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AUGMENTED PASCAL MATRIX AND ITS PROPERTY

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Abstract. We define a matrix which has a Pascal matrix block in its Cholesky factorization. We define augmented Pascal matrix by this factorization and we present a generalization and another version of this matrix. We obtain some properties of the generalized augmented Pascal matrix and present a relation with coefficients of Bernoulli polynomials. We also give a factorization of augmented Pascal matrix which has a connection with Fibonacci matrix and also obtain a factorization of the other version and we see that this factorization has entries related with generalized Stirling numbers.

1. Introduction

Pascal's triangle is a very suitable structure for exploring, formulating and proving mathematical patterns and for doing many interesting experiments. We denote the Pascal matrix of order n as P_n and its form is as following

$$P_n := \left[\binom{i}{j}\right]_{0\le i,j\le n-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 1 & 3 & 3 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \cdots & \binom{n}{n} \end{bmatrix}. \tag{1.1}$$

This magical structure has many features and deep knowledge that attract the attention of many researchers, see [1–9,11–19]. Most of the articles in the literature give generalizations, factorizations and combinatorial properties of Pascal's matrix. In these generalizations, authors put some parameters and variables in the Pascal matrix (1.1) and study some algebraic properties of these matrices. In [3], a matrix for two variables x and y is defined by removing the columns of Pascal matrix.

In this paper, we examine a symmetric matrix which has a similar construction with symmetric Pascal matrix for $x = y = 1$. The Cholesky factorization of this defined matrix is seen to be related both the Pascal matrix and in some manner the matrix defined in [3]. We call the lower triangular matrix in this factorization as augmented Pascal matrix and this matrix has interesting connections with the Bernoulli polynomials $B(n, x)$ which can be defined by a generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B(n, x)\frac{t^n}{n!}.$$

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We are interested in several properties and factorizations of the augmented Pascal matrix. One of the factorization of this matrix has a connection with the Fibonacci matrix. The Fibonacci matrix $\mathcal{F}_n[x]$ of order n is defined by

$$\mathcal{F}_n[x]_{i,j} = \begin{cases} F_{i-j+1}x^{i-j}, & \text{if } i-j+1 \geq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{1.2}$$

where F_n is the $n-$th Fibonacci number. We obtain another interesting matrix using augmented Pascal matrix and this matrix has some factorizations which has entries related with the generalized Stirling numbers defined by

$$a(n) = n! \sum_{k=0}^{n-1} \frac{k+1}{n-k}. \tag{1.3}$$

The first few terms of these numbers are $0, 1, 5, 26, 154, 1044, \ldots$.

2. Augmented Pascal Matrix

In this section, we define a matrix with indeterminate x and obtain some factorizations and properties of it and the matrices in its Cholesky factorization.

Definition 1. *A symmetric matrix Q_n of order $n+1$ with (i,j) entry $Q_n\{i,j\}$ defined by $Q_n\{i,j\} = 1 + \binom{i+j-2}{j-1}$.*

The matrix Q_n satisfy the following construction rule for $i, j = 1, 2, \ldots, n$

$$Q_n\{i,j\} = Q_n\{i-1,j\} + Q_n\{i-1,j-1\} - 1$$

and for $j = 0, 1, \ldots, n$

$$Q_n\{0,j\} = Q_n\{j,0\} = 1.$$

Example 1. *The matrix Q_5 is as follows*

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 7 & 11 & 16 \\ 1 & 2 & 5 & 11 & 21 & 36 \\ 1 & 2 & 6 & 16 & 36 & 71 \end{bmatrix}.$$

Lemma 1. *The Cholesky factorization of Q_n is given by $Q_n = A_n A_n^T$ where A_n is a matrix of the form*

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{1} & P_{n-1} \end{bmatrix}$$

with P_{n-1} is the well known Pascal matrix of order n.

In [3], the defined matrix is obtained by eliminating the columns of the Pascal matrix, but here we add a column and row to the Pascal matrix.

Definition 2. *The matrix A is called as augmented Pascal matrix.*

The row sums of the matrix A gives the expansion of

$$\frac{1-x-x^2}{(1-x)(1-2x)},$$

and the $k-$th power of A_n is given by

$$\begin{bmatrix} 1 & & & & & \\ B(1,1+k)-B(1,1) & 1 & & & & \\ \frac{B(2,1+k)-B(2,1)}{2} & k & 1 & & & \\ \frac{B(3,1+k)-B(3,1)}{3} & k^2 & 2k & 1 & & \\ \frac{B(4,1+k)-B(4,1)}{4} & k^3 & 3k^2 & 3k & 1 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix},$$

for Bernoulli polynomials $B(n,x)$. Let us consider a generalization of the matrix A_n for one variable x. Let $A_n[x]$ be a matrix of the form

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{X}_{n-1} & P_{n-1}[x] \end{bmatrix} \tag{2.1}$$

where $\mathbf{X}_{n-1} = \left[x, x^2, \ldots, x^n\right]^T$ and $P_{n-1}[x]$ is the generalization of the Pascal matrix P_{n-1} for one variable x. The following theorem gives the multiplication property of two matrices $A_n[x]$ and $A_n[y]$.

Theorem 1. *For $i,j \geq 1$, the (i,j) entry of the matrix $A_n[x]A_n[y]$ equals the (i,j) entry of the matrix $A_n[x+y]$ and the entries in the column zero of the product is given by* $\left[1, x+y, x^2+y(x+y), x^3+y(x+y)^2, \ldots, x^n+y(x+y)^{n-1}\right]^T$.

Proof. The matrix $A_n[x]$ is given in (2.1). Using the definition of multiplication of two matrices and the relation satisfied by Pascal matrices

$$P_{n-1}[x]P_{n-1}[y] = P_{n-1}[x+y],$$

we get the result. □

The inverse of the matrix $A_n[x]$ is given as follows.

Theorem 2. *For $i,j \geq 1$, the (i,j) entry of the matrix $A_n[x]^{-1}$ equals the (i,j) entry of the matrix $A_n[-x]$ and the entries in column zero of the inverse matrix is given by* $\left[1, -x, 0, \ldots, 0\right]^T$.

Proof. By Theorem 1, the column zero of $A_n[x]A_n[-x]$ is given by $[1,0,0,\ldots,0]^T$. Since

$$P_{n-1}[x]P_{n-1}[-x] = I_{n-1},$$

we obtain that $A_n[x]^{-1} = A_n[-x]$. □

Let $\mathrm{R_n}$ be a lower triangular matrix of order n with entries,

$$\begin{aligned} \mathrm{R_n}\{i,0\} &= [1,1,0,0,1,4,11,26,57,\ldots] \\ \mathrm{R_n}\{i,j\} &= -\binom{i-1}{j+1} - \binom{i-1}{j} + \binom{i-1}{j-1} \text{ for } i,j = 1,2,\ldots,n. \end{aligned}$$

Here, one can see that the numbers $\{0,0,1,4,11,26,57,\ldots\}$ in the zeroth column are the Eulerian numbers. Let F_n^* be a lower triangular matrix of order n defined by the entries

$$\begin{aligned} \mathrm{F}_n^*\{i,0\} &= [1,0,1,0,1,0,\ldots] \\ \mathrm{F}_n^*\{i,j\} &= \mathcal{F}_n[1]\{i,j\} \text{ for } i,j = 1,2,\ldots,n. \end{aligned}$$

Then we have the following result.

Theorem 3.

$$A_{n-1}[1] = \mathrm{R}_n \mathrm{F}_n^*.$$

Proof. By the definitions of the matrices $A_n[1], \mathrm{R}_n$ and F_n^*, we can easily prove that $A_{n-1}[1](\mathrm{F}_n^*)^{-1} = \mathrm{R}_n$. □

Let $Y_n[x]$ be the matrix with entries $Y_n[x]\{i,i\} = 1$ for $i \geq 0$, $Y_n[x]\{i,0\} = -\frac{(i-1)x^i}{i}$ for $i \geq 1$ and all others are 0. Let $Y_n[x]A_n[x] = \tilde{A}_n[x]$. Then we have that

$$\tilde{A}_n[x] = \begin{bmatrix} 1 & \mathbf{0} \\ \tilde{\mathbf{X}}_{n-1} & P_{n-1}[x] \end{bmatrix}$$

where $\tilde{\mathbf{X}}_{n-1} = \left[x, x^2/2, x^3/3, \ldots, x^n/n\right]^T$ and $P_{n-1}[x]$ is the generalization of the Pascal matrix P_{n-1} for one variable x.
Now, we investigate the properties of the matrix $\tilde{A}_n[x]$. The proofs can be done similary as in the proofs of Theorem 1 and Theorem 2.

Theorem 4. *For $i, j \geq 0$, we have $\tilde{A}_n[x]\tilde{A}_n[y] = \tilde{A}_n[x+y]$.*

The inverse of the matrix $\tilde{A}_n[x]$ is given as follows.

Theorem 5. *For $i, j \geq 0$, we have $\tilde{A}_n[x]^{-1} = \tilde{A}_n[-x]$.*

We will present some factorizations of the matrix $\tilde{A}_n[x]$. Let $U_n[x]$ be a lower triangular matrix defined by

$$\begin{aligned} U_n[x]\{i,0\} &= \tilde{A}_n[x]\{i,0\} \\ U_n[x]\{2,1\} &= 0 \text{ and } U_n[x]\{i,1\} = -\frac{a(i-2)x^{i-1}}{(i-1)!} \text{for } i \geq 3, \\ U_n[x]\{i,j\} &= x^{i-j} \text{ for } j \geq 2, \end{aligned}$$

where $a(n)$ is the $n-$th generalized Stirling number defined in (1.3), $\tilde{B}_n[x]$ be the matrix of the form

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \tilde{A}_n[x] \end{bmatrix},$$

and $F_m[x]$ be the matrix of the form

$$F_m[x] := \begin{bmatrix} I_{n-m-1} & \mathbf{0} \\ \mathbf{0} & U_m[x] \end{bmatrix} \text{ for } m = 1, 2, \ldots, n-1, \text{ and } F_n[x] := U_n[x].$$

Lemma 2. *For $m \geq 1$, $U_m[x]\tilde{B}_{m-1}[x] = \tilde{A}_m[x]$.*

Proof. For $m = 1$, we have $\tilde{B}_{m-1}[x]$ is the identity matrix and $\tilde{A}_m[x] = U_m[x]$. Let $m > 1$. One can see that $U_m[x]$ has a block matrix which is equal to the matrix $S_m[x]$ that corresponds to the Pascal matrix $P_m[x]$. The proof follows from the matrix product and properties of the matrices $S_m[x]$. □

Example 2.

$$\begin{aligned}
&U_5[x]\overline{B}_4[x]\\
&=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ x & 1 & 0 & 0 & 0 & 0\\ \frac{1}{2}x^2 & 0 & 1 & 0 & 0 & 0\\ \frac{1}{3}x^3 & -\frac{1}{2}x^2 & x & 1 & 0 & 0\\ \frac{1}{4}x^4 & -\frac{1}{6}5x^3 & x^2 & x & 1 & 0\\ \frac{1}{5}x^5 & -\frac{1}{24}26x^4 & x^3 & x^2 & x & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & x & 1 & 0 & 0 & 0\\ 0 & \frac{1}{2}x^2 & x & 1 & 0 & 0\\ 0 & \frac{1}{3}x^3 & x^2 & 2x & 1 & 0\\ 0 & \frac{1}{4}x^4 & x^3 & 3x^2 & 3x & 1 \end{bmatrix}\\
&=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ x & 1 & 0 & 0 & 0 & 0\\ \frac{1}{2}x^2 & x & 1 & 0 & 0 & 0\\ \frac{1}{3}x^3 & x^2 & 2x & 1 & 0 & 0\\ \frac{1}{4}x^4 & x^3 & 3x^2 & 3x & 1 & 0\\ \frac{1}{5}x^5 & x^4 & 4x^3 & 6x^2 & 4x & 1 \end{bmatrix}\\
&=\tilde{A}_5[x].
\end{aligned}$$

The following theorem is an immediate consequence of Lemma 2 and definition of the matrices $F_m[x]$.

Theorem 6. *The matrix $\tilde{A}_n[x]$ can be factorized by the matrices $F_n[x]$ as*

$$\tilde{A}_n[x] = F_n[x]F_{n-1}[x]\dots F_1[x].$$

Example 3. *Since*

$$\tilde{A}_5[x] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ x & 1 & 0 & 0 & 0 & 0\\ \frac{1}{2}x^2 & x & 1 & 0 & 0 & 0\\ \frac{1}{3}x^3 & x^2 & 2x & 1 & 0 & 0\\ \frac{1}{4}x^4 & x^3 & 3x^2 & 3x & 1 & 0\\ \frac{1}{5}x^5 & x^4 & 4x^3 & 6x^2 & 4x & 1 \end{bmatrix},$$

we can factorize this matrix as

$$F_5[x]F_4[x]F_3[x]F_2[x]F_1[x] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ x & 1 & 0 & 0 & 0 & 0\\ \frac{1}{2}x^2 & 0 & 1 & 0 & 0 & 0\\ \frac{1}{3}x^3 & -\frac{1}{2}x^2 & x & 1 & 0 & 0\\ \frac{1}{4}x^4 & -\frac{1}{6}5x^3 & x^2 & x & 1 & 0\\ \frac{1}{5}x^5 & -\frac{1}{24}26x^4 & x^3 & x^2 & x & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & x & 1 & 0 & 0 & 0\\ 0 & \frac{1}{2}x^2 & 0 & 1 & 0 & 0\\ 0 & \frac{1}{3}x^3 & -\frac{1}{2}x^2 & x & 1 & 0\\ 0 & \frac{1}{4}x^4 & -\frac{1}{6}5x^3 & x^2 & x & 1 \end{bmatrix}\times$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & x & 1 & 0 & 0\\ 0 & 0 & \frac{1}{2}x^2 & 0 & 1 & 0\\ 0 & 0 & \frac{1}{3}x^3 & -\frac{1}{2}x^2 & x & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & x & 1 & 0\\ 0 & 0 & 0 & \frac{1}{2}x^2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & x & 1 \end{bmatrix}$$

Let $V_n[x]$ be the lower triangular matrix defined by

$$\begin{aligned} V_n[x]\{i,i\} &= 1 \\ V_n[x]\{i,0\} &= -\tilde{A}_n[x]i,0 \text{ for } i \geq 1 \\ V_n[x]\{2,1\} &= \left[0,1,0,\frac{x^2}{2},\ldots,\frac{x^{n-1}}{n-1}\right]^T \\ V_n[x]\{i+1,i\} &= -x \text{ for } i \geq 2, \\ V_n[x]\{i,j\} &= 0 \text{ otherwise.} \end{aligned}$$

It is easy to see that $U_n[x]^{-1} = V_n[x]$. Then for the inverse of the matrix $\tilde{A}_n[x]$, we get

Lemma 3.

$$\begin{aligned} \tilde{A}_n[x]^{-1} &= F_1[x]^{-1}F_2[x]^{-1}\ldots F_n[x]^{-1} \\ &= G_1[x]^{-1}G_2[x]^{-1}\ldots G_n[x]^{-1} \end{aligned}$$

where

$$G_m[x] := \begin{bmatrix} I_{n-m-1} & \mathbf{0} \\ \mathbf{0} & V_m[x] \end{bmatrix} \text{ for } m = 1,2,\ldots,n-1, \text{ and } G_n[x] := V_n[x].$$

By Theorem 5, we have $\tilde{A}_n[x]^{-1}\{i,j\} = (-1)^{i+j}\tilde{A}_n[x]\{i,j\}$, we get the following result.

Theorem 7. *For a diagonal matrix* $J_n = \operatorname{diag}\left[1,-1,1,-1,\ldots,(-1)^n\right]$, *we have*

$$\tilde{A}_n[x]^{-1} = \tilde{A}_n[-x] = J_n\tilde{A}_n[x]J_n.$$

In the following result, we give another factorization of the matrix $\tilde{A}_n[x]$ with separating the variable x.

Theorem 8. *Let* $D_n[x] = \operatorname{diag}\left[1,x,x^2,x^3,\ldots,x^n\right]$ *be a diagonal matrix. For any positive integer* n *and any non-zero real number* x*, we have*

$$\tilde{A}_n[x] = D_n[x]\tilde{A}_n[1]D_n[x]^{-1}.$$

Now, we give a relation between the augmented Pascal matrix $\tilde{A}_n[x]$ and the Fibonacci matrix. Let $\mathrm{F}_n[x]$ be the matrix of order $n+1$ defined by

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathcal{F}_n[x] \end{bmatrix},$$

where $\mathcal{F}_n[x]$ is the matrix given in (1.2). For $\mathbf{Y}_{n-1} = \left[-x, x^2/2, -x^3/3, \ldots, (-1)^{i+1}x^n/n\right]^T$, which is also the $i-$th term of the vector $\tilde{\mathbf{X}}_{n-1}$ times $(-1)^{i+1}$, let $\mathrm{L}_n[x]$ be the matrix of order $n+1$ defined by

$$\begin{bmatrix} 1 & \mathbf{0} \\ \tilde{\mathbf{Y}}_{n-1} & \mathcal{L}_n[x] \end{bmatrix},$$

where $\mathcal{L}_n[x]$ is the Fibonacci Pascal triangle read by rows

$$\begin{aligned} \mathcal{L}_n[x]\{i,i\} &= 1 \\ \mathcal{L}_n[x]\{i,i-1\} &= -(i-1)x \\ \mathcal{L}_n[x]\{i,0\} &= \mathcal{L}_n[x]\{i-1,1\}x^2 \\ \mathcal{L}_n[x]\{i,k\} &= \mathcal{L}_n[x]\{i-1,k-1\} - \mathcal{L}_n[x]\{i-1,k\}x \quad \text{for } 0 < k < i-1. \end{aligned}$$

Then we have the following result.

Theorem 9.

$$\mathrm{F}_n[x] = \tilde{A}_{n-1}[x]\mathcal{L}_n[x].$$

Proof. By the definitions of the matrices $\tilde{A}_n[x], \mathcal{L}_n[x]$ and $\mathrm{F}_n[x]$, we can easily prove that $\tilde{A}_n[x]^{-1}\mathrm{F}_n[x] = \mathcal{L}_n[x]$. □

The analogous of the classical exponential function can also be defined as a matrix function for square matrices. This function is called matrix exponential. In other words, for any square matrix S , the exponential of S is defined to be the matrix

$$e^S = I + S + \frac{S^2}{2!} + \frac{S^3}{3!} + \ldots + \frac{S^k}{k!} + \ldots.$$

In the last part of this section, we will present the exponential of a special matrix and show a relation with $\tilde{A}_n[x]$. Let S_n be the matrix defined by

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & n-1 & 0 \end{bmatrix}. \tag{2.2}$$

Lemma 4. *For every nonnegative integer k, the entries of the matrix S_n^k are given by*

$$(S_n^k)_{i,j} = \begin{cases} \prod_{m=0}^{k-1} S_n\{m+j+1, m+j\}, & \text{if } i = j+k, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The proof will be done by induction on k. The case $k = 0$ follows straightforward. Let us assume the inductive hypothesis on $S_n^{k+1} = S_n S_n^k$. It is not hard to see for $i \neq j+k+1$, $(S_n^{k+1})_{i,j} = 0$. For $i = j+k+1$, we have

$$\begin{aligned} (S_n^{k+1})_{i,j} &= \left[S_n S_n^k\right]\{i,j\} = S_n\{j+k+1, j+k\} \prod_{m=0}^{k-1} S_n\{m+j+1, m+j\} \\ &= \prod_{m=0}^{k} S_n\{m+j+1, m+j\} \end{aligned}$$

and the proof is completed. □

Theorem 10. *For $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we have*

$$\tilde{A}_n[x] = e^{S_n x}.$$

Proof. Suppose that there is a matrix L_n such that $\tilde{A}_n[x] = e^{L_n x}$. Then we have

$$\frac{d}{dx}\tilde{A}_n[x] = L_n e^{L_n x} = L_n \tilde{A}_n[x]$$

and so

$$\frac{d}{dx}\tilde{A}_n[x] \mid_{x=0} = L_n.$$

Thus there is at most one matrix L_n such that $\frac{d}{dx}\tilde{A}_n[x] = e^{L_n x}$. It can be easily seen that $L_n = S_n$, where S_n is the matrix given in the definition 2.2, by calculating $\frac{d}{dx}\tilde{A}_n[x] \mid_{x=0}$. We conclude that $S_n^k = 0$ for $k \geq n+1$, thus

$$e^{S_n x} = \sum_{k=0}^{n} S_n^k \frac{x^k}{k!}.$$

For $i < j$, we see that $(e^{S_n x})\{i,j\} = 0$ and we also have $(e^{S_n x})\{i,i\} = 1$. Now, suppose that $i > j$ and let $i = j+k$.

$$(e^{S_n x})\{i,j\} = (S_n^k)_{i,j}\frac{x^k}{k!} = \frac{x^k}{k!}\prod_{m=0}^{k-1} S_n\{m+j+1, m+j\} = \tilde{A}_n[x]\{i,j\}.$$

Hence the proof is completed. □

Example 4. *We obtain the matrix $\frac{d}{dx}\tilde{A}_5[x]$ by taking the derivative of each entry of the matrix $\tilde{A}_5[x]$ with respect to x. Thus*

$$\frac{d}{dx}\tilde{A}_5[x] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 & 0 \\ x^2 & 2x & 2 & 0 & 0 & 0 \\ x^3 & 3x^2 & 6x & 3 & 0 & 0 \\ x^4 & 4x^3 & 12x^2 & 12x & 4 & 0 \end{bmatrix}.$$

Hence we have

$$S_5 = \frac{d}{dx}\tilde{A}_5[x] \mid_{x=0} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \end{bmatrix}$$

and

$$S_5^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1\times 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1\times 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\times 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3\times 4 & 0 & 0 \end{bmatrix},$$

$$S_5^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1\times 1\times 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1\times 2\times 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\times 3\times 4 & 0 & 0 & 0 \end{bmatrix},$$

$$S_5^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1\times 1\times 2\times 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1\times 2\times 3\times 4 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$S_5^5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1\times 1\times 2\times 3\times 4 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

At the end of this section, we will find the explicit inverse of the matrix $R_n[x] = [I_n - \lambda\tilde{A}_n[x]]^{-1}$ for all numbers $\mid \lambda \mid < 1$. So, we need the following result.

Lemma 5 ([10], Corollary 5.6.16)**.** *A matrix A of order n is nonsingular if there is a matrix norm $||\cdot||$ such that $||I - A|| < 1$. If this condition is satisfied,*

$$A^{-1} = \sum_{k=0}^{\infty}(I - A)^k.$$

Theorem 11. *The matrix $R_n[x]$ is defined for all numbers $\mid \lambda \mid < 1$. The entries of the matrix are*

$$(R_n[x])\{i,i\} = \frac{1}{1-\lambda}$$

and

$$(R_n[x])_{i,j} = (\tilde{A}_n[x])\{i,j\}Li_{j-i}(\lambda)$$

for $i > j$, where $Li_n(z)$ is the polylogarithm function

$$Li_n(z)) = \sum_{k=1}^{\infty}\frac{z^k}{k^n}.$$

Proof. The statement in Lemma 5 gives us that if $||\cdot||$ is a matrix norm and if $||A|| < 1$ for a square matrix of order n, then $I - A$ is invertible and $(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$. Then for $|\lambda| < 1$, we can write

$$(R_n[x])\{i,j\} = \sum_{k=0}^{\infty}(\tilde{A}_n[x])^k\lambda^k = \sum_{k=0}^{\infty}(\tilde{A}_n[xk])_{i,j}\lambda^k = (\tilde{A}_n[x])\{i,j\}\sum_{k=0}^{\infty}\lambda^k k^{i-j}.$$

We obtain the desired result by writing the sum for $i = j$ and $i > j$. □

Example 5.

$$\begin{aligned} I_4 - \lambda\tilde{A}_4[x] &= I_4 - \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 \\ x\lambda & \lambda & 0 & 0 & 0 \\ \frac{1}{2}\lambda x^2 & \lambda x & \lambda & 0 & 0 \\ \frac{1}{3}\lambda x^3 & \lambda x^2 & 2\lambda x & \lambda & 0 \\ \frac{1}{4}\lambda x^4 & \lambda x^3 & 3\lambda x^2 & 3\lambda x & \lambda \end{bmatrix} \\ &= \begin{bmatrix} 1-\lambda & 0 & 0 & 0 & 0 \\ -x\lambda & 1-\lambda & 0 & 0 & 0 \\ -\frac{1}{2}\lambda x^2 & -\lambda x & 1-\lambda & 0 & 0 \\ -\frac{1}{3}\lambda x^3 & -\lambda x^2 & -2\lambda x & 1-\lambda & 0 \\ -\frac{1}{4}\lambda x^4 & -\lambda x^3 & -3\lambda x^2 & -3\lambda x & 1-\lambda \end{bmatrix} \end{aligned}$$

The inverse of this matrix equals

$$\begin{bmatrix} \frac{1}{1-\lambda} & 0 & 0 & 0 & 0 \\ \frac{\lambda}{(1-\lambda)^2}x & \frac{1}{1-\lambda} & 0 & 0 & 0 \\ \frac{\lambda^2+\lambda}{2(1-\lambda)^3}x^2 & \frac{\lambda}{(1-\lambda)^2}x & \frac{1}{1-\lambda} & 0 & 0 \\ \frac{\lambda^3+4\lambda^2+\lambda}{3(1-\lambda)^4}x^3 & \frac{\lambda^2+\lambda}{(1-\lambda)^3}x^2 & \frac{\lambda}{(1-\lambda)^2}2x & \frac{1}{1-\lambda} & 0 \\ \frac{\lambda^4+11\lambda^3+11\lambda^2+\lambda}{4(1-\lambda)^5}x^4 & \frac{\lambda^3+4\lambda^2+\lambda}{(1-\lambda)^4}x^3 & \frac{\lambda^2+\lambda}{(1-\lambda)^3}3x^2 & \frac{\lambda}{(1-\lambda)^2}3x & \frac{1}{1-\lambda} \end{bmatrix}.$$

3. Conclusion

In the present paper we define a symmetric matrix and in its Cholesky factorization we see a matrix which has a connection with Pascal matrices. Based on the matrix in this factorization, we introduce two new matrices related to the Pascal matrix. We obtain several factorizations of these matrices and see the relations with Fibonacci matrices, Bernoulli polynomials and Stirling numbers.

Disclosure statement

We declare that we have no competing interests.

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