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Arithmetic Function Signed Graphs of Finite Groups

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Abstract

This paper is aimed to define and discuss a generalization of the order prime signed graph of a finite group, namely, arithmetic function signed graph with respect to an arithmetical function and present some results.

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1 Introduction

A signed graph can be created by adding a sign to each and every edge of a simple graph. In 1953, Harary presented the idea of signed graphs [1] in relation to a few social psychology issues. In 1982, Zaslavsky introduced matroids

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of signed graphs [12]. Signed graphs are very important as they are used in the study of complex systems of computer models and sociology. R. Rajendra et al. [4] defined two signed graphs associated with a finite group(FG) $\mathfrak{A}$, namely order prime signed graph of $\mathfrak{A}$ and general order prime signed graph of $\mathfrak{A}$, taking into consideration of the commutative property. Unless otherwise indicated, $\mathfrak{A}$ will represent a FG of order n throughout this paper.

Sattanathan and Kala [10] introduced the notion of order prime graph(OPG) of $\mathfrak{A}$. The OPG $OP(\mathfrak{A})$ of $\mathfrak{A}$ is the graph with node set $V(OP(\mathfrak{A})) = \mathfrak{A}$ and any two different nodes x and y are adjacent in $OP(\mathfrak{A})$ iff $\gcd(o(x), o(y)) = 1$. Rajendra and Reddy [2, 3, 6] proposed the notion of general order prime graph(GOPG) of $\mathfrak{A}$. The GOPG $GOP(\mathfrak{A})$ of $\mathfrak{A}$ is the graph with node set $V(GOP(\mathfrak{A})) = \mathfrak{A}$ and any two different nodes x and y are adjacent in $GOP(\mathfrak{A})$ iff $\gcd(o(x), o(y)) = 1$ or p, where p is a prime $< n$. It is obvious that a subgraph of $GOP(\mathfrak{A})$ is $OP(\mathfrak{A})$.

Sathyanaryana et al. [11] proposed the idea of a prime graph of an associative ring R and examined its features. The authors of [7] looked into a few characteristics of prime graphs of finite rings and developed a method that uses the gcd-sum function to find the number of edges in the prime graph of the ring of residue classes modulo n. They also demonstrate that, in the case when n is a prime, the prime graph of the ring $\mathbb{Z}_n$ is equal to the order prime graph defined in [10] of the additive group $\mathbb{Z}_n$, where n denotes a positive integer.

Let S be a positive integer set that is not empty. Rajendra et al. [5] introduced the notion of set-prime graph(S-PG) $G_S(\mathfrak{A})$ of $\mathfrak{A}$ with regard to S. It is a graph with node set $V(G_S(\mathfrak{A})) = \mathfrak{A}$ and any two distinct nodes x and y are adjacent in $G_S(\mathfrak{A})$ iff $\gcd(o(x), o(y)) \in S$. The OPG and GOPG are special cases of S-PGs, as noted by Rajendra et al. [5], who also looked at several aspects of S-PGs of FGs.

As a generalization of an OPG, Rajendra et al. [8] defined the arithmetic function graph(AFG) of a FG with respect to an arithmetical function(AF). The AFG $G_h(\mathfrak{A})$ of $\mathfrak{A}$ with respect to an AF h is defined to be a graph with node set $V(G_h(\mathfrak{A})) = \mathfrak{A}$ and two distinct nodes x and y are adjacent in $G_h(\mathfrak{A})$ iff $h(|x||y|) = h(|x|)h(|y|)$. It is noted that the OPG $OP(\mathfrak{A})$ is nothing but the AFG $G_\phi(\mathfrak{A})$ with respect to ϕ, the Euler's ϕ-function. The eigenvalues and energy of the AFGs of FGs are discussed in [9].

An ordered pair $\Sigma = (G, \sigma)$ is referred to as a signed graph [1, 12] where $G = (V, E)$ is the underlying graph of Σ and σ is a function from E to $\{+, -\}$.

If there are an even number of negative edges in each cycle of a signed graph Σ, then Σ is balanced [1]. If each and every edge in a signed graph has the same sign, then the signed graph is said to be homogenous; if not, it is said to be heterogeneous. The signed graph that results from reversing the signs on the edges of a signed graph Σ is called its negation of Σ and is denoted by $\neg\Sigma$.

A graph can be thought of as an all-positive signed graph, and signed graphs are a generalization of graphs in this sense. A signed graph's sign is equal to the product of the signs on its edges.

If the underlying undirected graphs of two signed graphs are isomorphic and the signs on the edges are retained, then the two signed graphs are isomorphic.

We assume that $\mathfrak{A}$ is a FG of order n. An edge with end vertices u and v in a graph is denoted by uv or $\overline{(u,v)}$. The notions of order prime signed graph(OPSG) and the general order prime signed graph(GOPSG) are as follows:

Definition 1.1 ([4]). The order prime signed graph(OPSG) $OPS(\mathfrak{A})$ of a FG $\mathfrak{A}$ is the signed graph $((OP(\mathfrak{A}),\sigma)$ where the function $\sigma : E(OP(\mathfrak{A})) \to \{+,-\}$ is given by

$$\sigma(\overline{(x,y)}) = \begin{cases} +, & \text{when } xy = yx; \\ -, & \text{otherwise.} \end{cases}$$

It can be shown from the Definition 1.1 that if a group $\mathfrak{A}$ is abelian, then all of $OPS(\mathfrak{A})$'s edges are of the $+$ sign, thus in this instance, $OPS(\mathfrak{A})$ is a balanced signed graph. If there is at least one edge with a $-$ sign in $OPS(\mathfrak{A})$, then $\mathfrak{A}$ is not an abelian group. However, in general, the converse of this statement is untrue. All of $OPS(\mathfrak{A})$'s edges for non-abelian groups of prime power order will have the sign '+.

Definition 1.2 ([4]). The general order prime signed graph(GOPSG) $GOPS(\mathfrak{A})$ of a FG $\mathfrak{A}$ is the signed graph $(GOP(\mathfrak{A}),\sigma)$, where the function $\sigma : E(GOP(\mathfrak{A})) \to \{+,-\}$ is given by

$$\sigma(\overline{(x,y)}) = \begin{cases} +, & \text{when } xy = yx; \\ -, & \text{otherwise.} \end{cases}$$

By the Definition 1.2, it is obvious that, if a group $\mathfrak{A}$ is abelian then all the edges of $GOPS(\mathfrak{A})$ are of '+' sign and so in this case $GOPS(\mathfrak{A})$ is a balanced signed graph. If $GOPS(\mathfrak{A})$ contains atleast one edge with '$-$' sign, then the group $\mathfrak{A}$ is non-abelian.

$OPS(\mathfrak{A})$ is a subgraph of $GOPS(\mathfrak{A})$ for every FG $\mathfrak{A}$, according to the Definitions 1.1 and 1.2. It follows that the graphs $OPS(\mathfrak{A})$ and $GOPS(\mathfrak{A})$ are connected since the identity element e in $\mathfrak{A}$ is the only element of order 1. Also, $d^+(e) = n-1$ and the maximum positive degree $\Delta^+(GOPS(\mathfrak{A})) = \Delta^+(OPS(\mathfrak{A})) = n-1$.

2 Arithmetic Function Signed Graph(AFSG)

This section is devoted to introduce the concept of arithmetic function signed graph(AFSG) of a FG $\mathfrak{A}$ with respect to an arithmetic function(AF) h and discuss some properties.

Definition 2.1. The arithmetic function signed graph(AFSG) $SG_h(\mathfrak{A})$ of a FG $\mathfrak{A}$ is the signed graph $(G_h(\mathfrak{A}), \sigma)$ where the function $\sigma : E(G_h(\mathfrak{A})) \to \{+, -\}$ is given by

$$\sigma(\overline{(x, y)}) = \begin{cases} +, & \text{when } xy = yx; \\ -, & \text{otherwise.} \end{cases}$$

Example 2.2. Consider the group $\mathbb{Z}_6$ under the operation addition modulo 6. The corresponding AF-ϕ signed graph $SG_\phi(\mathbb{Z}_6$ is shown in Figure 1. We see that $SG_\phi(\mathbb{Z}_6)$ homogeneous signed graph with + sign on all edges.

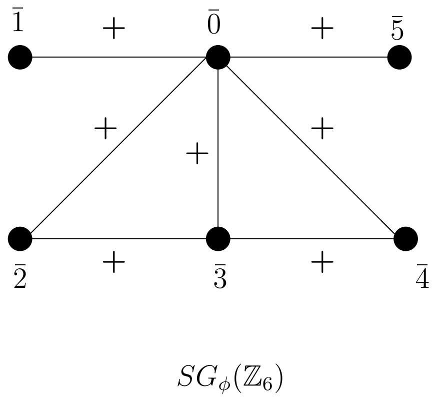
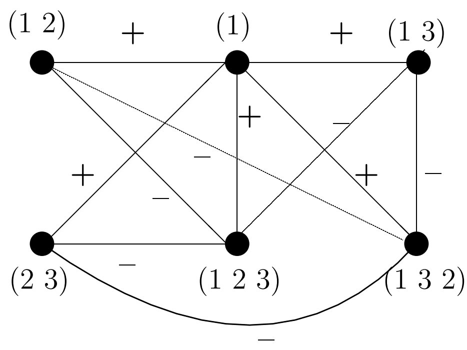


Figure 1: The signed graph $SG_\phi(\mathbb{Z}_6)$

Example 2.3. Consider the permutation group

$$S_3 = \{(1), (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$$

of 3 symbols. The corresponding order prime signed graph is shown in Figure 2. We see that $SG_\phi(S_3)$ is a heterogeneous unbalanced signed graph.



$SG_\phi(S_3)$

Figure 2: The signed graph $SG_\phi(S_3)$

Remark 2.4. By the Example 2.3, it follows that, an AFSG of a FG need not always be a triangulated signed graph and also it need not be balanced.

Proposition 2.5. Negation of an AFSG of a FG of order > 2 with respect to an AF h with $h(1) = 1$ will not be an AFSG.

Proof. Let Σ be an AFSG of a FG $\mathfrak{A}$ of order > 2 with respect to an AF h with $h(1) = 1$. Then there are exactly $o(\mathfrak{A}) - 1 > 1$ edges incident on e and all of them are assigned $+$ sign. The negation of Σ, that is $\neg\Sigma$, is the signed graph obtained by reversing signs on the edges of Σ. Note that in any AFSG of $\mathfrak{A}$ all the edges incident on e have positive sign. But, in $\neg\Sigma$, all the edges incident on e have negative sign. Therefore $\neg\Sigma$ is not be an AFSG. □

Proposition 2.6. If $\mathfrak{A}$ is a FG of order n, then for any AF h with $h(1) = 1$, $SG_h(\mathfrak{A})$ is a connected signed graph and the maximum positive degree $\Delta^+(SG_h(\mathfrak{A})) = n - 1$.

Proof. Suppose that h is an AF with $h(1) = 1$ and $\mathfrak{A}$ is a FG of order n. Then, we have

$$h(|e||x|) = h(1 \cdot |x|) = h(|x|) = 1 \cdot h(|x|) = h(|e|)h(|x|), \forall x \in \mathfrak{A}.$$

Also, the identity element e commutes with all other elements of $\mathfrak{A}$. Therefore, the vertex e is adjacent to every other vertex with $+$ sign on the edges in $SG_h(\mathfrak{A})$. Therefore, $d^+(e) = n - 1$ and $d^-(e) = 0$. Therefore $SG_h(\mathfrak{A})$ is connected. Since the underlying undirected graph of $SG_h(\mathfrak{A})$ i.e., $G_h(\mathfrak{A})$ is simple, it follows that, $\Delta^+(SG_h(\mathfrak{A})) = n - 1$. □

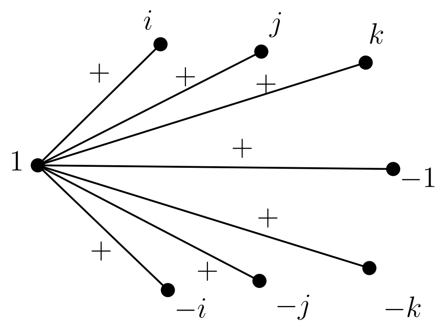
Proposition 2.7. If $\mathfrak{A}$ is abelian, then $SG_h(\mathfrak{A})$ is a homogeneous signed graph with positive sign on all edges (and hence it is a balanced signed graph).

Proof. If $\mathfrak{A}$ is abelian, then by Definition 2.1, it follows that, all the edges of $SG_h(\mathfrak{A})$ are of '+' sign and hence $SG_h(\mathfrak{A})$ is a homogeneous signed graph. □

Proposition 2.8. If $SG_h(\mathfrak{A})$ contains at least one edge with '−' sign (i.e., $SG_h(\mathfrak{A})$ is heterogeneous), then the group $\mathfrak{A}$ is non-abelian.

Proof. Follows by Definition 2.1. □

Remark 2.9. In general, the converse of Proposition 2.8 is not true. As an instance, consider the group of quaternions $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. The group Q_8 is non-abelian and $|x| = 2$, for all $x \neq 1$ in Q_8. All the edges of $SG_\phi(Q_8)$ are of '+' sign and hence $SG_\phi(Q_8)$ is a homogeneous (and also balanced) signed graph.



$SG_\phi(Q_8)$

Figure 3: Arithmetic function signed graph of Q_8 with respect to ϕ

Theorem 2.10. If h is a multiplicative function(MF), then for any non-abelian group of prime power order p^α, $SG_h(\mathfrak{A})$ is a homogeneous signed graph which is a star with '+' sign on all the edges (and so it is balanced).

Proof. Suppose h is a MF and $\mathfrak{A}$ be a non-abelian group of prime power order. Then $G_h(\mathfrak{A})$ is a star $K_{1,p^\alpha-1}$ with center e, the identity element. Since e commutes with every other element in $\mathfrak{A}$, it follows that, all the edges of $SG_h(\mathfrak{A})$ are assigned '+' sign in $SG_h(\mathfrak{A})$. Hence, $SG_h(\mathfrak{A})$ is a homogeneous signed graph with '+' sign on all the edges. □

Theorem 2.11. If h is a MF, then for any non-abelian group $\mathfrak{A}$ whose order is neither a prime nor a prime power, the signed graph $SG_h(\mathfrak{A})$ is heterogeneous and not balanced.

Proof. Suppose that h is a MF and $\mathfrak{A}$ be a non-abelian group whose order is neither a prime nor a prime power. Without sacrificing generality we assume that $o(\mathfrak{A}) = pq$, a product of two distinct primes p and q. Since p and q are prime divisors of $o(\mathfrak{A})$, by the well-known Cauchy's theorem, $\mathfrak{A}$ contains two elements x and y with $|x| = p$ and $|y| = q$. Since $\mathfrak{A}$ is not abelian, it follows that, $xy \neq yx$. Since $\gcd(p, q) = 1$, x and y are adjacent in $SG_h(\mathfrak{A})$ with $\sigma(\overline{(x, y)}) = -$. Then $SG_h(\mathfrak{A})$ contains a triangle with vertices e, x, y where the edge signing is

$$\sigma(\overline{(e, x)}) = +,\ \sigma(\overline{(e, y)}) = + \text{ and } \sigma(\overline{(x, y)}) = -.$$

Therefore $SG_h(\mathfrak{A})$ is heterogeneous and not balanced. □

Theorem 2.12. If h is a completely MF, then for any finite non-abelian group $\mathfrak{A}$, the signed graph $SG_h(\mathfrak{A})$ is heterogeneous and not balanced.

Proof. Suppose that h is a completely MF and $\mathfrak{A}$ be a finite non-abelian group. Then $o(\mathfrak{A})$ is composite and > 5. Then there exist two elements x and y in $\mathfrak{A}$ such that $xy \neq yx$. Since h is a completely MF, $G_h(\mathfrak{A})$ is a complete graph. So, the vertices x and y are adjacent in $G_h(\mathfrak{A})$ with $\sigma(\overline{(x, y)}) = -$. Then $SG_h(\mathfrak{A})$ contains a triangle with vertices e, x, y where the edge signing is

$$\sigma(\overline{(e, x)}) = +,\ \sigma(\overline{(e, y)}) = + \text{ and } \sigma(\overline{(x, y)}) = -.$$

Therefore $SG_h(\mathfrak{A})$ is heterogeneous and not balanced. □

3 f-balanced and f-unbalanced Groups

The following definition is due to the notions of balanced and unbalanced signed graphs, which allows us to classify the groups with respect to an AF f. In this section, we present a result, namely, Corollary 3.5 which classifies the groups with respect the Euler's ϕ-function.

Definition 3.1. Let f be an AF. A group $\mathfrak{A}$ is said to be f-balanced if the signed graph $SG_f(\mathfrak{A})$ is balanced; otherwise we say that $\mathfrak{A}$ is f-unbalanced.

Example 3.2. The AFSG $SG_\phi(Q_8)$ of the quaternion group Q_8 is ϕ-balanced (See Figure 3).

Example 3.3. The AFSG $SG_\phi(S_3)$ of the symmetric group S_3 is ϕ-unbalanced (See Figure 2).

Theorem 3.4. (i) Every finite abelian group is f-balanced for any AF f.

(ii) A non-abelian group of prime power order is f-balanced for any MF f.

(iii) A non-abelian group whose order is not a prime power, is f-unbalanced for any MF f.

(iv) A finite non-abelian group is f-unbalanced for any completely MF f.

Proof. (i) Obvious.
(ii) Follows from the Theorem 2.10.
(iii) Follows from the Theorem 2.11.
(iv) Follows from the Theorem 2.12. □

The following corollary is immediate:

Corollary 3.5. Let $\mathfrak{A}$ is a FG. Then $\mathfrak{A}$ is

$$\begin{cases} \phi\text{-balanced}, & \text{if } \mathfrak{A} \text{ is abelian or } \mathfrak{A} \text{ is a non-abelian group of prime power order;} \\ \phi\text{-unbalanced}, & \text{otherwise.} \end{cases}$$

From the Corollary 3.5, it follows that, the class of ϕ-unbalanced groups are nothing but the class of non-abelian groups whose orders are not prime powers.

The Corollary 3.5 can be stated in the notion of order prime signed graphs as follows:

Corollary 3.6. Let $\mathfrak{A}$ is a FG. Then $OPS(\mathfrak{A})$ is

$$\begin{cases} \text{balanced}, & \text{if } \mathfrak{A} \text{ is abelian or } \mathfrak{A} \text{ is a non-abelian group of prime power order;} \\ \text{unbalanced}, & \text{otherwise.} \end{cases}$$

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