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GROUP OF ONE-DIMENSIONAL BOUNDED PURE PSEUDOREPRESENTATIONS OF A SIMPLE LIE GROUP

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Abstract. We continue the study of the group of one-dimensional bounded pure pseudorepresentations of a group (see [1], [2]). This group is computed here for simple connected simply connected Lie groups.

§ 1. Introduction

For the definitions, notation, and generalities concerning pseudocharacters, quasicharacters, pseudorepresentations, and quasirepresentations, and also for the definition of the Guichardet–Wigner pseudocharacter on a connected simply connected Hermitian symmetric simple Lie group, see [3]–[5]. The groups of one-dimensional bounded pure pseudorepresentations of a group were introduced in [1] and studied in [2].

Recall that the (tensor or, equivalently, ordinary pointwise) product of two one-dimensional pseudorepresentations of a group is a one-dimensional quasirepresentation. Indeed, if π and ρ are one-dimensional pseudorepresentations of G with $|\pi(g)| \le C_\pi$ for all $g \in G$ and $|\pi(gh) - \pi(g)\pi(h)| \le \varepsilon_\pi$ (ε_π is the defect of π) and if $|\rho(g)| \le C_\rho$ for all $g \in G$ and $|\rho(gh) - \rho(g)\rho(h)| \le \varepsilon_\rho$, then $C_\pi = 1$, since the restriction of π to every cyclic subgroup is an ordinary character of the subgroup, and a direct calculation shows that $\Delta \le C_\pi\varepsilon_\rho + C_\rho\varepsilon_\pi + \varepsilon_\pi\varepsilon_\rho = \varepsilon_\rho + \varepsilon_\pi + \varepsilon_\pi\varepsilon_\rho$.

This means that the set of bounded one-dimensional pseudorepresentations of a given group G is a group with respect to the ordinary pointwise

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product, whose identity element is the identity character $\iota(g) = 1 \in \mathbb{C}$, $g \in G$, and the inverse one-dimensional pseudorepresentation to a given bounded one-dimensional pseudorepresentation π is the pseudorepresentation π^{-1} given by the formula $\pi^{-1}(g) = \pi(g^{-1})$, $g \in G$. As in [2], denote this group by $\mathrm{BODP}(G)$.

Note that the additional condition that the restriction of a pseudorepresentation to every cyclic group is an ordinary representation is obvious both for $\pi\rho$ (since the product of ordinary characters is a character) and for π^{-1} (since the inverse of an ordinary character is a character).

§ 2. Preliminaries

The group $\mathrm{BODP}(G)$ of bounded one-dimensional pseudorepresentations of a group G contains the subgroup $\mathrm{BODPP}(G)$ introduced in [1]. An additional information can be found in [2], and a description of one-dimensional pure pseudorepresentations of almost connected Lie groups was obtained in [3].

As follows from the results of [4] (Lemma 3.3.10), for a semisimple Hermitian symmetric Lie group G, every element of $\mathrm{BODP}(G)$ belongs to the subgroup $\mathrm{GBODP}(G)$ of $\mathrm{BODPP}(G)$, and thus $\mathrm{BODP}(G) = \mathrm{BODPP}(G) = \mathrm{GBODP}(G)$.

The group $\mathrm{BODP}(G)$ obviously contains the group $\widehat{G}$ of ordinary characters of G as a subgroup. The subgroup $\mathrm{BODPP}(G)$ coincides with $\widetilde{G}$ if G is amenable, see Corollary 3.3.8 of [4].

The universal covering group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ has nontrivial bounded one-dimensional pseudorepresentations [4–6] but has no nontrivial characters.

Let us describe the group $\mathrm{BODP}(G)$ for simple connected simply connected Lie groups.

§ 3. Main theorem

Theorem 1. *Let G be a simple connected simply connected Hermitian symmetric Lie group. Then the group* $\mathrm{BODP}(G)$ *consists of the one-dimensional pseudorepresentations of G of the form $f_t\colon g \mapsto \exp it\chi(g)$, $g \in G$, where $t \in \mathbb{R}$ and χ stands for the Guichardet–Wigner pseudocharacter of G.*

Thus, the group $\mathrm{BODP}(G)$ *is isomorphic to* $\mathbb{R}$.

Proof. The one-dimensional pseudorepresentations of a simple connected simply connected Hermitian symmetric Lie group are described in [2], where the explicit form of these pseudorepresentations is given in terms of the Guichardet–Wigner character on G.

Since the product of these pseudorepresentations corresponds to the addition of the values of the parameter t, it follows that $\mathrm{BODP}(G)$ is isomorphic to $\mathbb{R}$, which completes the proof.

The group G has no nontrivial ordinary characters, and thus the subgroup $\widetilde{G}$ of ordinary characters of G is the identity subgroup.

Theorem 2. *Let G be a simple connected Lie group with finite center Z. Let $\widetilde{G}$ be the universal covering group of G, let $\widetilde{Z}$ be the center of $\widetilde{G}$.*

(i) If $\widetilde{Z}$ is finite, then the group $\mathrm{BODP}(G)$ is the identity group.

(ii) Let $\widetilde{Z}$ be infinite, and let z_0 be a generator of the subgroup $N \cap \widetilde{Z} = N$, where N stands for the kernel of the canonical mapping $\widetilde{G} \to G$. Then the group $\mathrm{BODP}(G)$ consists of the one-dimensional pseudorepresentations of G of the form $f_t \colon g \mapsto \exp it\chi(g)$, $g \in G$, where $t \in \mathbb{R}$ is such that $\exp it\chi(z_0) = 1$ and χ stands for a Guichardet–Wigner pseudocharacter of G (defined uniquely up to a nonzero real factor). Thus, the group $\mathrm{BODP}(G)$ is isomorphic to $\mathbb{Z}$.

Proof. The assertion (i) is obvious. Let us prove (ii).

By [3], every one-dimensional pseudorepresentation of $\widetilde{G}$ is an exponential of a Guichardet–Wigner pseudocharacter on $\widetilde{G}$. This pseudorepresentation of the group $\widetilde{G}$ is automatically pure [3].

This pseudorepresentation can be regarded as a pseudorepresentation of G if and only if it is equal to one on the kernel N of the canonical mapping $\widetilde{G} \to G$. Indeed, this means that the pseudorepresentation is an exponential of a Guichardet–Wigner pseudocharacter which is trivial on N, and thus is defined by a pseudocharacter on $\widetilde{G}/N \sim G$ [4].

This completes the proof of the theorem.

§ 4. Concluding remarks

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