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The new type degenerate Fubini polynomials

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1 Abstract

Recently, T. Kim has investigated degenerate Fubini polynomials, revealing various theorems and diverse relationships with Euler polynomials. In this paper, we extend the exploration of properties associated with additional degenerate Fubini polynomials. We introduce new types of degenerate Fubini polynomials and examine their properties. Additionally, we will explore properties related to Bernoulli polynomials and estimate polynomial values by substituting specific values.

Key words: degenerate exponential, Fubini polynomials, Bernoulli numbers, new type Fubini polynomials, primitive polynomials

2 Introduction

For any $\lambda \in \mathbb{R}$ it is well known that degenerate exponentials are defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} \tag{1}$$

$$= \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} \quad (\text{See}[1]). \tag{2}$$

where $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\dots(x-(n-1)\lambda)$, $(x)_{0,\lambda} = 1$, here $n \in \mathbb{N}$ Note that $\lim_{\lambda\to 0} e_\lambda^x(t) = e^{xt}$ The Fubini numbers are

$$F_n = \sum_{k=0}^{n} k! S_2(n,k) \quad (\text{See}[2]). \tag{3}$$

It is the number of possible ways to write the Fubini formula for a summation of integration of order n.(See [2]).

$$F_n(x) = \sum_{k=0}^{n} k! S_2(n,k) x^k \quad (\text{See}[2]). \tag{4}$$

There are many variants of Fubini numbers and polynomials. It is well known that generating function of Fubini polynomials is defined by

$$\frac{1}{1-y(e^t-1)} = \sum_{n=0}^{\infty} F_n(y) \frac{t^n}{n!} \quad (\text{See}[2][3][4]). \tag{5}$$

And degenerate Fubini polynomials are defined by

$$\frac{1}{1-y(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} F_{n,\lambda}(y) \frac{t^n}{n!} \quad (\text{See}[2][3][4]). \tag{6}$$

Substituting -y instead of y in (6), we get

$$\frac{1}{1+y(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} F_{n,\lambda}(-y) \frac{t^n}{n!}. \tag{7}$$

3 Degenerate Fubini polynomials

In this section, we examine some properties which not investigated in [2]. We aim to examine several equations related to the degenerate Fubini polynomials (6). Firstly, as we have examined the equation with a specific value of y in the previous section, we attempt to substitute $y = -1$ into the original equation of the degenerate Fubini polynomials. At (6) we note that degnerate Fubini polynomials are defined by $\frac{1}{1-y(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} F_{n,\lambda}(y)\frac{t^n}{n!}$, let's substitution y to -1, then we get

$$\frac{1}{e_\lambda(t)} = \sum_{n=0}^{\infty} F_{n,\lambda}(-1)\frac{t^n}{n!}. \tag{8}$$

By equation (8), we get the following result:

$$\begin{aligned} 1 &= e_\lambda(t)\sum_{n=0}^{\infty} F_{n,\lambda}(-1)\frac{t^n}{n!} \\ &= (\sum_{l=0}^{\infty}\frac{(1)_{l,\lambda}}{l!}t^l)(\sum_{m=0}^{\infty} F_{m,\lambda}(-1)\frac{t^m}{m!}) \quad \text{(By(2))} \\ &= \sum_{n=0}^{\infty}(\sum_{m=0}^{n}\binom{n}{m}(1)_{n-m,\lambda}F_{m,\lambda}(-1)))\frac{t^n}{n!} \quad (n = m + l). \end{aligned} \tag{9}$$

Theorem 3.1 $\sum_{n=0}^{\infty}(\sum_{m=0}^{n}\binom{n}{m}(1)_{n-m,\lambda}F_{m,\lambda}(-1))\frac{t^n}{n!} = 1.$

From above theorem, $F_{0,\lambda}(-1) = 1$, $\sum_{n=1}^{\infty}(\sum_{m=0}^{n}\binom{n}{m}(1)_{n-m,\lambda}F_{m,\lambda}(-1))\frac{t^n}{n!} = 0.$

Theorem 3.2 $\frac{ty}{1-y^2(e_\lambda(t)-1)^2} = \sum_{n=0}^{\infty}(\sum_{m=0}^{n}\binom{n}{m}B_{n-m,\lambda}\cdot\frac{F_{m,\lambda}(y)-F_{m,\lambda}(-y)}{2})\frac{t^n}{n!}.$
Here $B_{n,\lambda}$ is degenerate Bernoulli number. (See [5]).

Also we can get some relation on Fubini polynomials and degenerate Bernoulli numbers.(See [5] [3]). We can easily prove that

$$\frac{y}{1-y^2(e_\lambda(t)-1)^2} = \frac{1}{2(e_\lambda(t)-1)}\sum_{m=0}^{\infty}(F_{m,\lambda}(y)-F_{m,\lambda}(-y))\frac{t^n}{n!}. \tag{10}$$

And in [5],[3] it is known that degenerate Bernoulli numbers are defined by

$$\frac{t}{e_\lambda(t)-1} = \sum_{l=0}^{\infty} B_{l,\lambda}\frac{t^l}{l!}. \tag{11}$$

Now, lets multiply both sides on (10) by t, then we get

$$\begin{aligned} \frac{ty}{1-y^2(e_\lambda(t)-1)^2} &= \frac{t}{2(e_\lambda(t)-1)}\sum_{m=0}^{\infty}(F_{m,\lambda}(y)-F_{m,\lambda}(-y))\frac{t^m}{m!} \\ &= (\sum_{l=0}^{\infty} B_{l,\lambda}\frac{t^l}{l!})(\sum_{m=0}^{\infty}\frac{F_{m,\lambda}(y)-F_{m,\lambda}(-y)}{2}\frac{t^m}{m!}) \\ &= \sum_{n=0}^{\infty}(\sum_{m=0}^{n}\binom{n}{m}B_{n-m,\lambda}\frac{F_{m,\lambda}(y)-F_{m,\lambda}(-y)}{2})\frac{t^n}{n!} \quad (n = m + l). \end{aligned}$$

By comparing the coefficients on both sides we get following theorem.

Theorem 3.3

$$\frac{ty}{1-y^2(e_\lambda(t)-1)^2}=\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\binom{n}{m}B_{n-m,\lambda}\frac{F_{m,\lambda}(y)-F_{m,\lambda}(-y)}{2}\Big)\frac{t^n}{n!}.$$

Through the above theorem, we have learned about the relationship between degenerate Fubini polynomials and degenerate Bernoulli numbers. By comparing on both sides we get a following result, we can observe that only one term remains and the rest of the terms become zero when $n = 1$. Therefore,

$$\Big(\binom{1}{0}B_{1,\lambda}\frac{F_{0,\lambda}(y)-F_{0,\lambda}(-y)}{2}+\binom{1}{1}B_{0,\lambda}\frac{F_{1,\lambda}(y)-F_{1,\lambda}(-y)}{2}\Big)t=\Big(B_{1,\lambda}\frac{F_{0,\lambda}(y)-F_{0,\lambda}(-y)}{2}+B_{0,\lambda}\frac{F_{1,\lambda}(y)-F_{1,\lambda}(-y)}{2}\Big)t \tag{12}$$

$$=\frac{ty}{1-y^2(e_\lambda(t)-1)^2}. \tag{13}$$

$$\therefore\ \frac{y}{1-y^2(e_\lambda(t)-1)^2}=\frac{1}{2}(B_{1,\lambda}(F_{0,\lambda}(y)-F_{0,\lambda}(-y))+B_{0,\lambda}(F_{1,\lambda}(y)-F_{1,\lambda}(-y))).$$

Furthermore, by substituting $y = 1$ into the above theorem, we can deduce the following result.

$$\frac{t}{1-(e_\lambda(t)-1)^2}=\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\binom{n}{m}B_{n-m,\lambda}\frac{F_{m,\lambda}(1)-F_{m,\lambda}(-1)}{2}\Big)\frac{t^n}{n!}. \tag{14}$$

And also

$$\begin{aligned}\frac{t}{1-(e_\lambda(t)-1)^2}&=-t\Big(\frac{1}{(e_\lambda(t)-1)^2-1}\Big)\\&=-t\Big(\frac{1}{e_\lambda(t)-1+1}\Big)\Big(\frac{1}{e_\lambda(t)-1-1}\Big)\\&=-t\Big(\frac{1}{e_\lambda(t)}\Big)\Big(\frac{1}{e_\lambda(t)-2}\Big)\\&=\frac{t}{2}\Big(\frac{1}{e_\lambda(t)}-\frac{1}{e_\lambda(t)-2}\Big).\end{aligned} \tag{15}$$

By (14), (15) and above equation, we obtain following result.

Theorem 3.4

$$\frac{1}{e_\lambda(t)}-\frac{1}{e_\lambda(t)-2}=\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\binom{n}{m}B_{n-m,\lambda}(F_{m,\lambda}(1)-F_{m,\lambda}(-1))\Big)\frac{t^{n-1}}{n!}. \tag{16}$$

We consider new type Fubini polynomials defined by

$$\frac{1}{1+y(e^t+1)}=\sum_{n=0}^{\infty}F_n^*(y)\frac{t^n}{n!}. \tag{17}$$

From (4) we note that

$$\begin{aligned}\frac{1}{1+y(e^t+1)}&=\frac{1}{1+y(e^t-1+2)}\\&=\frac{1}{1+2y+y(e^t-1)}\\&=\frac{1}{1+2y}\cdot\frac{1}{1+\frac{y}{1+2y}(e^t-1)}\\&=\frac{1}{1+2y}\cdot\sum_{n=0}^{\infty}F_n\Big(-\frac{y}{1+2y}\Big)\frac{t^n}{n!}.\end{aligned}$$

By (17) and above equation we obtain the following theorem.

Theorem 3.5 *For $n \geq 0$, we have*

$$F_n^*(y) = \frac{1}{1+2y} F_n(-\frac{y}{1+2y}). \tag{18}$$

4 The properties of new type degenrate Fubini polynomials

We define new type degenerate Fubini polynomials $G_{n,\lambda}(y)$. It's generating function is defined by

$$\frac{1}{1+y(e_\lambda(t)+1)} = \sum_{n=0}^{\infty} G_{n,\lambda}(y) \frac{t^n}{n!}. \tag{19}$$

We have examined the degenerate Fubini polynomials and the degenerate expression for $G_{n,\lambda}(y)$ thus far. Now, we aim to explore the correlation between these two expressions by means of the following approach:

$$\begin{aligned} \frac{1}{1+y(e_\lambda(t)+1)} &= \frac{1}{1+y(e_\lambda(t)-1+2)} \\ &= \frac{1}{1+2y+y(e_\lambda-1)} \\ &= \frac{1}{1+2y} \cdot \frac{1}{1+\frac{y}{1+2y}(e_\lambda(t)-1)} \\ &= \frac{1}{1+2y} \cdot \sum_{n=0}^{\infty} F_{n,\lambda}(-\frac{y}{1+2y}) \frac{t^n}{n!}. \end{aligned} \tag{20}$$

We can obtain the following result by comparing on both sides (19) and (20).

Theorem 4.1

$$G_{n,\lambda}(y) = \frac{1}{1+2y} F_{n,\lambda}(-\frac{y}{1+2y}). \tag{21}$$

In view of (19), we observe that

$$\begin{aligned} \frac{1}{1+y(e_\lambda(t)+1)} - \frac{1}{1-y(e_\lambda(t)+1)} &= \frac{-2y(e_\lambda(t)+1)}{1-y^2(e_\lambda(t)+1)^2} \\ &= \sum_{n=0}^{\infty} (G_{n,\lambda}(y) - G_{n,\lambda}(-y)) \frac{t^n}{n!}. \end{aligned} \tag{22}$$

And then we get

$$\frac{y(e_\lambda(t)+1)}{1-y^2(e_\lambda(t)+1)^2} = -\sum_{n=0}^{\infty} (\frac{G_{n,\lambda}(y) - G_{n,\lambda}(-y)}{2}) \frac{t^n}{n!}. \tag{23}$$

Also, we can derive the following theorem by modifying the equation (23):

$$\begin{aligned}
\frac{y(e_\lambda(t)+1)}{1-y^2(e_\lambda(t)+1)^2} &= \frac{1+y(e_\lambda(t)+1)-1}{1-y^2(e_\lambda(t)+1)^2} \\
&= \frac{1}{1-y(e_\lambda(t)+1)} - \frac{1}{1-y^2(e_\lambda(t)+1)^2} \\
&= \frac{1}{1-y(e_\lambda(t)+1)}(1-\frac{1}{1+y(e_\lambda(t)+1)}) \\
&= \sum_{l=0}^{\infty} G_{l,\lambda}(-y)\frac{t^l}{l!}(1-\sum_{m=0}^{\infty} G_{m,\lambda}(y)\frac{t^m}{m!}) \\
&= \sum_{l=0}^{\infty} G_{l,\lambda}(-y)\frac{t^l}{l!} - (\sum_{l=0}^{\infty} G_{l,\lambda}(-y)\frac{t^l}{l!})(\sum_{m=0}^{\infty} G_{m,\lambda}(y)\frac{t^m}{m!}) \\
&= \sum_{n=0}^{\infty} G_{n,\lambda}(-y)\frac{t^n}{n!} - \sum_{n=0}^{\infty}(\sum_{m=0}^{n} G_{n-m,\lambda}(-y)G_{m,\lambda}(y))\frac{t^n}{(n-m)!m!} \qquad (n=m+l) \\
&= \sum_{n=0}^{\infty} G_{n,\lambda}(-y)\frac{t^n}{n!} - \sum_{n=0}^{\infty}(\sum_{m=0}^{n} \binom{n}{m} G_{n-m,\lambda}(-y)G_{m,\lambda}(y))\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}(G_{n,\lambda}(y) - \sum_{m=0}^{n} \binom{n}{m} G_{n-m,\lambda}(-y)G_{m,\lambda}(y))\frac{t^n}{n!}. \qquad (24)
\end{aligned}$$

We get comparing on both sides on (23), (24)

$$-\frac{G_{n,\lambda}(y)-G_{n,\lambda}(-y)}{2} = G_{n,\lambda}(-y) - \sum_{m=0}^{n}\binom{n}{m} G_{n-m,\lambda}(-y)G_{m,\lambda}(y). \qquad (25)$$

So, we get the following theorem.

Theorem 4.2

$$\sum_{m=0}^{n}\binom{n}{m} G_{n-m,\lambda}(-y)G_{m,\lambda}(y) = \frac{G_{n,\lambda}(y)+G_{n,\lambda}(-y)}{2}. \qquad (26)$$

5 The properties of primitive polynomials of new type degenerate Fubini polynomails

Next, we aim to define another polynomial by utilizing integration to express the degenerate $G_{n,\lambda}(y)$ function. We have

$$\begin{aligned}
\int \frac{1}{1+y(e_\lambda(t)+1)}dy &= \frac{1}{e_\lambda(t)+1} log(1+y(e_\lambda(t)+1)) \qquad (27) \\
&= \sum_{n=0}^{\infty} U_{n,\lambda}(y)\frac{t^n}{n!}. \qquad (28)
\end{aligned}$$

By examining the correlation between the newly defined polynomials, the existing Fubini polynomials, and $G_{n,\lambda}(y)$, we can obtain the following results.

$$\begin{aligned}
\int \frac{1}{1+y(e_\lambda(t)+1)}dy &= \int \frac{1}{1+2y}\cdot\sum_{n=0}^{\infty} F_{n,\lambda}(-\frac{y}{1+2y})\frac{t^n}{n!}dy \\
&= \sum_{n=0}^{\infty}\frac{t^n}{n!}\int \frac{1}{1+2y}\cdot F_{n,\lambda}(-\frac{y}{1+2y})dy \\
&= \sum_{n=0}^{\infty} U_{n,\lambda}(y)\frac{t^n}{n!}.
\end{aligned}$$

and

$$\int \frac{1}{1+y(e_\lambda(t)+1)}dy = \int \sum_{n=0}^{\infty} G_{n,\lambda}(y)\frac{t^n}{n!} \tag{29}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \int G_{n,\lambda}.(y) \tag{30}$$

So by (28), (29), (30)

$$U_{n,\lambda}(y) = \int \frac{1}{1+2y} \cdot F_{n,\lambda}(-\frac{y}{1+2y})dy \tag{31}$$

$$= \int G_{n,\lambda}(y)dy. \tag{32}$$

By (27), (28), using the properties of $U_{n,\lambda}(y)$ and logarithm that we defined, we get

$$\int \sum_{m=0}^{n} \binom{n}{m} G_{n-m,\lambda}(-y)G_{m,\lambda}(y)dy = \int \frac{G_{n,\lambda}(y)+G_{n,\lambda}(-y)}{2}dy \tag{33}$$

$$= \frac{1}{2}\int G_{n,\lambda}(y)dy + \frac{1}{2}\int G_{n,\lambda}(-y)dy \tag{34}$$

$$= \frac{1}{2}U_{n,\lambda}(y) + \frac{1}{2}U_{n,\lambda}(-y). \tag{35}$$

And also

$$\begin{aligned}
\sum_{n=0}^{\infty}\int (\sum_{m=0}^{n} \binom{n}{m} G_{n-m,\lambda}(-y)G_{m,\lambda}(y)dy)\frac{t^n}{n!} &= \frac{1}{2}(\sum_{n=0}^{\infty} U_{n,\lambda}(y)\frac{t^n}{n!} + \sum_{n=0}^{\infty} U_{n,\lambda}(-y)\frac{t^n}{n!}) \\
&= \frac{1}{2}\cdot\frac{1}{e_\lambda(t)+1}log(1+y(e_\lambda(t)+1)) + \frac{1}{2}\cdot\frac{1}{e_\lambda(t)+1}log(1-y(e_\lambda(t)+1)) \\
&= \frac{1}{2(e_\lambda(t)+1)}log(1-y^2(e_\lambda(t)+1)^2) \\
&= \frac{1}{2(e_\lambda(t)+1)}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(-y^2(e_\lambda+1)^2)^n \qquad (log(1+t) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}t^n) \\
&= -\frac{1}{2(e_\lambda(t)+1)}\sum_{n=1}^{\infty}\frac{1}{n}(y^2(e_\lambda(t)+1)^2)^n \\
&= -\frac{1}{2(e_\lambda(t)+1)}\sum_{n=0}^{\infty}\frac{1}{n+1}(y^2(e_\lambda(t)+1)^2)^{n+1}. \qquad (36)
\end{aligned}$$

So, we obtain

Theorem 5.1

$$\int \sum_{m=0}^{n} \binom{n}{m} G_{n-m,\lambda}(-y)G_{m,\lambda}(y)dy = -\frac{1}{2(e_\lambda(t)+1)}\cdot\frac{(y^2(e_\lambda(t)+1)^2)^{n+1}}{n+1}. \tag{37}$$

By examining the degenerate $G_{n,\lambda}(y)$ polynomial, we derived the two theorems mentioned above. This will be helpful for future research. We have

$$\frac{1}{e_\lambda(t)+1}log(1+y(e_\lambda(t)+1)) - \frac{1}{e_\lambda(t)+1}log(1-y(e_\lambda(t)+1)) = \sum_{n=0}^{\infty}(U_{n,\lambda}(y)-U_{n,\lambda}(-y))\frac{t^n}{n!}. \tag{38}$$

and

$$\begin{aligned}\frac{1}{e_\lambda(t)+1}log(1+y(e_\lambda(t)+1)) - \frac{1}{e_\lambda(t)+1}log(1-y(e_\lambda(t)+1)) &= \frac{1}{e_\lambda(t)+1}(log(1+y(e_\lambda(t)+1)) - log(1-y(e_\lambda(t)+1))) \\ &= \frac{1}{e_\lambda(t)+1}log(\frac{1+y(e_\lambda(t)+1)}{1-y(e_\lambda(t)+1)}) \\ &= \frac{1}{e_\lambda(t)+1}log(1+\frac{2y(e_\lambda(t)+1)}{1-y(e_\lambda(t)+1)}) \\ &= \frac{1}{e_\lambda(t)+1}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(\frac{2y(e_\lambda(t)+1)}{1-y(e_\lambda(t)+1)})^n.\end{aligned}$$

$(\because\ log(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^(n+1)}{n}x^n)$

Theorem 5.2

$$\sum_{n=0}^{\infty}(U_{n,\lambda}(y) - U_{n,\lambda}(-y))\frac{t^n}{n!} = \frac{1}{e_\lambda(t)+1}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(\frac{2y(e_\lambda(t)+1)}{1-y(e_\lambda(t)+1)})^n.$$

6 Some values of polynomials

We aim to investigate the values of each polynomial obtained from the degenerate expressions by substituting y with specific constants, in order to examine their behavior. Let us begin by substituting y to 0 for the first time and observe the resulting values of the polynomials:

$$\begin{aligned}\frac{1}{e_\lambda+1}log(1+0\cdot(e_\lambda(t)+1)) &= \frac{1}{e_\lambda(t)+1}\cdot log(1) \\ &= 0 \\ &= \sum_{n=0}^{\infty}U_{n,\lambda}(0)\frac{t^n}{n!}.\end{aligned}$$

$\therefore\ U_{n,\lambda}(0) = 0(n \geq 0)$.
Also

$$\begin{aligned}\frac{1}{1+0(e_\lambda(t)+1)} &= 1 \\ &= \sum_{n=0}^{\infty}G_{n,\lambda}(0)\frac{t^n}{n!}.\end{aligned}$$

$\therefore\ G_{0,\lambda}(0) = 1,\ G_{n,\lambda}(0) = 0(n \geq 1)$.
Simillary

$$\begin{aligned}\frac{1}{1+0(e_\lambda(t)-1)} &= 1 \\ &= \sum_{n=0}^{\infty}F_{n,\lambda}(0)\frac{t^n}{n!}\end{aligned}$$

$\therefore\ F_{0,\lambda}(0) = 1,\ F_{n,\lambda}(0) = 0(n \geq 1)$
We have examined the values of the three polynomials when y is set to 0. Next, we will investigate the values

obtained by substituting y to -1. At first, on (5)

$$
\begin{aligned}
-\frac{1}{e_\lambda(t)} &= \sum_{n=0}^{\infty} G_{n,\lambda}(-1)\frac{t^n}{n!} \\
&= -\frac{1}{\sum\limits_{n=0}^{\infty}\frac{(1)_{n,\lambda}}{n!}t^n},
\end{aligned}
$$

$$
\begin{aligned}
-1 &= \Big(\sum_{l=0}^{\infty}\frac{(1)_{l,\lambda}}{l!}t^l\Big)\Big(\sum_{m=0}^{\infty}G_{m,\lambda}(-1)\frac{t^m}{m!}\Big) \\
&= \sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}(1)_{n-m,\lambda}\cdot G_{m,\lambda}(-1)\Big)\frac{t^n}{(n-m)!m!} \qquad (n=m+l) \\
&= \sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\binom{n}{m}(1)_{n-m,\lambda}\cdot G_{m,\lambda}(-1)\Big)\frac{t^n}{n!}.
\end{aligned}
$$

$\therefore\ G_{0,\lambda}(-1) = -1,\ \sum_{n=1}^{\infty}\big(\sum_{m=0}^{n}\binom{n}{m}(1)_{n-m,\lambda}\cdot G_{m,\lambda}(-1)\big)\frac{t^n}{n!} = 0.$

Simillary, we can prove $1 = \sum_{n=0}^{\infty}\big(\sum_{m=0}^{n}\binom{n}{m}(1)_{n-m.\lambda}\cdot F_{m,\lambda}(-1)\big)\frac{t^n}{n!}.$

$\therefore\ F_{0,\lambda}(-1) = 1,\ \sum_{n=1}^{\infty}\big(\sum_{m=0}^{n}\binom{n}{m}(1)_{n-m,\lambda}\cdot G_{m,\lambda}(-1)\big)\frac{t^n}{n!} = 0.$

I think we need to study the following polynomials a little more.

$\frac{1}{2-e_\lambda(t)} = \sum_{n=0}^{\infty} F_{n,\lambda}(1)\frac{t^n}{n!}$

$\frac{1}{2+e_\lambda(t)} = \sum_{n=0}^{\infty} G_{n,\lambda}(1)\frac{t^n}{n!}$

7 CONCLUSION

In recent years, there have been numerous studies on degenerate Bernoulli and Genocchi polynomials [2] resulting in significant findings. We have defined polynomials that resemble the forms of these degenerate Bernoulli polynomials and examined their properties. Furthermore, we derived explicit expressions for the polynomials when certain values are substituted in the formula we derived. Additionally, we found a relationship between the Bernoulli numbers and degenerate Fubini polynomials. Our future will work involve exploring new relationships by substituting various values into the polynomials we have discovered, which were not identified in this paper.

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