Advanced Studies in Contemporary Mathematics
34 (2024), No. 2, pp. 135 – 144

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2024.34.2.135

SUPERIOR TRANSCENDENTAL NUMBERS

HYUN SEOK LEE

ABSTRACT. From the Lindemann-Weierstrass Theorem, we can observe that e has more superior property than the usual definition of transcendental number. At this point, we can define the new type of transcendental number, that is, superior transcendental number. In Section 3, we will prove the generalization of Lindemann-Weiestrass theorem. Moreover, the Theorem 3.1 ensures that the numerous results of the open problems in the transcendental number theory.

1. INTRODUCTION

In [1], references to the existence of transcendental numbers go back many centuries. The “transcendental” comes from Leibniz in his 1682 paper where he proved $\sin x$ is not an algebraic function of x. Certainly Leibniz believed that, besides rational and irrational numbers (by “irrational” he meant algebraic irrational numbers in modern terminology), there also exist transcendental numbers. In [2], Liouville proved a fundamental theorem concerning approximations of algebraic numbers by rational numbers in 1853. This theorem gives first example of transcendental numbers.

Theorem 1.1 (J. Liouville, 1853)**.** *If α is algebraic of degree d, then there is a positive constant $C(\alpha)$, i.e. depending only on α, such that for all rationals $\frac{p}{q}$,*

$$\left|\alpha-\frac{p}{q}\right|>\frac{C(\alpha)}{q^d}.$$

From this theorem, we can find explicit examples of transcendental numbers.

Corollary 1.2. *The number*

$$\sum_{n=0}^{\infty}\frac{1}{2^{n!}}$$

is transcendental number.

In [3], there appeared Hermite’s epoch-making memoir entitled *Sur la fonction exponentielle* in which he established the transcendence of e, the natural base of logarithms. Liouville had shown in 1840, directly from the defining series, that in fact neither e nor e^2 could be rational or quadratic irrational; but Hermite’s work began a new era. In particular, within a decade, Lindemann succeeded in generalizing Hermite’s method and, in a classical paper, he proved that π is transcendental and solved thereby the ancient Greek problem concerning the quadrature of the circle. The work of Hermite and Lindemann was simplified by Weierstrass in 1885, and further simplified by Hilbert, Hurwitz and Gordan in 1893. In [4], the transcendence of e was first proved by Hermite in 1873 by using very different ideas and applying the approximation of analytic functions by rational functions.

2010 *Mathematics Subject Classification.* 11B68; 11B83;

Theorem 1.3 (C. Hermite, 1873)**.** *The number e is transcendental number.*

Theorem 1.4 (F. Lindemann, 1882)**.** *The number π is transcendental number.*

In [4], Lindemann stated more general results. One of them is Hermite-Lindemann Theorem:

Theorem 1.5 (Hermite-Lindemann)**.** *If β is a non-zero complex number. Then at least one of the two numbers β and e^{β} is transcendental.*

Thus, if β is algebraic, then e^{β} is transcendental number. Let α be non-zero algebraic number, and if λ is any non-zero determination of its logarithm, then λ is a transcendental number. Now, we define the set $\mathcal{L}$ of logarithm of non-zero algebraic numbers, that is the inverse image of the multiplicative group $\overline{\mathbb{Q}}^{\times}$ by the exponential map :

$$\mathcal{L} = \exp^{-1}(\overline{\mathbb{Q}}^{\times}) = \left\{\lambda \in \mathbb{C} : e^{\lambda} \in \overline{\mathbb{Q}}^{\times}\right\}.$$

The theorem of Hermite- Lindemann can be written $\overline{\mathbb{Q}} \cap \mathcal{L} = \{0\}$, that is, $\lambda\,(\neq 0) \in \mathcal{L}$ is transcendental number.

Theorem 1.6 (Lindemann-Weierstrass, 1885)**.** *If $\beta_1, \ldots, \beta_n$ are distinct algebraic numbers, then $e^{\beta_1}, \ldots, e^{\beta_n}$ are linearly independent over $\overline{\mathbb{Q}}$.*

In 1900, at the International Congress of Mathematicians held in Paris, Hilbert raised, as the seventh of his famous list of 23 problems, the question whether an irrational logarithms of an algebraic number to an algebraic base is transcendental. The question is capable of various alternative formulations; thus one can ask whether an irrational quotient of natural logarithms of algebraic number is transcendental, or whether α^{β} is transcendental for any algebraic number $\alpha \neq 0, 1$ and any algebraic irrational β.

Theorem 1.7 (Gelfond-Schneider, 1934)**.** *Suppose that $\alpha \neq 0, 1$ and that β is irrational. Then α, β and α^{β} cannot all be algebraic.*

In particular, $2^{\sqrt{2}}$ and $e^{\pi} = (-1)^{-i}$ are transcendental numbers. In the same year, Gelfond published extended his results [5] of the Gelfond-Schneider Theorem without proof. Actually, he does not published proofs, however in 1948 later he published weaker statements.

Conjecture 1.8 ([6], Gelfond's First conjecture)**.** *Let $\beta_1, \ldots, \beta_n$ be $\mathbb{Q}$-linearly independent algebraic numbers and if $\log\alpha_1, \ldots, \log\alpha_m$ $\mathbb{Q}$-linearly independent algebraic numbers. Then the numbers*

$$e^{\beta_1}, \ldots, e^{\beta_n},\ \log\alpha_1, \ldots, \log\alpha_m$$

are algebraically independent over $\mathbb{Q}$.

Conjecture 1.9 ([6], Gelfond' Second Conjecture)**.** *Let $\beta_1, \ldots, \beta_n$ be algebraic numbers with $\beta_i \neq 0$, and let $\log\alpha_1, \ldots, \log\alpha_m$ be logarithms of algebraic numbers with $\log\alpha_1 \neq 0$ and $\log\alpha_2 \neq 0$. Then the numbers*

$$e^{\beta_1 e^{\beta_2^{\cdot^{\cdot^{\beta_{n-1}e^{\beta_n}}}}}} \quad \text{and} \quad \alpha_1^{\alpha_2^{\alpha_3^{\cdot^{\cdot^{\alpha_n}}}}}$$

are transcendental, and there is no non-trivial algebraic relation between such numbers.

2. Superior Transcendental Numbers

This theorem is equivalent to the Lindemann-Weierstrass Theorem :

Theorem 2.1 (Lindemann-Weierstrass (`revisitied`)). *If $\beta_1,\dots,\beta_n$ are distinct algebraic numbers. Then*

$$\gamma_1 e^{\beta_1}+\cdots+\gamma_n e^{\beta_n}=0$$

for any algebraic numbers $\gamma_1,\dots,\gamma_n$ only if all $\gamma_i=0$.

Theorem 2.1 tell us that the number e has more superior property than the usual definition of transcendental number. From this point, we can naturally introduce the concept of *superior transcendental numbers*

Definition 2.2. *A transcendental number $\alpha\in\mathbb{C}$ is said to be a 'superior transcendental number' if $t_1,\dots,t_n$ are distinct algebraic numbers. Then*

$$\alpha^{t_1},\dots,\alpha^{t_n}$$

are linearly independent over $\overline{\mathbb{Q}}$. Otherwise, we say that the number α is 'inferior transcendental number'.

From the Theorem 2.1, we can know that the number e is superior transcendental number. On the other hand, the number $2^{\sqrt{2}}$ is inferior transcendental number, since

$$1\cdot\left(2^{\sqrt{2}}\right)^{\frac{1}{\sqrt{2}}}-2\cdot\left(2^{\sqrt{2}}\right)^{0}=0.$$

We can generalize this relation as follows :

If a and b are algebraic numbers with $a\neq 0,1$ and b is irrational number, then

$$1\cdot\left(a^{b}\right)^{\frac{1}{b}}-a\cdot\left(a^{b}\right)^{0}=0.$$

Thus, the number a^b is inferior transcendental number.

Theorem 2.3. *A number $\alpha\in\mathbb{C}$, then the following statements are equivalent:*

(a) α is superior transcendental number;
(b) α^{β} is a superior transcendental number for any $\beta\in\overline{\mathbb{Q}}\backslash\{0\}$;
(c) If $\beta_1,\dots,\beta_n$ are distinct algebraic numbers, and if $\gamma_1,\dots,\gamma_n$ are algebraic numbers not all zero. Then

$$\gamma_1\alpha^{\beta_1}+\cdots+\gamma_n\alpha^{\beta_n}$$

is a transcendental number.

Proof. Clearly, $(c)\Longrightarrow(b)\Longrightarrow(a)$. Thus, we enough to show that $(a)\Longrightarrow(c)$. Let

$$\delta=\gamma_1\alpha^{\beta_1}+\cdots+\gamma_n\alpha^{\beta_n}$$

where $\beta_1,\dots,\beta_n$ are distinct algebraic numbers, and $\gamma_1,\dots,\gamma_n$ are algebraic numbers not all zero. Suppose that δ is algebraic number, then $p(x)\in\mathbb{Z}[x]$ with $p(x)\neq 0$ such that $p(\delta)=0$. We say,

$$p(x)=\sum_{k=0}^{m}a_k x^k \quad (a_0a_m\neq 0).$$

Then

$$p(\delta)=\sum_{k=0}^{m}a_k\delta^k=\sum_{k=0}^{m}a_k\left(\sum_{l=0}^{n}\gamma_l\alpha^{\beta_l}\right)^k\neq 0.$$

Since $p(\delta)$ is a linear combination of algebraic numbers of distinct powers of algebraic numbers of α. Therefore, $p(\delta)$ can't be zero, since α is a superior transcendental number. □

Corollary 2.4. *For any non-zero $\alpha \in \overline{\mathbb{Q}}$, the numbers*

$$\sin\alpha,\ \cos\alpha,\ \tan\alpha,\ \sinh\alpha,\ \cosh\alpha,\ \tanh\alpha,\ \text{and}\ \log\alpha$$

are superior transcendental numbers.

3. Generalization of Lindemann-Weierstrass Theorem

In this section, we will prove the generalization of the Lindemann-Weierstras Theorem.

Theorem 3.1 (Generalization of the Lindemann-Weierstrass Theorem)**.** *If g is a transcendental function satisfying $g(x+y) = g(x)g(y)$ for all x, y and with $g(0) = 1$. Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers. Then*

$$g(\alpha_1),\ \ldots,\ g(\alpha_n)$$

are linearly independent over $\overline{\mathbb{Q}}$.

For the proof of Theorem 3.1, we need the generalization of Hermite's identity.

Lemma 3.2 (Generalization of Hermite's Identity)**.** *If g is a transcendental function with $g(0) = 1$. Also, $f \in \mathbb{C}[x]$ with $\deg f = m$. For $t \in \mathbb{C}$, define*

$$I(t, f) := \int_0^t g'(t-x) f(x)dx \tag{1}$$

where the integral is along the line segment from 0 to t.
Then

$$I(t, f) = g(t)\sum_{n=0}^{m} f^{(n)}(0) - \sum_{n=0}^{m} f^{(n)}(t). \tag{2}$$

Proof. Integration by parts by (1), we can get

$$I(t, f) = \int_0^t g'(t-x) f(x)dx = -f(t) + g(t)f(0) + I(t, f'). \tag{3}$$

We repeat this $(m-1)$-times, then we get the desired result. □

We can find the upper bound of $I(t, f)$ given by (1) :

$$\left|I(t, f)\right| \le |t| \max_{x\in[0,t]} \left|g'(t-x)\right| \max_{x\in[0,t]} \left|f(x)\right|. \tag{4}$$

Proof of Theorem 3.1. Suppose that

$$\beta_1 g(\alpha_1) + \cdots + \beta_n g(\alpha_n) = 0 \tag{5}$$

for some distinct algebraic numbers $\alpha_1, \ldots, \alpha_n$, and $\beta_1, \ldots, \beta_n$ are algebraic numbers not all zero. We can clearly assume, without loss of generality, that the β's are rational integers. This can be done by

$$\prod_{\sigma\in\mathrm{Gal}(\beta_1,\ldots,\beta_n)} \Big(\sigma(\beta_1)g(\alpha_1) + \cdots + \sigma(\beta_n)g(\alpha_n)\Big)$$

and then multiplying by a common denominator. Now, we claim that the set of α_i are ac complete set of conjugates, and then we have if $\alpha_j = \alpha_j$ are conjugates, then $\beta_i = \beta_j$. We can check this by choosing $p(x) \in \mathbb{Z}[x]\backslash\{0\}$ such that $p(\alpha_i) = 0$ for each $1 \le i \le n$. Now, $\alpha_{n+1}, \ldots, \alpha_N$ be the other roots, and we put $\beta_{n+1} = \cdots = \beta_N = 0$. Cleary, we have

$$\prod_{\sigma \in S_N} \Big(\beta_1 g\big(\alpha_{\sigma(1)}\big) + \cdots + \beta_N g\big(\alpha_{\sigma(N)}\big)\Big) = 0.$$

There are $N!$ factors in this product, by expanding this product, that is a sum of terms of the following form

$$g\big(h_1\alpha_1 + \cdots + h_N\alpha_N\big)$$

with integral coefficients, and $h_1 + \cdots + h_N = N!$. Note that the set of all such exponents forms a complete set of conjugates. Also, this product is not identically zero. Lastly, we can order in order that the conjugates of a particular α_i appear together.

Now, the rest part of the proof, we can assume that

$$\beta_1 g(\alpha_1) + \cdots + \beta_n g(\alpha_n) = 0$$

where β_i are rational integers, and that there are integers $0 = n_0 < n_1 < \cdots < n_r = n$ chosen such that

$$\alpha_{n_t+1}, \ldots, \alpha_{n_{t+1}}$$

forma complete set of Galois conjugates for each t. Also, we put

$$\beta_{n_t+1} = \beta_{n_t+2} = \cdots = \beta_{n_{t+1}}$$

Now, let d be a signify positive inteer such that $d\alpha_1, \ldots, d\alpha_n$ and $d\beta_1, \ldots, d\beta_n$ are algebraic integers, and set

$$f_i(x) = d^{np} \frac{(x-\alpha_1)^p \cdots (x-\alpha_n)^p}{(x-\alpha_i)^p}, \quad 1 \le i \le n,$$

where p denotes a large prime. Now, we define

$$J_i = \sum_{k=1}^{n} \beta_k I(\alpha_k, f_i), \quad (1 \le i \le n), \tag{6}$$

and $I(\alpha_k, f_i)$ is defined by the Lemma 3.2. From (2) and (6), we have

$$J_i = \sum_{k=1}^{n} \beta_k \left(g(\alpha_k) \sum_{n=0}^{m} f_i^{(n)}(0) - \sum_{n=0}^{m} f_i(\alpha_k) \right) = -\sum_{k=1}^{n} \beta_k \sum_{n=0}^{m} f_i^{(n)}(\alpha_k) \tag{7}$$

where $m = np - 1$. The last equality holds by our assumption. The rest part of the proof, we compute the derivatives of f_i. If $j \ne k$

$$f_j^{(l)}(\alpha_k) = \begin{cases} 0 & \text{if } l \le p-1, \\ \equiv 0 \pmod{p!} & \text{if } l \ge p, \end{cases}$$

and if $j = k$

$$f_j^{(l)}(\alpha_k) = \begin{cases} 0 & \text{if } l \le p-2, \\ d^{np}(p-1)! \prod_i (\alpha_k - \alpha_i)^p & \text{if } l = p-1, \\ \equiv 0 \pmod{p!} & \text{if } l \ge p, \end{cases}$$

Thus, we can note that each J_i is an algebraic integer divisible by $(p-1)!$ but not by $p!$. Moreover, by the initial our assumptions, we have

$$J_i = -\sum_{j=0}^{m} \sum_{t=0}^{r-1} \beta_{n_t+1} \Big(f_i^{(j)}(\alpha_{n_t+1}) + \cdots + f_i^{(j)}(\alpha_{n_{t+1}}) \Big),$$

Clearly, since $\alpha_1,\dots,\alpha_n$ is a complete set of Galois conjugates, the coefficient of $f_i^{(j)}(x)$ can be expressed in this form. Thus, $J_1\cdots J_n$ is rational. Since we assume that d was large enough to cancel all denominators, hence we have

$$|J_1\cdots J_n| \geq (p-1)!. \tag{8}$$

However,

$$|J_r| \leq c_r^p.$$

for some c_r independent of p. Therefore, we have

$$(p-1)! \leq |J_1\cdots J_n| \leq C^p$$

for some constant C. This inequalities are inconsistent if p is sufficiently large. □

Now, we takes $g(z) = e^{\alpha z}$, $g(z) = \pi^{\alpha z}$ and $g(z) = (\log\alpha)^z$. Then we get the following results. The below examples are restatement of the Lindemann-Weiestras Theorem.

Proposition 3.3. *The following numbers are transcendental numbers :*

(ii) e^α *is superior transcendental number for every* $\alpha \in \overline{\mathbb{Q}}\backslash\{0\}$.
(ii) π^β *is superior transcendental number for every* $\beta \in \overline{\mathbb{Q}}\backslash\{0\}$.
(iii) $\log\gamma$ *is superior transcendental number for every* $\gamma \in \overline{\mathbb{Q}}\backslash\{0\}$*, for any determination of the logarithm.*

It is not known that whether the numbers e^e, e^{e^2}, e^{e^3}, e^{e^4} are transcendental numbers. There are a few of results known that at least one of the numbers e^e, e^{e^2}, e^{e^3}, e^{e^4} is transcendental by the Six Exponentials Theorem.

Theorem 3.4 (Six Exponentials)**.** *If* $x_1,\dots,x_d$ *are complex numbers which are linearly independent over* $\mathbb{Q}$*, and if* $y_1,\dots,y_l$ *are complex numbers which are linearly independent over* $\mathbb{Q}$*. Suppose that* $dl > d+l$*, Then one at least of the* dl*-numbers*

$$\exp(x_i y_j), \quad (1 \leq i \leq d,\ 1 \leq j \leq l)$$

is transcendental number.

By taking $g(z) = e^{ze^\alpha}$ or $g(z) = e^{ze^{\pi^\alpha}}$, then we can get the following corollaries.

Corollary 3.5. *The following numbers are transcendental numbers :*

(i) *The numbers* e^{e^α} *are superior transcendental numbers for all* $a \in \overline{\mathbb{Q}}$.
(ii) *The numbers* e^{π^α} *are superior transcendental numbers for all* $\alpha \in \overline{\mathbb{Q}}$ *with* $\alpha \neq 1$.

In particular, the number e^{π^2} is a superior transcendental number. Now, we can prove the transcendence of the number π^e. The transcendence of π^e is one of the important open problems in this area. By considering the $g(z) = \pi^{(\alpha+e^\beta)z}$, where $\alpha,\beta \in \overline{\mathbb{Q}}$, then we have

Corollary 3.6. *The numbers* $\pi^{(\alpha+e^\beta)}$ *where* $\alpha,\beta \in \overline{\mathbb{Q}}$ *are superior transcendental numbers.*

Now, we denote $\mathcal{G}$ the set all Gelfond-Schneider type of transcendental numbers, say,

$$\mathcal{G} = \{\alpha^\beta : \alpha \in \overline{\mathbb{Q}}\backslash\{0,1\},\ \beta \in \overline{\mathbb{Q}}\backslash\mathbb{Q}\}.$$

Theorem 3.7. *A number* γ *is superior transcendental number if and only if* $\gamma \notin \mathcal{G}$.

Proof. Let $\gamma \in \mathcal{G}$. Since $\gamma^{\frac{1}{\beta}} - \alpha\gamma^0 = 0$ is a non-trivial linear combination of $\overline{\mathbb{Q}}$, so we can see that γ is not superior transcendental number. Conversely, if γ is transcendental number with $\gamma \notin \mathcal{G}$. Then the function

$$\gamma^z = e^{z\log\gamma}$$

is a transcendental function with $g(0) = 1$. By the Theorem 3.1, let $\alpha_1, \ldots, \alpha_n$ be distinct algebraic numbers, then

$$\gamma^{\alpha_1}, \ldots, \gamma^{\alpha_n}$$

are linearly independent over $\overline{\mathbb{Q}}$. Thus, we can get the desired conclusion that γ is superior transcendental number. □

Now, we introduce the following definition.

Definition 3.8. *A complex number is said to be 'inferior' if this number is either an algebraic number or the type of Gelfond-Schneider transcendental number. A complex number is 'superior' if it is not 'inferior'.*

Now, we denote by $\mathcal{I}$ is the set all 'inferior complex numbers' and by $\mathcal{C}$ its complements. From the definition, we can get

$$\mathcal{I} = \overline{\mathbb{Q}} \bigcup \mathcal{G}.$$

Also, the set of superior complex numbers conicides to the set superior transcendental numbers. Thus, we can get the following theorem.

Theorem 3.9. *A complex number is either inferior complex number or a superior transcendental number. The complex number field $\mathbb{C}$ is disjoint unions of $\overline{\mathbb{Q}}$, $\mathcal{G}$, and $\mathcal{C}$, that is,*

$$\mathbb{C} = \overline{\mathbb{Q}} \bigcup \mathcal{G} \bigcup \mathcal{C}.$$

Note that the set $\mathcal{G}$ is inferior transcendental numbers is countable, so is the set $\mathcal{I} = \overline{\mathbb{Q}} \cup \mathcal{G}$. Therefore, most complex numbers are superior transcendental numbers. Now, we denote the set

$$\mathcal{L}_{\mathcal{G}} := \left\{\beta \log\alpha : \alpha \in \overline{\mathbb{Q}}\backslash\{0,1\},\ \beta \in \overline{\mathbb{Q}}\backslash\mathbb{Q}\right\}.$$

If $\delta \notin \mathcal{L}_{\mathcal{G}}$, then $g(z) = e^{\delta z}$ is a transcendental function such that $g(0) = 1$. We can get the following theorem. The converse also holds.

Theorem 3.10. *Let $\delta \notin \mathcal{L}_{\mathcal{G}}$, then the number $e^{\delta} \in \mathcal{S}$.*

The results of Theorem 3.10 ensure that the following more stronger result :

Corollary 3.11. *If α and β are non-zero algebraic numbers. Then $e^{\alpha^\beta} \in \mathcal{S}$. In particular, the number $e^{e^\pi} (= e^{i^{-2i}})$ is a superior transcendental number.*

Proof. We enough to show that $\alpha^\beta \neq \beta' \log\alpha'$ for every $\alpha', \beta' \in \overline{\mathbb{Q}}\backslash\{0\}$, since the left hand side is a inferior complex number, however the right-hand side is a superior transcendental number. By the Theorem 3.9. we can get that the number e^{α^β} is superior transcendental number. □

4. Linear Independence of the Superior Transcendental numbers

In this section, we will derive the linear independence of superior transcendental numbers. These results provide seminal open problems in transcendental number theory.

Theorem 4.1. *If* α *and* β *are non-zero algebraic numbers, then the numbers* e^{β} *and* $\log\alpha$ *is linearly independent over* $\overline{\mathbb{Q}}$.

Proof. Suppose that they are linearly dependent over $\overline{\mathbb{Q}}$, say

$$\gamma_1 e^{\beta} + (-\gamma_2)\log\alpha = 0$$

where γ_1, γ_2 are non-zero algebraic numbers. Then we have,

$$\alpha^{\frac{\gamma_2}{\gamma_1}} = e^{e^{\beta}}.$$

It is a contradiction, since $\alpha^{\frac{\gamma_2}{\gamma_1}}$ is a inferior complex number, however $e^{e^{\beta}}$ is superior transcendental number. □

Theorem 4.2. *Let* α, β *be non-zero algebraic numbers, then the numbers* π^{β} *and* $\log\alpha$ *are* $\overline{\mathbb{Q}}$*-linearly independent*

Proof. Suppose that

$$\delta_1\pi^{\beta} + (-\delta_2)\log\alpha = 0$$

for some non-zero algebraic numbers δ_1, δ_2. From this, we can get

$$\alpha^{\frac{\delta_2}{\delta_1}} = e^{\pi^{\beta}}.$$

However, it is contradicts the fact that the number $e^{\pi^{\beta}}$ is superior transcendental number, however the number $\alpha^{\frac{\delta_2}{\delta_1}}$ is inferior transcendental number. □

In the same way, we can prove the following results:

Theorem 4.3. *If* β *is non-zero algebraic numbers, then* $\{\pi, \log\pi^{\beta}\}$ *is linearly independent over* $\overline{\mathbb{Q}}$.

Now, we can answer that one of seminal conjecture in this area. The transcendence of $e\cdot\pi$ and $\frac{\pi}{e}$.

Theorem 4.4. *The numbers* $\pi\cdot e$, $\frac{\pi}{e}$ *are transcendental numbers*

Proof. First, we can suppose that $e\cdot\pi = \alpha$ is algebraic number. Then $\pi = a\cdot e^{-1}$ and $e^{\pi} = \left(e^{e^{-1}}\right)^{\alpha}$. From the Theorem 3.10, we can get $e^{-1} \notin \mathcal{L}_{\mathcal{G}}$. So, $e^{e^{-1}} \notin \mathcal{L}_{\mathcal{G}}$, and consequently $\left(e^{e^{-1}}\right)^{a}$ is a superior transcendental number. This is contradiction, since e^{α} is an inferior transcendental number. Secondly, we suppose that $\frac{\pi}{e} = \beta$ is algebraic number. Then $\pi = \beta\cdot e$ and $e^{\pi} = \left(e^{e}\right)^{\beta}$. By the Theorem 3.10, $e \notin \mathcal{L}_{\mathcal{G}}$. Thus, $e^{e} \notin \mathcal{L}_{\mathcal{G}}$, and hence $\left(e^{e}\right)^{\beta}$ is a superior transcendental number. It is contradict the fact that the number e^{π} is inferior transcendental number. □

The next theorem claim that the similar conclusion can be stated if in the above theorems, if we can places the distinct powers of e with just distinct superior transcendental number.

Theorem 4.5. *Let* $\eta_1, \ldots, \eta_n$ *be distinct superior transcendental numbers, then the numbers* $1, \eta_1, \ldots, \eta_n$ *are linearly independent over* $\overline{\mathbb{Q}}$.

Proof. Assume that

$$d_0 + d_1\eta_1 + \cdots + d_n\eta_n = 0 \tag{9}$$

where $d_0, \ldots_d$ are algebraic numbers not all zero. Now, we remove for all i such that $d_i = 0$, we can assume that all coefficients are different from zero. Observe that if $d_i \neq 0$, then we can multiply by d_i^{-1} to obtain equation same form of (9). Also, we can assume that $d_0 \neq 0$.

Now, we consider the polynomial

$$f(x) = x^{p-1}(x-1)^p \cdots (x-n)^p.$$

Then for $0 < x < n$, we have

$$\left|f(x)\right| \leq n^{np+p-1}.$$

For $t = 1, \ldots, n$, we set

$$I(t,f) := \int_0^t \left(\log\eta_t\right)\eta_t^{(t-x)} f(x)dx.$$

Integration by parts, we have

$$I(t,f) = \int_0^t f(x)d\left(-\eta_t^{(t-x)}\right) = -f(t) + \eta_i f(0) + I(t,f').$$

By repeating this procedure, we have

$$I(t,f) = \eta_t \sum_{j=0}^{np+p+1} f^{(j)}(0) - \sum_{j=0}^{np+p-1} f^{(j)}(t)$$

Also, we can note that

$$\left|I(t,f)\right| \leq \int_0^t \left|\left(\log\eta_t\right)\eta_t^{b(t-x)} f(x)\right|dx \leq \left|\log\eta_t\right| e^{|b\log\eta_t|} n^{n(p+1)-1}. \tag{10}$$

Now, we define

$$J := d_1 I(1,f) + \cdots + d_n I(n,f). \tag{11}$$

Then, we have

(12)

$$J = \sum_{i=1}^n d_i\left(\tau_i \sum_{j=0}^{np+p-1} f^{(j)}(0) - \sum_{j=0}^{np+p-1} f^{(j)}(i)\right) = -d_0 \sum_{j=0}^{np+p1} f^{(j)}(0) - \sum_{i=1}^n d_i \sum_{j=0}^{np+p-1} f^{(j)}(i)$$

where the last equality follows from $d_1\tau_1 + \cdots + d_n\tau_n = -d_0$. The rest part of the proof, we compute the derivatives of f. Note that $f^{(j)}(0) = 0$ for $j < p-1$; $f^{(p-1)}(0) = (-1)^{np}(p-1)!(n!)^p$ is not divisible by $p!$ for $p > n$ and $f^{(j)}(0)$ is an integer divisible by $p!$ for $j \geq p$. Also, $f^{(j)}(i) = 0$ for $j < p$ and $f^{(j)}(i)$ is an integer divisible by $p!$ for $j \geq p$. Therefore, it follows that if $p > |d_0|$, J is a non-zero integer divisible by $(p-1)!$ and

$$|J| \geq (p-1)!$$

From the (10) and (11), it follows that

$$|J| \leq \sum_{t=1}^n \left|d_i\right| \cdot \left|I(t,f)\right| \leq n^{n(p+1)-1}\left(\sum_{t=1}^n |d_i|\left|\log\eta_t\right| e^{|b\log\eta_t|}\right) < C^p. \tag{13}$$

for some independent p. From (12) and (13) are inconsistent for sufficiently large prime p, and we get the contradiction. □

From the our results, we can know that the numbers e and π are superior transcendental numbers. Using the Theorem 4.5, we can prove that the long-standing conjecture of transcendence of $e+\pi$ and $e-\pi$.

Corollary 4.6. *The numbers $e+\pi$ and $e-\pi$ are transcendental numbers.*

We note that e^{α} and π^{β} are superior transcendental numbers for non-zero algebraic numbers α and β. Assume that $e^{\alpha}=\pi^{\beta}$, then $e^{\frac{\alpha}{\beta}}=\pi$ and $e^{e^{\frac{\alpha}{\beta}}}=e^{\pi}$. It is contradict the fact the number e^{π} is inferior complex number, however $e^{\frac{\alpha}{\beta}}$ is superior transcendental number. We apply Theorem 4.5 to the set $\{e^{\alpha},\pi^{\beta}\}$, then can get the result of the long-standing conjecture of the algebraic independence of the numbers e and π.

Corollary 4.7. *The numbers 1, e^{α} and π^{β} is linearly independent over $\overline{\mathbb{Q}}$ for non-zero algebraic numbers α,β. Also, any $\overline{\mathbb{Q}}$-linear combination of the numbers e^{α} and π^{β} is transcendental number.*

Now, Gelfond's First and Second conjectures are proved by the Corollary 3.5, Corollary 3.11, Theorem 4.1 and Theorem 4.5.

REFERENCES

[1] Fel'dman, Naum I., and Yu V. Nesterenko. *Number theory IV: transcendental numbers. Vol. 44.* Springer Science & Business Media, 2013.

[2] Murty, Maruti Ram, and Purusottam Rath. *Transcendental numbers.* New York: Springer, 2014.

[3] Baker, Alan. *Transcendental number theory.* Cambridge university press, 1974.

[4] Natarajan, Saradha, and Ravindranathan Thangadurai. *Pillars of transcendental number theory.* Springer Singapore, 2020.

[5] Gelfond, A. O. *Sur quelques rsultats nouveaux dans la théorie des nombres transcendants.* CR Acad. Sc. Paris, Sér. A 199 (1934): 259.

[6] Waldschmidt, Michel. *Diophantine approximation on linear algebraic groups: transcendence properties of the exponential function in several variables.* Vol. 326. Springer Science & Business Media, 2013.

DEPARTMENT OF MATHEMATICS, KWANGWOON UNIVERSITY, SEOUL 139-701, REPUBLIC OF KOREA

Email address: `luciasconstant@kw.ac.kr`