, \\ z_k = \beta_k x_k + (1 - \beta_k)T y_k, \\ x_{k+1} = \alpha_k x_k + (1 - \alpha_k)T z_k, \end{cases}$$

for all $k \geq 1$ and $\{\alpha_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$ are sequences in $(0, 1)$. This iterative scheme is a generalized form of the Mann iteration scheme and Ishikawa iteration scheme. Mann iteration, Ishikawa iteration, Noor iteration are further generalized and other several iterative schemes have been developed by my authors to study fixed point of nonexpansive type mappings.

The following iteration scheme is called M iteration scheme, introduced by Ullah et al. [23] as follows:

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n T^n x_n, \\ y_n = T^n z_n, \\ x_{n+1} = T^n y_n. \end{cases}$$

Ullah et al. [23] obtained weak and strong convergence results of M iteration schene for nonexpansive type mappings in the framework of Banach space.

The class of hyperbolic spaces contains normed spaces, CAT(0) spaces, and many more. There are so many examples in the literature that shows hyperbolic spaces are more general than Banach spaces (for detail, refer to [1, 13]).

Following is the version of M iteration scheme in hyperbolic spaces for asymptotically nonexpansive mappings:
Let (X, d, W) be a hyperbolic space and $K \subset X$ is non-empty closed convex. Let $T : K \to K$ be an asymptotically nonexpansive mapping and $\{x_n\}$ is a sequence in K defined by

$$\begin{cases} z_n = W(x_n, T^n x_n, \alpha_n) \\ y_n = T^n z_n, \\ x_{n+1} = T^n y_n, \end{cases} \tag{1}$$

where $\{\alpha_k\}$ is a sequences in $(0, 1)$.

Sharma [3] studied approximate common fixed points of nearly asymptotically nonexpansive mappings by using modified $SP-$iteration process in the setting of $CAT(k)$ spaces and established strong and $\Delta-$convergence theorems. Here, we extend the results of Sharma [3] to approximate fixed points of nearly asymptotically nonexpansive mappings in complete hyperbolic spaces by using iteration scheme (1).

2. Preliminaries

Definition 2.1. [3] *Let K be a non-empty subset of a metric space (X, d). Then a mapping $T : K \to K$ is said to be*

- *Nonexpansive if $d(Tx, Ty) \le d(x, y)$, for all $x, y \in K$;*
- *Asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that $d(T^n x, T^n y) \le (1 + k_n)d(x, y)$, for all $x, y \in K$, $n \ge 1$;*
- *Asymptotically quasi-nonexpansive if $F(T) \ne \emptyset$ and there exists a sequence $\{k_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that $d(T^n x, p) \le (1 + k_n)d(x, p)$, for all $x \in K$, $p \in F(T)$, $n \ge 1$;*
- *Uniformly $L-$Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \le Ld(x, y)$, for all $x, y \in K$, $n \ge 1$;*
- *A sequence $\{x_n\}$ in K is called an approximating fixed point sequence for T if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

Sahu [4] introduced an important generalization of the class of Lipschitzian mappings namely class of nearly Lipschitzian mappings.

Definition 2.2. [3] *Let K be a non-empty subset of a metric space (X, d). Fix a sequence $\{s_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} s_n = 0$. A mapping $T : K \to K$ is said to be nearly Lipschitzian with respect to $\{s_n\}$ if for all $n \geq 1$, there exists a constant $k_n \geq 0$ such that*

$$d(T^n x, T^n y) \leq k_n[d(x, y) + s_n],$$

for all $x, y \in K$. The infimum of constant k_n for which above inequality holds, is denoted by $\zeta(T^n)$ and called nearly Lipschitz constant. Note that

$$\zeta(T^n) = \sup\{\frac{d(T^n x, T^n y)}{d(x, y) + s_n : x, y \in K, x \neq y}\}$$

A nearly Lipschitzian mapping T with sequence $\{s_n, \zeta(T^n)\}$ is said to be

- *Nearly nonexpansive if $\zeta(T^n) = 1$ for all $n \geq 1$;*
- *Nearly asymptotically nonexpansive if $\zeta(T^n) \geq 1$ for all $n \geq 1$ and $\lim_{n\to\infty} \zeta(T^n) = 1$;*
- *Nearly uniform $k-$Lipschitzian if $\zeta(T^n) \leq k$ for all $n \geq 1$.*

Remark. [3] *Every asymptotically nonexpansive mapping is nearly asymptotically nonexpansive.*

Definition 2.3. [3] *Let (X, d) be a metric space and K a subset of X. A mapping $T : K \to K$ with non-empty fixed point set $F(T)$ in K will be said to satisfy Condition (I), if there is a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in K$, where $d(x, F(T)) = \inf\{||x - z|| : z \in F(T)\}$.*

Definition 2.4. [19] *Let X be a Banach space. X satisfies Opial's condition if for each $x \in X$ and each sequence $\{x_n\}$ weakly convergent to x,*

$$\liminf ||x_n - y|| > \liminf ||x_n - x||,$$

holds for $y \neq x$.

Definition 2.5. [5] *A hyperbolic space (X, d, W) is a metric space (X, d) together with a convexity mapping $W : X \times X \times [0, 1] \to X$ such that for all $x, y, z \in X$ and $\alpha, \beta \in [0, 1]$, we have*

(i) $d(u, W(x, y, \alpha)) \leq (1 - \alpha)d(u, x) + \alpha d(u, y)$,
(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$,
(iii) $W(x, y, \alpha) = W(y, x, 1 - \alpha)$,
(iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha)d(x, y) + \alpha d(z, w)$.

Example 2.1. [15] *Let $X = \mathbb{R}$ be a Banach space. Let $d : X \times X \to [0, \infty)$ be a mapping defined by*

$$d(x, y) = ||x - y||.$$

It is clear that d is metric on X. Let $K = [0, 1]$ be a subset of X. Further we define a mapping $W : X \times X \times [0, 1]$ by

$$W(x, y, \alpha) = \alpha x + (1 - \alpha)y,$$

for all $x, y \in X$ and $\alpha \in [0, 1]$. Then (X, d, W) is hyperbolic space.

Definition 2.6. [7] *A non-empty subset K of a hyperbolic space X is said to be convex if $W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$.*

Leustean [6] introduced the concept of uniformly convex hyperbolic spaces. Later [9] defined uniformly convex hyperbolic spaces in the following way:

Definition 2.7. [9] *A hyperbolic space X is said to be uniformly convex if for any $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that for all $x, y, z \in X$,*

$$d(W(x, y, \frac{1}{2}), z) \leq (1 - \delta)r,$$

provided $d(x, z) \leq r$, $d(y, z) \leq r$ and $d(x, y) \geq \varepsilon r$.

Definition 2.8. [8] *Let K be a non-empty subset of a metric space X and $\{x_k\}$ be any bounded sequence in K. For $x \in X$, there is a continuous functional $r(., \{x_k\}) : X \to [0, \infty)$ defined by*

$$r(x, \{x_k\}) = \limsup_{k\to\infty} d(x_k, x).$$

The asymptotic radius $r(K, \{x_k\})$ of $\{x_k\}$ with respect to K is given by

$$r(K, \{x_k\}) = \inf\{r(x, \{x_k\}) : x \in K\}.$$

A point $x \in K$ is said to be an asymptotic center of the sequence $\{x_k\}$ with respect to K, if

$$r(x, \{x_k\}) = \inf\{r(y, \{x_k\}) : y \in K\}.$$

The set of all asymptotic centres of $\{x_k\}$ with respect to K is denoted by $A(K, \{x_k\})$.

Remark. *In uniformly convex Banach spaces and $CAT(0)$ spaces, bounded sequences have unique asymptotic center with respect to closed convex subset.*

Definition 2.9. [8] *A sequence $\{x_k\}$ in X is said to be $\Delta-$ converge to $x \in X$ if x is the unique asymptotic center of $\{x_{k_n}\}$ of $\{x_k\}$. In this case $\Delta - \lim_{k\to\infty} x_k = x$.*

Lemma 2.10. [10, 11] *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be a sequences of non-negative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \; n \geq 1.$$

If $\sum \delta_n < \infty$, $\sum b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to 0, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.11. [12] *Let X be a Banach space satisfying Opial's condition and let K be a non-empty closed convex subset of X. Let $T : K \to K$ be an asymptotically nonexpansive mapping. Then $(I - T)$ is demiclosed at zero.*

3. Main Results

Theorem 3.1. *Let (X, d, W) be a complete uniformly convex hyperbolic space with uniformly convexity η and $K \subset X$ is non-empty closed convex. Let $T : K \to K$ be a nearly asymptotically nonexpansive mapping with a sequence $\{s_n, \zeta(T^n)\}$ such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in K defined by (1) with the assumption that*

- $\lim_{n\to\infty} \inf \alpha_n(1 - \alpha_n) > 0$;
- $\sum s_n < \infty$;
- $\sum(\zeta(T^n) - 1) < \infty$.

Then $\lim_{n\to\infty} d(x_n, p)$ *exists.*

Proof. Since $F(T) \neq \emptyset$, let $p \in F(T)$. Also T is nearly asymptotically nonexpansive mapping, we have

$$\begin{aligned}
d(z_n, p) &= d(W(x_n, T^n x_n, \alpha_n), p) \\
&\leq (1-\alpha_n) d(x_n, p) + \alpha_n d(T^n x_n, p) \\
&\leq (1-\alpha_n) d(x_n, p) + \alpha_n d(T^n x_n, T^n p) \\
&\leq (1-\alpha_n) d(x_n, p) + \alpha_n [\zeta(T^n)(d(x_n, p) + s_n)] \\
&\leq \zeta(T^n)[d(x_n, p) + s_n], \\
d(y_n, p) &= d(T^n z_n, T^n p) \\
&\leq \zeta(T^n)[d(z_n, p) + s_n] \\
&\leq \zeta(T^n)[\zeta(T^n)(d(x_n, p) + s_n) + s_n] \\
&= \zeta(T^n)^2 d(x_n, p) + \zeta(T^n)^2 s_n + \zeta(T^n) s_n, \\
d(x_{n+1}, p) &= d(T^n y_n, T^n p) \\
&\leq \zeta(T^n)[d(y_n, p) + s_n] \\
&\leq \zeta(T^n)[\zeta(T^n)^2 d(x_n, p) + [\zeta(T^n)^2 s_n) + [\zeta(T^n) s_n + s_n] \\
&= \zeta(T^n)^3 d(x_n, p) + \zeta(T^n)^3 s_n + \zeta(T^n)^2 s_n + \zeta(T^n) s_n \\
&= \zeta(T^n)^3 d(x_n, p) + [\zeta(T^n)^3 + \zeta(T^n)^2 + \zeta(T^n] s_n \\
&= \zeta(T^n)^3 d(x_n, p) + \beta_n \\
& (1+\delta_n) d(x_n, p) + \beta_n,
\end{aligned}$$

where $\beta_n = [\zeta(T^n)^3 + \zeta(T^n)^2 + \zeta(T^n] s_n$,
and

$$\begin{aligned}
(1+\delta_n) &= \zeta(T^n)^3 \\
\delta_n &= \zeta(T^n)^3 - 1 \\
&= (\zeta(T^n) - 1)(\zeta(T^n)^2 + \zeta(T^n + 1).
\end{aligned}$$

Since $\sum(\zeta(T^n) - 1) < \infty$, $\sum s_n < \infty$, hence $\sum \alpha_n < \infty$, $\sum \beta_n < \infty$. Hence from Lemma 2.10, we have $\lim_{n\to\infty} d(x_n, p)$ exists. □

Lemma 3.2. *Let* (X, d, W) *be a complete uniformly convex hyperbolic space with uniformly convexity* η *and* $K \subset X$ *is non-empty closed convex. Let* $T : K \to K$ *be a nearly asymptotically nonexpansive mapping with a sequence* $\{s_n, \zeta(T^n)\}$ *such that* $F(T) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence in* K *defined by (1) with the assumption that*

- $\lim_{n\to\infty} \inf \alpha_n(1-\alpha_n) > 0$;
- $\sum s_n < \infty$;
- $\sum(\zeta(T^n) - 1) < \infty$.

Then $\lim_{n\to\infty} d(T^n x_n, x_n) = 0$

Proof. Since $\{x_n\}$ is bounded sequence, there exists $R > 0$ such that $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ are subsets of $B_R(p)$ for all $n \geq 1$. Using (1), we have

$$\begin{aligned}
d^2(z_n,p) &= d^2(W(x_n,T^nx_n,\alpha_n),p)\\
&= d^2((1-\alpha_n)x_n+\alpha_nT^nx_n,p)\\
&\leq (1-\alpha_n)d^2(x_n,p)+\alpha_nd^2(T^nx_n,p)-\frac{R}{2}(1-\alpha_n)\alpha_nd^2(T^nx_n,x_n)\\
&= (1-\alpha_n)d^2(x_n,p)+\alpha_n[\zeta(T^n)d^2(x_n,p)+s_n]\\
&-\frac{R}{2}(1-\alpha_n)\alpha_nd^2(T^nx_n,x_n)\zeta(T^n)^2d^2(x_n,p)\\
&= -\frac{R}{2}(1-\alpha_n)\alpha_nd^2(T^nx_n,x_n)+As_n,
\end{aligned}$$

where $A = \zeta(T^n)^2[s_n + 2d(x_n,p)]$.

$$\begin{aligned}
d^2(y_n,p) &= d^2(T^nz_n,T^np)\\
&\leq \zeta(T^n)^2[d^2(z_n,p)]\\
&= \zeta(T^n)^2[\zeta(T^n)^2d^2(x_n,p)-\frac{R}{2}(1-\alpha_n)\alpha_nd^2(T^nx_n,x_n)+As_n]\\
&= \zeta(T^n)^4d^2(x_n,p)+\zeta(T^n)^2As_n-\zeta(T^n)^2\frac{R}{2}(1-\alpha_n)\alpha_nd^2(T^nx_n,x_n)\\
&= \zeta(T^n)^4d^2(x_n,p)+Bs_n-\frac{R'}{2}(1-\alpha_n)\alpha_nd^2(T^nx_n,x_n),
\end{aligned}$$

where $B = \zeta(T^n)^2A$, $R^{'} = \zeta(T^n)^2R$.

$$\begin{aligned}
d^2(x_{n+1},p) &= \zeta(T^n)^2d^2(y_n,p)\\
&= \zeta(T^n)^2[\zeta(T^n)^4d^2(x_n,p)+\zeta(T^n)^2As_n-\zeta(T^n)^2\frac{R}{2}(1-\alpha_n)\\
&\alpha_nd^2(T^nx_n,x_n)]\zeta(T^n)^6d^2(x_n,p)+\zeta(T^n)^4As_n\\
&= -\zeta(T^n)^4\frac{R}{2}(1-\alpha_n)\alpha_nd^2(T^nx_n,x_n)\\
&= \zeta(T^n)^6d^2(x_n,p)+Cs_n-\frac{R^{"}}{2}(1-\alpha_n)\alpha_nd^2(T^nx_n,x_n),
\end{aligned}$$

where $C = \zeta(T^n)^2B$, $R^{"} = \zeta(T^n)^2R^{'}$.

$$\begin{aligned}
d^2(x_{n+1},p) &= [1+\zeta(T^n)^6-1]d^2(x_n,p)+Cs_n-\frac{R^{"}}{2}(1-\alpha_n)\alpha_nd^2(T^nx_n,x_n)\\
&= [1+(\zeta(T^n)-1)\delta]d^2(x_n,p)+Cs_n-\frac{R^{"}}{2}(1-\alpha_n)\alpha_nd^2(T^nx_n,x_n),
\end{aligned}$$

where $\delta = \zeta(T^n)^5 + \zeta(T^n)^4 + ... + 1$.

$$\begin{aligned}
\frac{R^{"}}{2}(1-\alpha_n)\alpha_nd^2(T^nx_n,x_n) &\leq [1+(\zeta(T^n)-1)\delta]d^2(x_n,p)-d^2(x_{n+1},p)+Cs_n\\
&\leq d^2(x_n,p)-d^2(x_{n+1},p)+(\zeta(T^n)-1)\delta]d^2(x_n,p)\\
&+Cs_n.
\end{aligned}$$

Since $\sum s_n < \infty$, $\sum(\zeta(T^n) - 1) < \infty$, $d(x_n,p) < R$ implies that $\frac{R^{"}}{2}(1 - \alpha_n)\alpha_nd^2(T^nx_n,x_n) < \infty$. Hence $\lim_{n\to\infty} d(T^nx_n,x_n) = 0$. □

Theorem 3.3. *Let (X, d, W) be a complete uniformly convex hyperbolic space with uniformly convexity η and $K \subset X$ is non-empty closed convex. Let $T : K \to K$ be a nearly asymptotically nonexpansive mapping with a sequence $\{s_n, \zeta(T^n)\}$ such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in K defined by (1). Then $\lim_{n\to\infty} d(x_n, p) = 0$.*

Proof.

$$\begin{aligned} d(x_n, p) &= d(x_n, T^n p) \\ &\leq d(x_n, T^n x_n) + d(T^n x_n, T^n, p) \\ &\leq d(x_n, T^n x_n) + k_n[d(x_n, p) + s_n] \\ (1 - k_n)d(x_n, p) &\leq k_n s_n + d(x_n, T^n x_n). \end{aligned}$$

This implies that $d(x_n, p) \to 0$ as $n \to \infty$. □

Theorem 3.4. *Let (X, d, W) be a complete uniformly convex hyperbolic space with uniformly convexity η and $K \subset X$ is non-empty closed convex. Let $T : K \to K$ be a nearly asymptotically nonexpansive mapping with a sequence $\{s_n, \zeta(T^n)\}$ such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in K defined by (1) with the assumption that*

- $\lim_{n\to\infty} \inf \alpha_n(1 - \alpha_n) > 0$;
- $\sum s_n < \infty$;
- $\sum(\zeta(T^n) - 1) < \infty$.

Then $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

Proof. Since

$$d^2(z_n, p) \leq \zeta(T^n)^2 d^2(x_n, p) - \frac{R}{2}(1 - \alpha_n)\alpha_n d^2(T^n x_n, x_n) + As_n$$

$$d^2(y_n, p) \leq \zeta(T^n)^6 d^2(x_n, p) + Cs_n - \frac{R''}{2}(1 - \alpha_n)\alpha_n d^2(T^n x_n, x_n).$$

Taking $\lim_{n\to\infty}$, we have $d^2(z_n, p) = 0$, $d^2(y_n, p) = 0$. Now

$$d(y_n, z_n) \leq d(y_n, p) + d(p, z_n)$$

This implies that $\lim_{n\to\infty} d(y_n, z_n) = 0$.
Similarly $d(z_n, x_n) = 0$. Now

$$d(x_{n+1}, y_n) \leq d(x_{n+1}, p) + d(p, y_n)$$

Hence, $\lim_{n\to\infty} d(x_{n+1}, y_n) = 0$.

$$\begin{aligned} d(x_n, Tx_n) \leq\ & d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_{n+1}, T^{n+1}x_n) \\ & + d(T^{n+1}x_n, Tx_n) \\ & \leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + \zeta(T^{n+1})d(x_n, x_{n+1}) \\ & + s_{n+1} + d(T^{n+1}x_n, Tx_n) \\ & = [1 + \zeta(T^{n+1})]d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_n, Tx_n) \\ & + s_{n+1}. \end{aligned}$$

This implies that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. □

Theorem 3.5. *Let (X, d, W) be a complete uniformly convex hyperbolic space with uniformly convexity η and $K \subset X$ is non-empty closed convex. Let $T : K \to K$ be a nearly asymptotically nonexpansive mapping with a sequence $\{s_n, \zeta(T^n)\}$ such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in K defined by (1). Then $\{x_n\}$ $\Delta-$converges to a fixed point of T.*

Proof. Since $\{x_n\}$ is bounded and from Lemma 3.2, $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$. Since X is uniformly convex, we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to $x^* \in K$. Now, we show that $\{x_k\}$ has a unique weak subsequential limit in $F(T)$. Let x^* and u are two weak limits of the subsequence $\{x_{n_k}\}$ and $\{x_{n_m}\}$ of $\{x_n\}$. Now suppose that $x^* \neq u$.

$$\begin{aligned}
\limsup d(x_n, x^*) &= \limsup d(x_{n_k}, x^*)\\
&< \limsup d(x_{n_k}, u)\\
&= \limsup d(x_n, u)\\
&< \limsup d(x_{n_m}, u)\\
&= \limsup d(x_{k_m}, x^*)\\
&= \limsup d(x_k, x^*),
\end{aligned}$$

which is a contradiction. Hence $x^* = u$. □

Theorem 3.6. *Let (X, d, W) be a complete uniformly convex hyperbolic space with uniformly convexity η and $K \subset X$ is non-empty closed convex. Let $T : K \to K$ be a nearly asymptotically nonexpansive mapping with a sequence $\{s_n, \zeta(T^n)\}$ such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in K defined by (1). Then $\{x_n\}$ converges strongly to $p \in F(T)$ if and only if $\lim_{k\to\infty} d(x_n, F(T)) = 0$, where $d(x_n, F(T)) = \inf\{d(x_n, p) : p \in F(T)\}$.*

Proof. From Theorem 3.3, we have $\lim_{n\to\infty} d(x_n, p) = 0$ for all $p \in F(T)$. Therefore $\lim_{k\to\infty} d(x_n, F(T)) = 0$.
Conversely, suppose that $\lim_{k\to\infty} d(x_n, F(T)) = 0$. Next we will show that $\{x_n\}$ is Cauchy sequence in K. Since

$$\begin{aligned}
\lim_{k\to\infty} d(x_n, F(T)) &= 0\\
\Rightarrow d(x_n, F(T)) &< \frac{\epsilon}{2}\\
d(x_n, p) &< \frac{\epsilon}{2},\ p \in F(T).
\end{aligned}$$

For all $n, m \geq n_0$,

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq d(x_{n+m}, p) + d(p, x_n)\\
&< \epsilon.
\end{aligned}$$

This implies that $\{x_n\}$ is Cauchy sequence in K. Since K is closed subset of complete hyperbolic space X, hence $\lim_{k\to\infty} x_n = q$ for some $q \in K$. As $\lim_{k\to\infty} d(x_n, F(T)) = 0 \Rightarrow d(q, F(T)) = 0$. It concludes that $q \in F(T)$. □

Theorem 3.7. *Let (X, d, W) be a complete uniformly convex hyperbolic space with uniformly convexity η and $K \subset X$ is non-empty closed convex. Let $T : K \to K$ be a nearly asymptotically nonexpansive mapping with a sequence $\{s_n, \zeta(T^n)\}$ such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in K*

defined by (1). Suppose that T satisfies Condition(I). Then $\{x_n\}$ converges strongly to a point of $F(T)$.

Proof. Since $\lim_{k\to\infty} d(x_n, F(T))$ exists and $\lim_{k\to\infty} d(x_n, Tx_n) = 0$, therefore by using condition(I), we have

$$\lim_{k\to\infty} g(d(x_n, F(T))) \leq \lim_{k\to\infty} d(x_n, Tx_n) = 0.$$

Therefore $\lim_{k\to\infty} d(x_n, F(T)) = 0$, and result follows from the Theorem 3.6. □

Theorem 3.8. *Let (X, d, W) be a complete uniformly convex hyperbolic space with uniformly convexity η and $K \subset X$ is non-empty closed convex. Let $T : K \to K$ be a nearly asymptotically nonexpansive mapping with a sequence $\{s_n, \zeta(T^n)\}$ such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in K defined by (1). Suppose that T^m is compact for some $m \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. $\lim_{k\to\infty} d(x_n, Tx_n) = 0$, and T is uniformly continuous, we have

$$d(x_n, T^m x_n) \leq d(x_n, Tx_n) + d(Tx_n, T^2 x_n) + ... + d(T^{m-1}x_n, T^m x_n).$$

This implies that $\lim_{n\to\infty} d(x_n, T^m x_n) = 0$.
Since T^m is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{n\to\infty} x_{n_k} = p$, $p \in K$. Again by uniform continuity of T, we have

$$d(p, Tp) \leq d(Tp, Tx_{n_k}) + d(Tx_{n_k}, x_{n_k}) + d(x_{n_k}, p).$$

This implies that $\lim_{n\to\infty} x_{n_k} = p$, i.e., $p \in F(T)$. Since $\lim_{n\to\infty} d(x_n, p)$ exists, hence p is strong limit of the sequence $\{x_n\}$ itself. □

Theorem 3.9. *Let (X, d, W) be a complete uniformly convex hyperbolic space with uniformly convexity η that satisfies Opial's condition and $K \subset X$ is non-empty closed convex. Let $T : K \to K$ be a nearly asymptotically nonexpansive mapping with a sequence $\{s_n, \zeta(T^n)\}$ such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in K defined by (1). Then $\{x_n\}$ converges weakly to some fixed point of T.*

Proof. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ weakly converges to some $q \in K$. Since $\lim_{k\to\infty} d(x_{n_k}, Tx_{n_k}) = 0$, and $(I - T)$ is demiclosed at 0, we have $(I - T)q = 0 \Rightarrow q \in F(T)$. We claim that $\{x_n\}$ converges weakly to q. If possible suppose that $\{x_n\}$ does not converges weakly to q, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to r \neq q$, $r \in F(T)$. Using Opial's condition, we have

$$\begin{aligned}\liminf d(x_{n_k}, q) &< \liminf d(x_{n_k}, r)\\ &\liminf d(x_n, r)\\ &= \liminf d(x_{n_k}, q),\end{aligned}$$

which is a contradiction. Hence $\{x_n\}$ converges weakly to q. □

4. Numerical Examples

Example 4.1. *Let* $X = \mathbb{R}$, $K = [0, 1]$ *and* $T : K \to K$ *is defined by*

$$Tx = \begin{cases} \frac{1}{5}, x \in [0, \frac{1}{5}]; \\ 0, x \in (\frac{1}{5}, 1]. \end{cases}$$

Let d *be a metric on* X *defined by* $d(x, y) = |x - y|$. *Here* $F(T) = \frac{1}{5}$ *and* T *is not continuous, hence* T *is non-Lipschitzian. However* T *is nearly nonexpansive mapping and hence nearly asymptotically nonexpansive mapping with a sequence* $\{(s_n, \eta(T^n)\} = \{\frac{1}{5^n}, 1\}$. *In-fact for a sequence* $\{s_n\}$ *with* $s_1 = \frac{1}{5}$ *and* $\lim_{n\to\infty} s_n = 0$, *we have*

$$d(Tx, Ty) \leq d(x, y) + s_1, \ \forall x, y \in K,$$

and

$$d(T^n x, T^n y) \leq d(x, y) + s_n, \ \forall x, y \in K, \ n \geq 2.$$

References

[1] B. Ali, *Convergence theorems for finite families of total asymptotically nonexpansive mappings in hyperbolic spaces*, Fixed Point Theory Appl. 2016 (1) (2016), 1-13.

[2] K. Goebel and W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, J. Proc. Amer. Math. Soc. 35 (1) (1972), 171-174.

[3] A. Sharma, *Approximating fixed points of nearly asymptotically nonexpansive mappings in $CAT(k)$ spaces*, Arab J. Math. Sci. 24 (2) (2018), 166-181.

[4] D.R. Sahu, *Fixed points of demicontinuous nearly Lipschitzian mappings in Banach spaces*, Comment. Math. Univ. Carolin 46 (4) (2005), 653-666.

[5] U. Kohlenbach, *Some logical meta theorems with applications in functional analysis*, Tran. Amer. Math. Soc. 357 (1) (2004), 89-128.

[6] L. Leustean, *A quadratic rate of asymptotic regularity for CAT(0) spaces*, J. Math. Anal. Appl. 325 (2007), 386-399.

[7] S.S. Chang, G. Wang, L. Wang, Y.K. Tang and Z. Ma, $\Delta-$ *convergence theorems for multi-valued nonexpansive mappings in hyperbolic spaces*, Appl. Math. Comp. 249 (2014), 535-540.

[8] A.U. Bello, C.C. Okeke and C. Izuchukw, *Approximating common fixed point for family of multi-valued mean nonexpansive mappings in hyperbolic spaces*, Adv. Fixed Point Theory 7 (4) (2017), 524-543.

[9] C. Suanoom and C. Klin-eam, *Remark on fundamentally nonexpansive mappings in hyperbolic spaces*, Bull. Aust. J. Nonlinear Sci. Appl. 9 (2016), 1952-1956.

[10] M.O. Osilike and S.S. Aniagbosor, *Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings*, Mathematical and computer modelling 32 (2000), 1181-1191.

[11] A. Kasloom, M. Rashid, T.C. Suan and et al., *Fixed points of monotone total asymptotically nonexpansive mapping in hyperbolic space via new algorithm*, Journal of function spaces 2021 (2021), 10 pages.

[12] J. Gornicki, *weak convergence theorems for asymptotically nonexpansive mappings in uniformly convex Banach spaces*, Comment. Math. Univ. Carolin 30 (1989), 249-252.

[13] W. Takahashi, *A convexity in metric space and nonexpansive mappings*, Kodai Math. J. 22 (2) (1970), 142-149.

[14] Q.H. Liu, *Iterative sequences for asymptotically quasi-nonexpansive mappings with error member*, J. Math. Anal. Appl. 259 (2001), 18-24.

[15] K. Dewangan, N. Gurudwan and L. Rathour, *Fixed points of multi-valued Osilike-Berinde nonexpansive mappings in hyperbolic spaces*, Nonlinear Fun. Anal. Appli. 28 (3) (2023), 685-702. http://nfaa.kyungnam.ac.kr/journal-nfaa

[16] G.S. Saluja, *Strong convergence theorems for two asymptotically quasi-nonexpansive mappings with errors in Banach space*, Tamkang J. Math. 38 (1) (2007), 85-92.

[17] G.S. Saluja, *Convergence ressults of $(L,\alpha)-$uniformly Lipschitz asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces*, Jnanabha 38 (2008), 41-48.

[18] J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansoive mappings*, Bull. Aus. Math. Soc. 43 (1) (1991), 153-159.

[19] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. 73 (1967), 591-597.

[20] W.R. Mann, *Mean value methods in iterations*, Proc. Amer. Math. Soc. 4 (1953), 171-174.

[21] M.A. Noor, *New approximation schemes for general variation inequality*, J. Math. Anal. Appl. 251 (2000), 221-229.

[22] S. Ishikawa, *Fixed point by new iteration method*, Proc. Amer. Math. Soc. 44 (1974), 147-150.

[23] K. Ullah and M. Arshad, *Numerical reckoning fixed points for Suzuki's generalized nonexpansive mappings via new operation process*, Filomat 32 (1) (2018), 187–196.

[24] K.K. Tan and H.K. Xu, *Fixed point iteration processes for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 122 (1994), 733-739.

[25] B.L. Xu and M.A. Noor, *Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. 267 (2) (2002), 444-453.

[26] L.C. Ceng, D.R. Sahu and J.C. Yao, *Implicit iterative algorithm for asymptotically nonexpansive mappings in the intermediate sense and Lipschitz-continuous monotone mappings*, J. Comput. Appl. Math. 233 (11) (2010), 2902-2915.

[27] A. Chandola, R.M. Pandey, R. Agarwal, L. Rathour and V.N. Mishra, *On some properties and applications of the generalized $\boldsymbol{m}$-parameter Mittag-Leffler function*, Adva. Math. Models. Appl. 7 (2) (2022), 130-145.

[28] K. Kumar, L. Rathour, M.K. Sharma and V.N. Mishra, *Fixed point approximation for Suzuki generalized nonexpansive mapping using $B_{(\delta,\mu)}$ condition*, Appl. Math. 13 (2) (2022), 215-227.

[29] P. Shahi, L. Rathour and V.N. Mishra, *Expansive fixed point theorems for tri-simulation functions*, J. Engi. Exact Sci. 8 (3) (2022), 1-8. DOI: https://doi.org/10.18540/jcecvl8iss3pp14303-01e

[30] A.G. Sanatee, L. Rathour, V.N. Mishra and V. Dewangan, *Some fixed point theorems in regular modular metric spaces and application to Caratheodory's type anti-periodic boundary value problem*, J. Anal. 31 (2023), 619-632. DOI: https://doi.org/10.1007/s41478-022-00469-z

[31] M.M.A. Metwali and V.N. Mishra, *On the measure of noncompactness in $L_p(\mathbb{R}^+)$ and applications to a product of n-integral equations*, Turkish J. Math. 47 (1) (2023), 372-386. DOI: https://doi.org/10.55730/1300-0098.3365

[32] V.K. Pathak, L.N. Mishra and V.N. Mishra, *On the solvability of a class of nonlinear functional integral equations involving Erdêlyi-Kober fractional operator*, Math. Meth. Appl. Sci. 46 (13) (2023), 14340-14352. DOI: https://doi.org/10.1002/mma.9322

[33] S.K. Paul, L.N. Mishra, V.N. Mishra and D. Baleanu, *An effective method for solving nonlinear integral equations involving the Riemann-Liouville fractional operator*, AIMS Mathematics 8 (8) (2023), 17448-17469. DOI: https://doi.org/10.3934/math.2023891.

[34] S.K. Paul, L.N. Mishra, V.N. Mishra and D. Baleanu, *Analysis of mixed type nonlinear Volterra–Fredholm integral equations involving the Erdélyi–Kober fractional operator*, J. King Saud Univ. 35 (2023), 1-9. DOI: https://doi.org/10.1016/j.jksus.2023.102949

[35] I.A. Bhat, L.N. Mishra, V.N. Mishra, C. Tunc and O. Tunc, *Precision and efficiency of an interpolation approach to weakly singular integral equations*, Inter. J. Numer. Metho. Heat & Fluid Flow, (2024). DOI: 10.1108/HFF-09-2023-0553

[36] N. Sharma, L.N. Mishra, V.N. Mishra and S. Pandey, *Solution of Delay Differential equation via N_1^v iteration algorithm*, Euro. J. Pure Appl. Math. 13 (5) (2020), 1110-1130. DOI: https://doi.org/10.29020/nybg.ejpam.v13i5.3756.

[37] C.E. Chidume, E.U. Ofoedu, and H. Zegeye, *Strong and weak convergence theorems for asymptotically nonexpansive mappings*, J. Math. Anal. Appl. 280 (2003), 364-374.
[38] S. H. Khan and N. Hussain, *Convergence theorems for non-self asymptotically nonexpansive mappings*, Math. Appl. 55 (11) (2008), 2544-2553.
[39] H. K. Pathak, Y. J. Cho, and S. M. Kang, *Strong and weak convergence theorems for non-self asymptotically perturbed nonexpansive mappings*, Nonlinear Anal.: Theory Methods Appl. 70 (5) (2009), 1929-1938.
[40] L. Wang, *Strong and weak convergence theorems for common fixed point of non-self asymptotically nonexpansive mappings*, J. Math. Anal. Appl. 323 (1) (2006), 550-557.
[41] Y. Alber, C. E. Chidume, and H. Zegeye, *Approximating fixed points of total asymptotically nonexpansive mappings*, Fixed Point Theory Appli. 2006(1) (2006), Article ID 10673.

Department of Mathematics, Government Dudhadhari Bajrang Girls Postgraduate Autonomous College, Raipur-492001, Chhattisgarh, India
Email address: dewangan.kiran@gmail.com

Department of Mathematics, National Institute of Technology, Chaltlang, Aizawl-796012, Mizoram, India
Email address: laxmirathour817@gmail.com

Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore-632014, Tamilnadu, India
Email address: lakshminarayanmishra04@gmail.com

Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur-484887, Madhya Pradesh, India
Email address: vishnunarayanmishra@gmail.com