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SOME RESULT ON r-TRUNCATED LAH NUMBERS AND r-TRUNCATED LAH-BELL POLYNOMIALS

SI-HYEON LEE AND LEE CHAE JANG

ABSTRACT. In this paper, we define r-truncated Lah numbers and Lah-Bell polynomials, and derived some properties of such numbers and polynomials. Furthermore, We investigate some identities related to a Poisson random variable and r-truncated Lah-Bell polynomials.

1. INTRODUCTION

It is well known that the Stirling number of the first kind $S_1(n,k)$,which counts the number of permutations of n elements consisting of k disjoint cycle is given by

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k, \quad (\text{see}[1-7]). \tag{1}$$

and

$$\frac{1}{k!}(log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \quad (\text{see}[1-7]). \tag{2}$$

Where

$$(x)_0 = 1(x)_n = (x)(x-1)\cdots(x-n+1), (n \geq 1).$$

The Stirling number of the second kind $S_2(n,k), (n \geq k \geq 0)$,which is the number of ways to partition a set with n elements into k non-empty subsets, is given by

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \quad (\text{see}[1-7]). \tag{3}$$

and

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \quad (\text{see}[1-7]). \tag{4}$$

A random variable X, taking on one of the values 0, 1, 2,. . . , is said to be the Poisson random variable with parameter $\alpha(>0)$, if the probability mass function of X is given by

$$p(i) = P\{X = i\} = e^{-\alpha}\frac{\alpha^i}{i!}(i = 0,1,2,\dots), \quad (\text{see}[1,6,9]). \tag{5}$$

Recall that Poisson random variable indicates how many events occurred within a given period of time. Let f be a real valued function, and let X be a Poisson random variable. Then we have

$$E[f(X)] = \sum_{i=0}^{\infty} f(i)p(i), \quad (\text{see}[1,6,9]), \tag{6}$$

where p is the probability mass function of X.

For $n,k \geq 0$,the unsigned Lah numbers $L(n,k)$ is given by

(7) $$L(n,k) = \binom{n-1}{k-1}\frac{n!}{k!}, \quad (\text{see}[1,2,3,5,8]).$$

The generating function of Lah numbers $L(n,k)$ is given by

(8) $$\frac{1}{k!}\left(\frac{t}{1-t}\right) = \sum_{n=k}^{\infty} L(n,k)\frac{t^n}{n!}, (k \geq 0), \quad (\text{see}[1,2,3,5,8]).$$

The Lah-Bell polynomials is given by

(9) $$e^{x\left(\frac{1}{1-t}-1\right)} = \sum_{n=0}^{\infty} B_n^L(x)\frac{t^n}{n!}, \quad (\text{see}[1,2,3,5]).$$

When $x = 1$,$B_n^L = B_n^L(1)$ are called Lah-Bell numbers.

The aim of this paper is to study r-truncated Lah numbers and r-truncated Lah-Bell polynomials. The outline of this paper is as follows. In Section 1, we introduce the Stirling numbers of the first kind, the Stirling numbers of second kind, Poisson random variable and the Lah numbers. In Section 2 , we define r-truncated Lah numbers and r-truncated Lah-Bell polynomials and we obtain some identities of them. Furthermore,we obtain some related identities to Poisson random variable and r-truncated Lah-Bell polynomials. In section 3, we specifically explain the meaning and value of the results we have obtained.

2. r-TRUNCATED LAH NUMBERS AND LAH-BELL POLYNOMIALS

In this section we define r-truncated Lah numbers. For $r,k \in$ Z, with $r \geq 0$, r-truncated Lah numbers which are defined by

(10) $$\frac{1}{k!}\left(\frac{t^r}{1-t}\right)^k = \sum_{n=kr}^{\infty} L^{(r)}(n,k)\frac{t^n}{n!}.$$

When $r = 1$, we note that $L^{(1)}(n,k) = L(n,k)$.

From (10), we note that

(11) $$\begin{aligned}\frac{1}{k!}\left(\frac{t^r}{1-t}\right)^k &= \sum_{n=0}^{\infty}\binom{n+k-1}{n}t^n \\ &= \frac{t^r}{k!}\sum_{n=0}^{\infty}\binom{n+k-1}{k-1}t^n \\ &= \frac{t^{r-k}}{k!}\sum_{n=0}^{\infty}\binom{n+k-1}{k-1}t^n \\ &= \frac{t^{r-k}}{k!}\sum_{n=kr}^{\infty}\binom{n-kr+k-1}{k-1}t^{n-kr} \\ &= \sum_{n=kr}^{\infty}\frac{n!}{k!}\binom{n+k(1-r)-1}{k-1}\frac{t^n}{n!}.\end{aligned}$$

Thus, For $r \in$ N and $k \geq 0, n \geq kr$,we have

(12) $$L^{(r)}(n,k) = \frac{n!}{k!}\binom{n+k(1-r)-1}{k-1}.$$

From (10), we have

$$\frac{1}{k!}\left(\frac{t^r}{1-t}\right)^k = \frac{t^{k(r-1)}}{k!}\left(\frac{t}{1-t}\right)^k = t^{k(r-1)}\sum_{n=k}^{\infty}L(n,k)\frac{t^n}{n!} = \sum_{n=k}^{\infty}L(n,k)\frac{t^{n+k(r-1)}}{n!} = \sum_{n=kr}^{\infty}L(n-k(r-1),k)\frac{n!}{(n-k(r-1))!}\frac{t^n}{n!}. \tag{13}$$

Thus, by comparing the coefficients of (10) and (13), we have the following theorem.

Theorem 1. *For $r \in \mathrm{N}$ and $k \geq 0$, we have*

$$L^{(r)}(n,k) = \begin{cases} L(n-k(r-1),k)\frac{n!}{(n-k(r-1))!} & \text{if } n \geq kr, \\ 0, & \text{if } 0 < n < kr. \end{cases}$$

By (10),we have

$$\frac{1}{k!}\left(\frac{t^r}{1-t}\right)^k = \frac{t^{rk}}{k!}\left(\frac{1}{1-t}\right)^k = \frac{t^{rk}}{k!}(1-t)^{-k} = \frac{t^{rk}}{k!}\sum_{n=0}^{\infty}\langle k\rangle_n\frac{t^n}{n!} = \frac{1}{k!}\sum_{n=rk}^{\infty}\langle k\rangle_{n-rk}(n)_{rk}\frac{t^n}{n!}. \tag{14}$$

Comparing the coefficients of both sides (10) and (14),we have the following theorem.

Theorem 2. *For $r \in \mathrm{N}$, $n,k \geq 0$,we have*

$$L^{(r)}(n,k) = \begin{cases} \frac{1}{k!}\langle k\rangle_{n-rk}(n)_{rk} & \text{if } n \geq kr, \\ 0, & \text{if } 0 < n < kr. \end{cases}$$

From Theorem 2 and (1),we note that

$$L^{(r)}(n,k) = \frac{1}{k!}\sum_{l=0}^{rk}S_1(rk,l)n^l\langle k\rangle_{n-rk}. \tag{15}$$

Replacing t by $1-e^{-t}$ in (10), we get

$$\sum_{k=lr}^{\infty}L^{(r)}(k,l)\frac{(1-e^{-t})^k}{k!} = \frac{1}{e^{-t}}\frac{(1-e^{-t})^{rl}}{l!}. \tag{16}$$

From the left hand side of (16),we note

$$\sum_{k=rl}^{\infty} L^{(r)}(k,l)\frac{(1-e^{-t})^k}{k!} = \sum_{k=rl}^{\infty} L^{(r)}(k,l)(-1)^k \sum_{n=k}^{\infty}(-1)^n S_2(n,k)\frac{t^n}{n!}$$
$$= \sum_{n=rl}^{\infty}\sum_{k=rl}^{n}(-1)^{n+k} L^{(r)}(k,l) S_2(n,k)\frac{t^n}{n!}. \tag{17}$$

From the other hand side of (16), we observe

$$\frac{1}{e^{-t}}\frac{(1-e^{-t})^{rl}}{l!} = e^t \frac{(rl)!}{l!}\frac{(1-e^{-t})^{rl}}{(rl)!}$$
$$= \frac{(rl)!}{l!}\sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{k=rl}^{\infty}(-1)^{rl+k}S_2(k.rl)\frac{t^k}{k!}$$
$$= \frac{(rl)!}{l!}\sum_{n=rl}^{\infty}\sum_{k=rl}^{n}(-1)^{rl+k}S_2(k,rl)\frac{n!}{k!(n-k)!}\frac{t^n}{n!}$$
$$= \sum_{n=rl}^{\infty}\sum_{k=rl}^{n}(-1)^{rl+k}\binom{n}{k}S_2(k,rl)\frac{t^n}{n!}. \tag{18}$$

By comparing the both sides (17) and (18), we have the following theorem.

Theorem 3. *For $n,k,l \geq 0$,we have*

$$(-1)^{n+k}L^{(r)}(k,l)S_2(n,k) = (-1)^{rl+k}\binom{n}{k}S_2(k,rl).$$

Now, the r-truncated Lah-Bell polynomials are defined by

$$e^{x(\frac{t^r}{1-t})} = \sum_{n=kr}^{\infty} B_n^{L,r}(x)\frac{t^n}{n!}. \tag{19}$$

From (10) and (19),we observe

$$e^{x(\frac{t^r}{1-t})} = \sum_{k=0}^{\infty}\frac{x^k}{k!}\left(\frac{t^r}{1-t}\right)^k$$
$$= \sum_{k=0}^{\infty}x^k\sum_{n=kr}^{\infty}L^{(r)}(n,k)\frac{t^n}{n!}$$
$$= \sum_{n=kr}^{\infty}\left(\sum_{k=0}^{[\frac{n}{r}]}L^{(r)}(n,k)\right)x^k\frac{t^n}{n!}. \tag{20}$$

By (19) and (20), we have the following theorem.

Theorem 4. *For $n,k \geq 0$,we have*

$$B_n^{L,r}(x) = \sum_{k=0}^{[\frac{r}{n}]}L^{(r)}(n,k)x^k.$$

Let X be the Poisson random variable with parameter $\alpha(>0)$,and we note that

$$(21)\qquad E\left[\left(\frac{t^r}{1-t}\right)^X\right]=\sum_{i=0}^{\infty}\left(\frac{t^r}{1-t}\right)^i e^{-\alpha}\frac{\alpha^i}{i!}$$
$$=e^{\alpha(\frac{t^r}{1-t}-1)}$$
$$=e^{-\alpha}\sum_{n=rX}^{\infty}B_n^{L,r}(\alpha)\frac{t^n}{n!}.$$

From the left hand side of (21), we observe that

$$(22)\qquad E\left[\left(\frac{t^r}{1-t}\right)^X\right]=E\left[t^{rX}\left(\frac{1}{1-t}\right)^X\right]$$
$$=\sum_{n=0}^{\infty}E\left[\langle X\rangle_n\right]\frac{t^{n+rX}}{n!}$$
$$=\sum_{n=rX}^{\infty}E\left[\langle X\rangle_{n-rX}\right]\frac{n!}{(n-rX)!}\frac{t^n}{n!}$$
$$=\sum_{n=rX}^{\infty}E\left[\langle X\rangle_{n-rX}\right](n)_{rX}\frac{t^n}{n!}.$$

By comparing the coefficients on both sides of (21) and (22),we have the following theorem.

Theorem 5. *Let X be the Poisson random variable with parameter $\alpha(>0)$. For $r\in\mathrm{N}$, and $n\geq 0$,we have*

$$B_n^{L,r}(\alpha)=e^{\alpha}E\left[\langle X\rangle_{n-rX}\right](n)_{rX}.$$

From Theorem 4 and Theorem 5 ,we note that

$$(23)\qquad \sum_{k=0}^{[\frac{n}{r}]}x^kL^{(r)}(n,k)=e^{\alpha}E\left[\langle k\rangle_{n-rk}\right](n)_{rk}.$$

Let X be a poisson random variable with parameter $\alpha \geq 0$, and we calculate the variance $Var\left(\left(\frac{t^r}{1-t}\right)^k\right)$ as follows.

$$(24)\qquad Var\left(\left(\frac{t^r}{1-t}\right)^k\right) = \sum_{i=0}^{\infty}\left(\frac{t^r}{1-t}\right)^{2i} e^{-\alpha}\frac{\alpha^i}{i!} - \left(\sum_{i=0}^{\infty}\left(\frac{t^r}{1-t}\right)^{i} e^{-\alpha}\frac{\alpha^i}{i!}\right)^2$$
$$= e^{\alpha\left(\left(\frac{t^r}{1-t}\right)^2-1\right)} - \left(e^{\alpha\left(\left(\frac{t^r}{1-t}\right)-1\right)}\right)^2$$
$$= e^{-\alpha}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{t^r}{1-t}\right)^{2k}\alpha^k - \sum_{n=kr}^{\infty} B_n^{L,r}(2\alpha)\frac{t^n}{n!}\right)$$
$$= e^{-\alpha}\left(\sum_{k=0}^{\infty}\frac{(2k)!}{k!}\frac{1}{(2k)!}\left(\frac{t^r}{1-t}\right)^{2k}\alpha^k - \sum_{n=kr}^{\infty} B_n^{L,r}(2\alpha)\frac{t^n}{n!}\right)$$
$$= e^{-\alpha}\left(\sum_{k=0}^{\infty}\frac{(2k)!}{k!}\sum_{n=2kr}^{\infty} L^{(r)}(n,2k)\alpha^k\frac{t^n}{n!} - \sum_{n=kr}^{\infty} B_n^{L,r}(2\alpha)\frac{t^n}{n!}\right)$$
$$= e^{-\alpha}\left(\sum_{n=kr}^{\infty}\sum_{k=0}^{\left[\frac{r}{2n}\right]}\frac{(2k)!}{k!}L^{(r)}(n,2k)\alpha^k\frac{t^n}{n!} - \sum_{n=kr}^{\infty} B_n^{L,r}(2\alpha)\frac{t^n}{n!}\right)$$
$$= e^{-\alpha}\sum_{n=kr}^{\infty}\left(\sum_{k=0}^{\left[\frac{r}{2n}\right]}\frac{(2k)!}{k!}L^{(r)}(n,2k)\alpha^k - B_n^{L,r}(2\alpha)\right)\frac{t^n}{n!}.$$

Thus we obtain the following theorem.

Theorem 6. *Let X be a poisson random variable with parameter $\alpha \geq 0$, For $k \geq 0$,we have*

$$Var\left(\left(\frac{t^r}{1-t}\right)^k\right) = e^{-\alpha}\sum_{n=kr}^{\infty}\left(\sum_{k=0}^{\left[\frac{r}{2n}\right]}\frac{(2k)!}{k!}L^{(r)}(n,2k)\alpha^k - B_n^{L,r}(2\alpha)\right)\frac{t^n}{n!}.$$

3. Conclusion

First, we defined r-truncated Lah numbers and r-Lah-Bell polynomials and obtained. In Theorem 1,2, we obtained some identities r-truncated Lah numbers.we also investigated Theorem 3 related to r-truncated Lah number and the Stirling numbers of the second kind. Second, we defined r-truncated Lah-Bell polynomials and obtained Theorem 4 related to Lah-Bell polynomials and r-truncated Lah numbers. Third, we obtained Theorem 5 related to r-truncated Lah-Bell polynomials and the expectation of the Poisson random variable. Finally, we investigated Theorem 6 related to the variance of Poisson random variable and, r-truncated Lah numbers and r-truncated Lah-Bell polynomials.

References

[1] Kim T, Kim DS, Dolgy DV, et al. Degenerate binomial and poisson random variables associated with degenerate Lah-Bell polynomials. Open Math.2021;19(1):1588–1597. doi:10.1515/math-2021-0116.

[2] Kim DS, Kim T. r-extended Lah-Bell numbers and polynomials associated with r-Lah numbers.Proc Jangjeon Math Soc.2021;24:1–10.

[3] Ma Y, Kim DS, Kim T, et al. Some identities of Lah-Bell polynomials. Adv Differ Equ.2020;2020(510):1–10.doi: 10.1186/s13662-020-02966-6.

[4] Kim T, Kim DS. Some identities on truncated polynomials associated with degenerate Bell polynomials. Russ J Math Phys.2021;28:342–355. doi:10.1134/S1061920821030079
[5] Kim, D.S. Kim, T. Lah-Bell numbers and polynomials, Proc. Jangjeon Math. Soc. 23 (2020), no. 4.
[6] Kim T, Kim DS, Kwon J, et al. A note on identities involving special numbers andmoments of poisson random variable. Adv Stud Contemp Math.2022;32(4):457–464.doi:10.17777/ascm2022.32.4.457.
[7] Kim T, Kim DS, Lee H, et al. A study on properties of degenerate and zero-truncated degenerate poisson random variables. Mathematics.2021;2021:1–10.doi: 10.48550/arXiv.2106.13481.
[8] Nyul, G.; R ´acz, G. The r-Lah numbers, Discrete Math. 338 (2015), no. 10, 1660–1666.
[9] S. M. Ross, Introduction to Probability Models, 20th edition, Academic Press, London, 2019.

KWANGWOON UNIVERSITY, SEOUL 139-701, REPUBLIC OF KOREA
Email address: ugug11@naver.com

GRADUATE SCHOOL OF EDUCATION, KONKUK UNIVERSITY,SEOUL,05029,REPUBLIC OF KOREA
Email address: lcjang@konkuk.ac.kr