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CONFORMAL QUASI HEMI-SLANT RIEMANNIAN SUBMERSIONS FROM COSYMPLECTIC MANIFOLDS

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Abstract: In this paper, we introduce some geometric properties of conformal quasi hemi-slant Riemannian submersions from a cosymplectic manifold to a Riemannian manifold. We obtain the necessary and sufficient conditions for the integrability conditions of distributions. We also search for totally geodesicity on the base manifold of the submersions. Finally, we give an explicit example of this type of submersions.

1. INTRODUCTION

In Riemannian geometry, there are few appropriate maps among Riemannian manifolds that compare their geometric properties. In this direction, as a generalization of the notions of isometric immersions and Riemannian submersions, the Riemannian map between Riemannian manifolds was initiated by Fischer [10], while isometric immersions and Riemannian submersions were widely studied in [7] and [25], respectively. However, the notion of Riemannian maps is a new research topic for geometers. In [29], other prominent basic maps for comparing geometric structures between Riemannian manifolds are studied by O'Neill. O'Neill defined a Riemannian submersion, which is the "dual" notion of isometric immersion, and obtained some fundamental equations corresponding to those in Riemannian submanifold geometry, that is, Gauss, Codazzi, and Ricci equations. This notion is related to physics and has some applications in Yang-Mills theory [5, 33], supergravity and superstring theories [12, 16], and Kaluza-Klein theory [6, 11]. On the other hand, Riemannian submersions were considered between almost complex manifolds by Watson [32] under the name of almost Hermitian submersions. For Riemannian submersions between almost-contact manifolds, Chinea [8] studied them under the name of almost-contact submersions.

As a natural generalization of holomorphic submersions and totally real submersions, B. Sahin introduced the notion of slant submersions [23] and semi-invariant submersions [22] from almost Hermitian manifolds onto arbitrary Riemannian manifolds. The different kinds of Riemannian submersions between Riemannian manifolds endowed with different structures were studied by several geometers [3, 19, 13, 20, 26, 24]. As a generalization of invariant submersions and slant submersions, Park and Prasad [18] defined and studied the notion of semi-slant submersions from an almost Hermitian manifold onto a Riemannian manifold. Also, bi-slant submersions in complex geometry [21] is studied by Sayar et al., 2020. As a generalization of slant submersions and anti-invariant submersions, B. Sahin introduced the notion of

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hemi-slant Riemannian submersions [28] from almost Hermitian manifolds onto Riemannian manifolds. He gave a decomposition theorem for such submersions. Also, Sahin gave some main results about Riemannian submersions and an application to robotic theory [26]. Therefore, a new vision on submersions by applying conformality conditions was presented by Akyol and Sahin [1, 2]. Riemannian submersions have many applications, such as texture mapping, remeshing and simulation [15], computer graphics and medical imaging fields [30] and brain mapping research [31].

The purpose of this paper is to study conformal quasi-hemi-slant Riemannian submersion from a cosymplectic manifold, which includes the classes of conformal hemi-slant submersion, conformal semi-invariant submersion and conformal semi-slant submersion.

2. Preliminaries

In this section, we give several definitions and results to be used throughout the study for conformal quasi-hemi slant Riemannian submersion.

A (2n + 1)-dimensional C^∞-manifold is said to have an almost contact structure on Σ_m if there exist a tensor field Ω of type $(1,1)$, a vector field ξ, and a 1-form η satisfying

$$\Omega^2 = -I + \eta \otimes \xi, \;\; \Omega\xi = 0, \;\; \eta \circ \Omega = 0, \;\; \eta(\xi) = 1. \tag{2.1}$$

There always exists a Riemannian metric g_1 on a cosymplectic manifold Σ_m satisfying the following conditions:

$$g_1(\Omega P, \Omega Q) = g_1(P, Q) - \eta(P)\eta(Q), \tag{2.2}$$

where $P,\ Q\ \in \Gamma(T\Sigma_m))$. The immediate consequence of (2.2) is

$$\eta(P) = g_1(P, \xi) \text{ and } g_1(\Omega P, Q) + g_1(P, \Omega Q) = 0. \tag{2.3}$$

An almost contact structure (Ω, ξ, η, g_1) is said to be normal if the almost complex structure J_1 on the product manifold $\Sigma_m \times R$ is given by

$$J_1(P, f\frac{d}{dt}) = (\Omega P - f\xi, \eta(P)\frac{d}{dt}),$$

where f is a C^∞-function on $\Sigma_m \times R$ having no torsion, i.e., J_1 is integrable. The condition for normality in terms of Ω, ξ, and η is $[\Omega, \Omega] + 2d\eta \otimes \xi = 0$ on Σ_m, where $[\Omega, \Omega]$ is the Nijenhuis tensor of Ω. Finally, the fundamental two-form Φ is defined as $\Phi(P, Q) = g_1(P, \Omega Q)$. An almost contact metric structure (Ω, ξ, η, g_1) is said to be cosymplectic manifold ([4, 27]) if it is normal and both Φ and η are closed, and the structure equation of a cosymplectic manifold is given by

$$(\nabla_P \Omega)Q = 0 \tag{2.4}$$

for any $P, Q \in \Gamma(T\Sigma_m)$, where ∇ denotes the Riemannian connection of the metric g_1 on Σ_m. Moreover, for a cosymplectic manifold, we have

$$\nabla_P \xi = 0. \tag{2.5}$$

The covariant derivative of Ω is defined as

$$(\nabla_P \Omega)Q = \nabla_P \Omega Q - \Omega \nabla_P Q.$$

If Σ_m is a cosymplectic manifold, then we have

$$\Omega \nabla_P Q = \nabla_P \Omega Q. \tag{2.6}$$

Let $(\Sigma_m, \Omega, \xi, \eta, g_1)$ be a cosymplectic manifold and (Σ_n, g_2) be a Riemannian manifold. Let $\gamma : (\Sigma_m, \Omega, \xi, \eta, g_1) \to (\Sigma_n, g_2)$ be a smooth map, then the second fundamental form of γ is given by

$$(\nabla\gamma_*)(P, Q) = \nabla^{\gamma}_{P}\gamma_*(Q) - \gamma_*(\nabla_P Q), \text{ for all } P, Q \in \Gamma(T\Sigma_m). \tag{2.7}$$

The second fundamental form $\nabla\gamma_*$ is symmetric [17]. Here γ_* is differential map of γ from tangent space of Σ_m at a point $x \in \Sigma_m$ to tangent space of Σ_n at $\gamma(x)$ such that $\gamma_* : T_x\Sigma_m \to T_{\gamma(x)}\Sigma_n$.

A smooth map $\gamma : (\Sigma_m, \Omega, \xi, \eta, g_1) \to (\Sigma_n, g_2)$ between Riemannian manifolds is called a Riemannian submersion, if γ has maximal rank and the differential γ_* preserves the lengths of horizontal vectors. On the other hand, let $(\Sigma_m, \Omega, \xi, \eta, g_1)$ be an m-dimensional cosymplectic manifold and (Σ_n, g_2) be an n-dimensional Riemannian manifold, and let $\gamma : (\Sigma_m, \Omega, \xi, \eta, g_1) \to (\Sigma_n, g_2)$ be a differentiable map between them and $x \in \Sigma_m$, then γ is called horizontally weakly conformal or semi-conformal at x if either $(\gamma_*)_x = 0$, or $(\gamma_*)_x$ is surjective and there exists a number $\chi(x) \neq 0$ such that

$$g_2(\gamma_* P, \gamma_* Q) = \chi(x) g_1(P, Q), \quad \text{for all } P, Q \in ((\ker\gamma_*)_x)^{\perp}. \tag{2.8}$$

We say that point x is a critical point if it satisfies $(\gamma_*)_x = 0$ and we shall call the point x a regular point if $(\gamma_*)_x$ is surjective. At a critical point, $(\gamma_*)_x$ has rank 0; at a regular point, $(\gamma_*)_x$ has rank n and γ is submersion. Furthermore, $\chi(x)$ is called the square dilation of γ at x, and its square root is $\lambda(x) = \sqrt{\chi(x)}$ is called the dilation of γ at x. The map γ is called horizontally weakly conformal or semi-conformal on Σ_m if it is horizontally weakly conformal at every point on Σ_m. If γ has no critical point, then it is said to be a (horizontally) conformal submersion [3].

A vector field E on Σ_m is called projectiable if there exists a vector field E' on Σ_n such that $\gamma_*(E_x) = E'_{\gamma(x)}$ for any $x \in \Sigma_m$. In this case, E and E' are called $\gamma-$related. If E is both a horizontal and a projectiable vector field, we say E is a basic vector field on Σ_m. From now on, when we mention a horizontal vector field, we always consider a basic vector field [3].

The fundamental tensors $\mathcal{T}$ and $\mathcal{A}$ defined by O'Neill's for vector fields E and F on Σ_m such that

$$\mathcal{A}_E F = \mathcal{H}\nabla^M_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla^M_{\mathcal{H}E}\mathcal{H}F, \tag{2.9}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla^M_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla^M_{\mathcal{V}E}\mathcal{H}F \tag{2.10}$$

where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections respectively. Note that the tensor field $\mathcal{T}$ is symmetric on the vertical distribution [29]. On the other hand, from equations (2.9) and (2.10) we have

$$\nabla_L K = \mathcal{T}_L K + \widehat{\nabla}_L K, \tag{2.11}$$

$$\nabla_L P = \mathcal{H}\nabla_L P + \mathcal{T}_L P, \tag{2.12}$$

$$\nabla_P L = \mathcal{A}_P L + \mathcal{V}\nabla_P L, \tag{2.13}$$

$$\nabla_P Q = \mathcal{H}\nabla_P Q + \mathcal{A}_P Q \tag{2.14}$$

for all $L, K \in \Gamma(\ker\gamma_*)$ and $P, Q \in \Gamma(\ker\gamma_*)^{\perp}$, where $\mathcal{V}\nabla_L K = \widehat{\nabla}_L K$ [9]. If P is basic, then $\mathcal{A}_P K = \mathcal{H}\nabla_P K$.

Lemma 2.1. *Let* $\gamma : (\Sigma_m, \Omega, \xi, \eta, g_1) \to (\Sigma_n, g_2)$ *be a horizontal conformal submersion, then for any horizontal vector fields* P, Q *and vertical vector fields* L, K [3], *we have*

$$(\nabla\gamma_*)(P,Q) = P(\ln\lambda)\gamma_*(Q) + Q(\ln\lambda)\gamma_*(P) - g_1(P,Q)\gamma_*(grad\ln\lambda), \tag{2.15}$$

$$(\nabla\gamma_*)(L,K) = -\gamma_*(\mathcal{T}_L K), \tag{2.16}$$

$$(\nabla\gamma_*)(P,L) = -\gamma_*(\nabla^M_P L) = -\gamma_*(\mathcal{A}_P L). \tag{2.17}$$

Here, λ *is the dilation of* γ *at a point* $x \in \Sigma_m$ *and it is a continuous function as* $\lambda : \Sigma_m \to [0,\infty)$.

3. Conformal Quasi Hemi-slant Riemannian Submersions

In this section, we define and study conformal quasi hemi-slant Riemannian submersions from a cosymplectic manifold to a Riemannian manifold.

Definition 3.1. $\gamma : (\Sigma_m, \Omega, \xi, \eta, g_1) \to (\Sigma_n, g_2)$ *be a conformal submersion such that its vertical distribution* $\ker\gamma_*$ *admits four mutually orthogonal distributions* D, D_θ, $D^\perp$ *and* $<\xi>$. *Where* D *is invariant* ($\Omega(D) = D$), D^θ *is slant (the angle* θ *between* D_θ *and* $\Omega(D_\theta)$ *is a constant) and* $D^\perp$ *is anti-invariant* ($\Omega(D^\perp) \subseteq (\ker\gamma_*)^\perp$), *i.e.,*

$$\ker\gamma_* = D \oplus D_\theta \oplus D^\perp \oplus <\xi> . \tag{3.1}$$

Then we say γ *is a conformal quasi hemi-slant Riemannian submersion and the angle* θ *is called the quasi hemi-slant angle of the map.*

Here, we have some particular cases:

(i) If the distribution $D = \{0\}$ then the map γ is a conformal hemi-slant submersion.

(ii) If the distribution $D_\theta = \{0\}$ then the map γ is a conformal semi-invariant submersion.

(iii) If the distribution $D^\perp = \{0\}$ then the map γ is a conformal semi-slant submersion.

Let γ be a conformal quasi hemi-slant Riemannian submersion from a Cosymplectic manifold $(\Sigma_m, \Omega, \xi, \eta, g_1)$ onto a Riemannian manifold (Σ_n, g_2), then we have

$$T\Sigma_m = (\ker\gamma_*) \oplus (\ker\gamma_*)^\perp. \tag{3.2}$$

A vertical vector field U can be written as

$$U = f_1U + f_2U + f_3U + \eta(U)\xi, \tag{3.3}$$

where f_1, f_2 and f_3 are projections onto D, D_θ and $D^\perp$ respectively.

For all $U \in \Gamma(\ker\gamma_*)$, we have

$$\Omega U = \mu_1 U + \mu_2 U, \tag{3.4}$$

where $\mu_1 U$ and $\mu_2 U$ are vertical and horizontal components of ΩU respectively. From (3.3), (3.4) and Definition 3.1, we obtain $\mu_2 f_1 U = 0$, $\mu_1 f_3 U = 0$ and

$$\Omega U = \mu_1 f_1 U + \mu_1 f_2 U + \mu_2 f_2 U + \mu_2 f_3 U. \tag{3.5}$$

Hence, we can write

$$\Omega(\ker\gamma_*) = D \oplus \mu_1 D_\theta \oplus \mu_2 D_\theta \oplus \Omega(D^\perp). \tag{3.6}$$

Using (3.6), we obtained

$$(\ker\gamma_*)^\perp = \mu_2 D_\theta \oplus \Omega(D^\perp) \oplus \mu \tag{3.7}$$

where μ is the orthogonal complement distribution of $\mu_2 D_\theta \oplus \Omega(D^\perp)$ in $(\ker\gamma_*)^\perp$ and μ is the invariant with respect to Ω. Lastly, for a horizontal vector field P, we have

$$\Omega P = \nu_1 P + \nu_2 P \tag{3.8}$$

where $\nu_1 P \in \Gamma(\mu_2 D_\theta \oplus \Omega(D^\perp))$ and $\nu_2 P \in \Gamma(\mu)$.

Lemma 3.2. *Let γ be a conformal quasi hemi-slant Riemannian submersion from a cosymplectic manifold $(\Sigma_m, \Omega, \xi, \eta, \mathrm{g}_1)$ onto a Riemannian manifold (Σ_n, g_2), then we have*

$$(a)\ \ \mu_1 D_\theta = D_\theta, \quad (b)\ \ \mu_1 D^\perp = \{0\},$$
$$(c)\ \ \nu_1\mu_2 D_\theta = D^\theta, \quad (d)\quad \nu_1\Omega D^\perp = D^\perp, \quad (e)\quad \mu_2 D = \{0\}.$$

Lemma 3.3. *Let $(\Sigma_m, \Omega, \xi, \eta, \mathrm{g}_1)$ be a cosymplectic manifold and (Σ_n, g_2) be a Riemannian manifold. If $\gamma : (\Sigma_m, \Omega, \xi, \eta, \mathrm{g}_1) \to (\Sigma_n, \mathrm{g}_2)$ is a conformal hemi-slant Riemannian submersion, then*

$${\mu_1}^2 X + \nu_1\mu_2 X = -X + \eta(X)\xi, \qquad \mu_2\mu_1 X + \nu_2\mu_2 X = 0,$$
$$\mu_1\nu_1 Z + \nu_1\nu_2 Z = 0, \qquad \mu_2\nu_1 Z + {\nu_2}^2 Z = -Z$$

for all $X \in \Gamma(\ker\pi_*)$ and $Z \in \Gamma(\ker\pi_*)^\perp$.

Lemma 3.4. *Let γ be a conformal quasi hemi-slant Riemannian submersion from cosymplectic manifold $(\Sigma_m, \Omega, \xi, \eta, \mathrm{g}_1)$ onto a Riemannian manifold (Σ_n, g_2), then we have*

$$-{\mu_1}^2 = \cos^2\theta U, \tag{3.9}$$

$$\mathrm{g}_1(\mu_1 U, \mu_1 V) = \cos^2\theta \mathrm{g}_1(U, V), \tag{3.10}$$

$$\mathrm{g}_1(\mu_2 U, \mu_2 V) = \sin^2\theta \mathrm{g}_1(U, V) \tag{3.11}$$

for $U, V \in \Gamma(D_\theta)$.

Lemma 3.5. *Let γ be a conformal quasi hemi-slant Riemannian submersion from a cosymplectic manifold $(\Sigma_m, \Omega, \xi, \eta, \mathrm{g}_1)$ onto a Riemannian manifold (Σ_n, g_2), then we have*
(i) $\mathrm{g}_1(\nabla_M N, \xi) = 0$,
(ii) $\mathrm{g}_1([M, N], \xi) = 0$,
where $M, N \in (D \oplus D_\theta \oplus D^\perp)$.

Throughout this section, we give necessary and sufficient conditions to be integrability for distributions.

Theorem 3.6. *Let γ be a conformal quasi hemi-slant Riemannian submersion from a cosymplectic manifold $(\Sigma_m, \Omega, \xi, \eta, \mathrm{g}_1)$ onto a Riemannian manifold (Σ_n, g_2), then the distribution D_θ is integrable if and only if*

$$\begin{aligned} & \mathrm{g}_2((\nabla\gamma_*)(N, \mu_1 f_1 Z), \gamma_*(\mu_2 M)) - \mathrm{g}_2((\nabla\gamma_*)(M, \mu_1 f_1 Z), \gamma_*(\mu_2 N)) \\ = \ & \lambda^2\{\mathrm{g}_1(\widehat{\nabla}_N \mu_1 f_1 Z + \mathcal{T}_N \mu_2 f_3 Z, \mu_1 M) - \mathrm{g}_1(\mathcal{H}\nabla_M \mu_2 f_3 Z, \mu_2 N) \\ & + \mathrm{g}_1(\mathcal{H}\nabla_N \mu_2 f_3 Z, \mu_2 M) - \mathrm{g}_1(\widehat{\nabla}_M \mu_1 f_1 Z + \mathcal{T}_M \mu_2 f_3 Z, \mu_1 N)\} \end{aligned}$$

for $M, N \in \Gamma(D_\theta)$ and $Z \in \Gamma(D \oplus D^\perp)$.

Proof. We have, from Lemma 3.5 $g_1([M,N],\xi)=0$. Thus D_θ is integrable if and only if $g_1([M,N],Z)=0$. Since Σ_m is a cosymplectic manifold, we have

$$g_1(\nabla_M N, Z) = -g_1(\nabla_M \Omega Z, \Omega N) \text{ for } M,N\in\Gamma(D_\theta) \text{ and } Z\in\Gamma(D\oplus D^\perp).$$

So, we get from (2.11), (2.12), (3.3) and (3.4)

$$\begin{aligned} -g_1(\nabla_M\Omega Z,\Omega N) &= -g_1(\nabla_M\mu_1 f_1 Z+\nabla_M\mu_2 f_3 Z,\mu_1 N+\mu_2 N)\\ &= -g_1(\widehat{\nabla}_M\mu_1 f_1 Z+\mathcal{T}_M\mu_2 f_3 Z,\mu_1 N)\\ &\quad - g_1(\mathcal{T}_M\mu_1 f_1 Z+\mathcal{H}\nabla_M\mu_2 f_3 Z,\mu_2 N). \end{aligned} \tag{3.12}$$

Changing the roles of M and N in (3.12), we have second part of $g_1([M,N],Z)$. Hence from (2.16) we obtain

$$\begin{aligned} g_1([M,N],Z) &= g_1(\widehat{\nabla}_N\mu_1 f_1 Z+\mathcal{T}_N\mu_2 f_3 Z,\mu_1 M) - g_1(\widehat{\nabla}_M\mu_1 f_1 Z+\mathcal{T}_M\mu_2 f_3 Z,\mu_1 N)\\ &\quad + g_1(\mathcal{H}\nabla_N\mu_2 f_3 Z,\mu_2 M) - g_1(\mathcal{H}\nabla_M\mu_2 f_3 Z,\mu_2 N)\\ &\quad + \frac{1}{\lambda^2}\{g_2((\nabla\gamma_*)(M,\mu_1 f_1 Z),\gamma_*(\mu_2 N)) - g_2((\nabla\gamma_*)(N,\mu_1 f_1 Z),\gamma_*(\mu_2 M))\}. \end{aligned} \tag{3.13}$$

The proof is completed from (3.13). □

In a similar way, we have the following theorem.

Theorem 3.7. *Let γ be a conformal quasi hemi-slant Riemannian submersion from a cosymplectic manifold $(\Sigma_m,\Omega,\xi,\eta,g_1)$ onto a Riemannian manifold (Σ_n,g_2), then the distribution D is integrable if and only if*

$$f_1(\widehat{\nabla}_M\mu_1 f_2 Z+\mathcal{T}_M\mu_2 Z)=0 \quad \textit{and} \quad f_1(\widehat{\nabla}_N\mu_1 f_2 Z+\mathcal{T}_N\mu_2 Z)=0$$

for $M,N\in\Gamma(D)$ and $Z\in\Gamma(D_\theta\oplus D^\perp)$.

Proof. We have, from Lemma 3.5, $g_1([M,N],\xi)=0$. Thus D is integrable if and only if $g_1([M,N],Z)=0$. Using (2.4), (2.11), (2.12) and (3.5), we have

$$\begin{aligned} g_1(\nabla_M N,Z) &= -g_1(\nabla_M(\mu_1 f_2 Z+\mu_2 f_2 Z+\mu_2 f_3 Z),\Omega N)\\ &= -g_1(\widehat{\nabla}_M\mu_1 f_2 Z+\mathcal{T}_M\mu_2 f_2 Z+\mathcal{T}_M\mu_2 f_3 Z,\Omega N), \end{aligned} \tag{3.14}$$

for $M,N\in\Gamma(D)$ and $Z\in\Gamma(D_\theta\oplus D^\perp)$. Using $\mu_2(f_2 Z+f_3 Z)=\mu_2 Z$ and equation (3.14) we obtain

$$g_1([M,N],Z) = g_1(\widehat{\nabla}_N\mu_1 f_2 Z+\mathcal{T}_N\mu_2 Z,\Omega M) - g_1(\widehat{\nabla}_M\mu_1 f_2 Z+\mathcal{T}_M\mu_2 Z,\Omega N). \tag{3.15}$$

Since D is an invariant distribution, we have $\Omega M,\Omega N\in\Gamma(D)$. Therefore, we obtain the proof from (3.15). □

Here, integrability condition of the anti-invariant distribution $D^\perp$ is same as the condition for hemi-slant Riemannian submersion in [14]. In addition, we know that the vertical distribution of a submersion is always integrable. Hence, we lastly give integrability condition for the horizontal distribution $(\ker\gamma_*)^\perp$.

Theorem 3.8. *Let γ be a conformal quasi hemi-slant Riemannian submersion from a cosymplectic manifold $(\Sigma_m,\Omega,\xi,\eta,g_1)$ onto a Riemannian manifold (Σ_n,g_2), then* *the distribution* $(\ker\gamma_*)^\perp$ *is integrable if and only if*

$$\begin{aligned} & g_2((\nabla\gamma_*)(P,\nu_1Q)-(\nabla\gamma_*)(Q,\nu_1P)+\nabla^{\gamma}_{Q}\gamma_*(\nu_2P)-\nabla^{\gamma}_{P}\gamma_*(\nu_2Q),\gamma_*(\mu_2Z)) \\ = \ & \lambda^2\{g_1(\mathcal{V}\nabla_P\nu_1Q-\mathcal{V}\nabla_Q\nu_1P,\mu_1Z)+\lambda^2 g_1(\mathcal{A}_P\nu_2Q-\mathcal{A}_Q\nu_2P,\mu_1Z)-\nu_2Q(\ln\lambda)g_1(P,\mu_2Z) \\ & +\mu_2Z(\ln\lambda)g_1(P,\nu_2Q)+\nu_2P(\ln\lambda)g_1(Q,\mu_2Z)-\mu_2Z(\ln\lambda)g_1(Q,\nu_2P) \end{aligned}$$

for $P,Q\in\Gamma(\ker\gamma_*)^\perp$ *and* $Z\in\Gamma(\ker\gamma_*)$.

Proof. Firstly, from (2.13), (2.14), (3.4) and (3.8), we have

$$g_1(\nabla_PQ,Z)=g_1(\mathcal{A}_P\nu_1Q+\mathcal{H}\nabla_P\nu_2Q,\mu_2Z)+g_1(\mathcal{A}_P\nu_2Q+\mathcal{V}\nabla_P\nu_1Q,\mu_1Z) \tag{3.16}$$

for $P,Q\in\Gamma(\ker\gamma_*)^\perp$ and $Z\in\Gamma(\ker\gamma_*)$. Now, changing the roles of P and Q in (3.16), we get

$$\begin{aligned} g_1([P,Q],Z) &= g_1(\mathcal{A}_P\nu_1Q+\mathcal{H}\nabla_P\nu_2Q-\mathcal{A}_Q\nu_1P-\mathcal{H}\nabla_Q\nu_2P,\mu_2Z) \\ &\quad + g_1(\mathcal{A}_P\nu_2Q+\mathcal{V}\nabla_P\nu_1Q-\mathcal{A}_Q\nu_2P-\mathcal{V}\nabla_Q\nu_1P,\mu_1Z). \end{aligned} \tag{3.17}$$

Hence, using equations (2.7), (2.15) and (2.17) in (3.17), and since μ is orthogonal to $\mu_2D_\theta\oplus\Omega(D^\perp)$ therefore, we obtain

$$\begin{aligned} 0 &= g_1(\mathcal{V}\nabla_P\nu_1Q+\mathcal{A}_P\nu_2Q-\mathcal{V}\nabla_Q\nu_1P-\mathcal{A}_Q\nu_2P,\mu_1Z) \\ &\quad+\frac{1}{\lambda^2}g_2((\nabla\gamma_*)(Q,\nu_1P)-(\nabla\gamma_*)(P,\nu_1Q),\gamma_*(\mu_2Z)) \\ &\quad+\frac{1}{\lambda^2}g_2(\nabla^{\gamma}_{P}\gamma_*(\nu_2Q)-\nabla^{\gamma}_{Q}\gamma_*(\nu_2P),\gamma_*(\mu_2Z)) \\ &\quad-\nu_2Q(\ln\lambda)g_1(P,\mu_2Z)+\mu_2Z(\ln\lambda)g_1(P,\nu_2Q) \\ &\quad+\nu_2P(\ln\lambda)g_1(Q,\mu_2Z)-\mu_2Z(\ln\lambda)g_1(Q,\nu_2P). \end{aligned} \tag{3.18}$$

One can see the proof from (3.18). □

4. Totally Geodesicness on Distributions

In this section, we present conditions for certain distributions and the map γ to define totally geodesic foliations on Σ_m.

Theorem 4.1. *Let* γ *be a conformal quasi hemi-slant Riemannian submersion from a cosymplectic manifold* $(\Sigma_m,\Omega,\xi,\eta,g_1)$ *onto a Riemannian manifold* (Σ_n,g_2), *then the distribution* D *defines totally geodesic foliations on* Σ_m *if and only if*

(*i*) $\lambda^2 g_1(\widehat{\nabla}_M\Omega N,\mu_1f_2Y)=g_2((\nabla\gamma_*)(M,\Omega N),\gamma_*(\mu_2Y))$,
(*ii*) $\lambda^2 g_1(\widehat{\nabla}_M\Omega N,\nu_1Z)=g_2((\nabla\gamma_*)(M,\Omega N),\gamma_*(\nu_2Z))$
for $M,N\in\Gamma(D)$, $Y\in\Gamma(D_\theta\oplus D^\perp)$ *and* $Z\in\Gamma(\ker\gamma_*)^\perp$.

Proof. Firstly, from (2.11), (2.16), and (3.4) we have

$$\begin{aligned} g_1(\nabla_MN,Y) &= g_1(\mathcal{T}_M\Omega N,\mu_2Y)+g_1(\widehat{\nabla}_M\Omega N,\mu_1Y) \\ &= -\frac{1}{\lambda^2}g_2((\nabla\gamma_*)(M,\Omega N),\gamma_*(\mu_2Y))+g_1(\widehat{\nabla}_M\Omega N,\mu_1Y) \end{aligned} \tag{4.1}$$

for $M,N\in\Gamma(D)$ and $Y\in\Gamma(D_\theta\oplus D^\perp)$. On the other hand, from (2.11), (2.16) and (3.8) we have

$$\begin{aligned} g_1(\nabla_MN,Z) &= g_1(\mathcal{T}_M\Omega N,\nu_2Z)+g_1(\widehat{\nabla}_M\Omega N,\nu_1Z) \\ &= -\frac{1}{\lambda^2}g_2((\nabla\gamma_*)(M,\Omega N),\gamma_*(\nu_2Z))+g_1(\widehat{\nabla}_M\Omega N,\nu_1Z). \end{aligned} \tag{4.2}$$

for $M, N \in \Gamma(D)$ and $Z \in \Gamma(\ker \gamma_*)^{\perp}$.
We obtain (i) and (ii) from (4.1) and (4.2), respectively. □

Theorem 4.2. *Let γ be a conformal quasi hemi-slant Riemannian submersion from a cosymplectic manifold $(\Sigma_m, \Omega, \xi, \eta, \mathrm{g}_1)$ onto a Riemannian manifold (Σ_n, g_2), then the distribution D_θ defines totally geodesic foliations on Σ_m if and only if*

$$(i)\ -\lambda^2\{\cos^2\theta \mathrm{g}_1(\widehat{\nabla}_M f_2 N, Y) + \mathrm{g}_1(\mathcal{H}\nabla_M \mu_2 f_2 N, \mu_2 f_3 Y)\} = \mathrm{g}_2((\nabla\gamma_*)(M, Y), \gamma_*(\mu_2\mu_1 f_2 N)) + \mathrm{g}_2((\nabla\gamma_*)(M, \mu_1 f_1 Y), \gamma_*(\mu_2 f_2 N)),$$

$$(ii)\ \lambda^2\{\mathrm{g}_1(\mathcal{H}\nabla_M \mu_2\mu_1 f_2 N, Z) + \mathrm{g}_1(\mathcal{H}\nabla_M \mu_2 f_2 N, \nu_2 Z)\} = \cos^2\theta \mathrm{g}_2((\nabla\gamma_*)(M, f_2 N), \gamma_*(Z)) - \mathrm{g}_2((\nabla\gamma_*)(M, \nu_1 Z), \gamma_*(\mu_2 f_2 N))$$

for $M, N \in \Gamma(D_\theta)$, $Y \in \Gamma(D \oplus D^{\perp})$ and $Z \in \Gamma(\ker \gamma_*)^{\perp}$.

Proof. From equations (2.11), (2.12), (2.16), (3.9) and skew-symmetric properties of $\mathcal{T}$ we have

$$\begin{aligned}
\mathrm{g}_1(\nabla_M N, Y) &= \cos^2\theta \mathrm{g}_1(\nabla_M f_2 N, Y) + \mathrm{g}_1(\mathcal{T}_M \mu_2\mu_1 f_2 N, Y) \\
&\quad + \mathrm{g}_1(\mathcal{T}_M \mu_2 f_2 N, \mu_1 f_1 Y) + \mathrm{g}_1(\mathcal{H}\nabla_M \mu_2 f_2 N, \mu_2 f_3 Y) \\
&= \cos^2\theta \mathrm{g}_1(\nabla_M f_2 N, Y) - \mathrm{g}_1(\mathcal{T}_M Y, \mu_2\mu_1 f_2 N) \\
&\quad - \mathrm{g}_1(\mathcal{T}_M \mu_1 f_1 Y, \mu_2 f_2 N) + \mathrm{g}_1(\mathcal{H}\nabla_M \mu_2 f_2 N, \mu_2 f_3 Y) \\
&= \cos^2\theta \mathrm{g}_1(\nabla_M f_2 N, Y) + \mathrm{g}_1(\mathcal{H}\nabla_M \mu_2 f_2 N, \mu_2 f_3 Y) \\
&\quad + \frac{1}{\lambda^2}\mathrm{g}_2((\nabla\gamma_*)(M, Y), \gamma_*(\mu_2\mu_1 f_2 N)) \\
&\quad + \frac{1}{\lambda^2}\mathrm{g}_2((\nabla\gamma_*)(M, \mu_1 f_1 Y), \gamma_*(\mu_2 f_2 N))
\end{aligned} \tag{4.3}$$

for $M, N \in \Gamma(D_\theta)$ and $Y \in \Gamma(D \oplus D^{\perp})$. In a similar way, from (3.8) we have

$$\begin{aligned}
\mathrm{g}_1(\nabla_M N, Z) &= \cos^2\theta \mathrm{g}_1(\nabla_M f_2 N, Z) + \mathrm{g}_1(\mathcal{H}\nabla_M \mu_2\mu_1 f_2 N, Z) \\
&\quad - \mathrm{g}_1(\mathcal{T}_M \nu_1 Z, \mu_2 f_2 N) + \mathrm{g}_1(\mathcal{H}\nabla_M \mu_2 f_2 N, \nu_2 Z) \\
&= \cos^2\theta \mathrm{g}_1(\mathcal{T}_M f_2 N, Z) + \mathrm{g}_1(\mathcal{H}\nabla_M \mu_2\mu_1 f_2 N, Z) \\
&\quad + \frac{1}{\lambda^2}\mathrm{g}_2((\nabla\gamma_*)(M, \nu_1 Z), \gamma_*(\mu_2 f_2 N)) + \mathrm{g}_1(\mathcal{H}\nabla_M \mu_2 f_2 N, \nu_2 Z) \\
&= -\cos^2\theta \frac{1}{\lambda^2}\mathrm{g}_2((\nabla\gamma_*)(M, f_2 N), \gamma_*(Z)) + \mathrm{g}_1(\mathcal{H}\nabla_M \mu_2\mu_1 f_2 N, Z) \\
&\quad + \frac{1}{\lambda^2}\mathrm{g}_2((\nabla\gamma_*)(M, \nu_1 Z), \gamma_*(\mu_2 f_2 N)) + \mathrm{g}_1(\mathcal{H}\nabla_M \mu_2 f_2 N, \nu_2 Z)
\end{aligned} \tag{4.4}$$

$M, N \in \Gamma(D_\theta)$ and $Z \in \Gamma(\ker \gamma_*)^{\perp}$. We obtain (i) and (ii) from (4.3) and (4.4), respectively. □

Theorem 4.3. *Let γ be a conformal quasi hemi-slant Riemannian submersion from a cosymplectic manifold $(\Sigma_m, \Omega, \xi, \eta, \mathrm{g}_1)$ onto a Riemannian manifold (Σ_n, g_2), then the distribution $D^{\perp}$ defines totally geodesic foliations on Σ_m if and only if*

(i) $-\lambda^2 \mathrm{g}_1(\mathcal{H}\nabla_M \Omega N, \mu_2 f_2 Y) = \mathrm{g}_2((\nabla\gamma_*)(M, \mu_1 Y), \gamma_*(\Omega N))$,

(ii) $-\lambda^2 \mathrm{g}_1(\mathcal{H}\nabla_M \nu_2 Z, \Omega N) = \mathrm{g}_2((\nabla\gamma_*)(M, \nu_1 Z), \gamma_*(\Omega N))$

for $M, N \in \Gamma(D^{\perp}), Y \in \Gamma(D \oplus D_\theta)$ and $Z \in \Gamma(\ker \gamma_*)^{\perp}$.

Proof. Since the distribution D is invariant. Hence, from (3.3) and (3.4) we have $\Omega Y = \mu_1 Y + \mu_2 f_2 Y$. So, we get using skew-symmetric properties of $\mathcal{T}$, (2.12) and (2.16)

$$\begin{aligned} g_1(\nabla_M N, Y) &= g_1(\mathcal{T}_M \Omega N, \mu_1 Y) + g_1(\mathcal{H}\nabla_M \Omega N, \mu_2 f_2 Y) \\ &= -g_1(\mathcal{T}_M \mu_1 Y, \Omega N) + g_1(\mathcal{H}\nabla_M \Omega N, \mu_2 f_2 Y) \\ &= \frac{1}{\lambda^2} g_2((\nabla\gamma_*)(M, \mu_1 Y), \gamma_*(\Omega N)) + g_1(\mathcal{H}\nabla_M \Omega N, \mu_2 f_2 Y) \end{aligned} \tag{4.5}$$

$M, N \in \Gamma(D^\perp)$ and $Y \in \Gamma(D \oplus D_\theta)$. Similarly, from (2.11), (2.12), (3.4) and (3.8) we get

$$\begin{aligned} g_1(\nabla_M N, Z) &= -g_1(\mathcal{T}_M \nu_1 Z + \mathcal{H}\nabla_M \nu_2 Z, \Omega N) \\ &= \frac{1}{\lambda^2} g_2((\nabla\gamma_*)(M, \nu_1 Z), \gamma_*(\Omega N)) + g_1(\mathcal{H}\nabla_M \nu_2 Z, \Omega N) \end{aligned} \tag{4.6}$$

for $M, N \in \Gamma(D^\perp)$ and $Z \in \Gamma(\ker\gamma_*)^\perp$. We obtain (i) and (ii) from (4.5) and (4.6), respectively. □

Theorem 4.4. *Let γ be a conformal quasi hemi-slant Riemannian submersion from a cosymplectic manifold $(\Sigma_m, \Omega, \xi, \eta, g_1)$ onto a Riemannian manifold (Σ_n, g_2), then the vertical distribution $\ker\gamma_*$ defines totally geodesic foliations on Σ_m if and only if*

$$\begin{aligned} &\lambda^2\{g_1(\mathcal{H}\nabla_M \mu_2 f_2 N + \mathcal{T}_M \mu_2 f_3 N, \nu_2 Z) - g_1(\mathcal{H}\nabla_M \mu_2\mu_1 f_2 N, Z) \\ &+ g_1(\widehat{\nabla}_M \mu_1 f_1 N + \mathcal{T}_M \mu_2 f_2 N + \mathcal{V}\nabla_M \mu_2 f_3 N, \nu_1 Z)\} \\ &= \cos^2\theta g_2((\nabla\gamma_*)(M, f_2 N), \gamma_*(Z)) + g_2((\nabla\gamma_*)(M, \mu_1 f_1 N), \gamma_*(\nu_2 Z)) \end{aligned}$$

for $M, N \in \Gamma(\ker\gamma_*)$ and $Z \in \Gamma(\ker\gamma_*)^\perp$.

Proof. We calculate the case of $g_1(\nabla_M N, Z) = 0$ for $M, N \in \Gamma(\ker\gamma_*)$ and $Z \in \Gamma(\ker\gamma_*)^\perp$. So, from (2.11), (2.12) and (3.5) we have

$$\begin{aligned} g_1(\nabla_M N, Z) &= g_1(\nabla_M(\mu_1 f_1 N + \mu_1 f_2 N + \mu_2 f_2 N + \mu_2 f_3 N), \Omega Z) \\ &= g_1(\mathcal{T}_M \mu_1 f_1 N + \mathcal{H}\nabla_M \mu_2 f_2 N + \mathcal{T}_M \mu_2 f_3 N, \nu_2 Z) \\ &+ g_1(\widehat{\nabla}_M \mu_1 f_1 N + \mathcal{T}_M \mu_2 f_2 N + \mathcal{V}\nabla_M \mu_2 f_3 N, \nu_1 Z) \\ &- g_1(\nabla_M \mu_1^2 f_2 N + \nabla_M \mu_2\mu_1 f_2 N, Z). \end{aligned} \tag{4.7}$$

Here, we use equations (2.16) and (3.9) in (4.7). Hence, we obtain

$$\begin{aligned} 0 &= g_1(\mathcal{T}_M \mu_1 f_1 N + \mathcal{H}\nabla_M \mu_2 f_2 N + \mathcal{T}_M \mu_2 f_3 N, \nu_2 Z) \\ &+ g_1(\widehat{\nabla}_M \mu_1 f_1 N + \mathcal{T}_M \mu_2 f_2 N + \mathcal{V}\nabla_M \mu_2 f_3 N, \nu_1 Z) \\ &+ \cos^2\theta g_1(\mathcal{T}_M f_2 N, Z) - g_1(\mathcal{H}\nabla_M \mu_2\mu_1 f_2 N, Z) \\ &= -\frac{1}{\lambda^2} g_2((\nabla\gamma_*)(M, \mu_1 f_1 N), \gamma_*(\nu_2 Z)) \\ &+ g_1(\mathcal{H}\nabla_M \mu_2 f_2 N + \mathcal{T}_M \mu_2 f_3 N, \nu_2 Z) \\ &+ g_1(\widehat{\nabla}_M \mu_1 f_1 N + \mathcal{T}_M \mu_2 f_2 N + \mathcal{V}\nabla_M \mu_2 f_3 N, \nu_1 Z)\} \\ &- \frac{1}{\lambda^2}\cos^2\theta g_2((\nabla\gamma_*)(M, f_2 N), \gamma_*(Z)) - g_1(\mathcal{H}\nabla_M \mu_2\mu_1 f_2 N, Z). \end{aligned} \tag{4.8}$$

The proof is completed from (4.8). □

Theorem 4.5. *Let γ be a conformal quasi hemi-slant Riemannian submersion from a cosymplectic manifold $(\Sigma_m, \Omega, \xi, \eta, g_1)$ onto a Riemannian manifold (Σ_n, g_2), then the horizontal distribution $(\ker\gamma_*)^\perp$ defines totally geodesic foliations on Σ_m if and only if*

$$\frac{1}{\lambda^2}\{g_2((\nabla\gamma_*)(M,\nu_1 N),\gamma_*(\mu_2 Z)) - g_2((\nabla\gamma_*)(M,\mu_1 Z),\gamma_*(\nu_2 N))\} = g_1(\mathcal{V}\nabla_M\nu_1 N,\mu_1 Z) + \nu_2 N(\ln\lambda)g_1(M,\mu_2 Z) - \mu_2 Z(\ln\lambda)g_1(M,\nu_2 N)$$

for $M, N \in \Gamma(\ker\gamma_*)^\perp$ and $Z \in \Gamma(\ker\gamma_*)$.

Proof. Using equations (2.4), (2.13), (2.14) and (3.4), we get

$$\begin{aligned} g_1(\nabla_M N, Z) &= g_1(\nabla_M\nu_1 N + \nabla_M\nu_2 N, \mu_1 Z + \mu_2 Z) \\ &= g_1(\mathcal{A}_M\nu_1 N + \mathcal{H}\nabla_M\nu_2 N, \mu_2 Z) \\ &\quad + g_1(\mathcal{V}\nabla_M\nu_1 N + \mathcal{A}_M\nu_2 N, \mu_1 Z) \end{aligned} \tag{4.9}$$

for $M, N \in \Gamma(\ker\gamma_*)^\perp$ and $Z \in \Gamma(\ker\gamma_*)$. Here, we apply (2.15), (2.17), (3.8) to (4.9) and from skew-symmetric properties of $\mathcal{A}$, we obtain

$$\begin{aligned} g_1(\nabla_M N, Z) = &-\frac{1}{\lambda^2}g_2((\nabla\gamma_*)(M,\nu_1 N),\gamma_*(\mu_2 Z)) + \nu_2 N(\ln\lambda)g_1(M,\mu_2 Z) \\ &- \mu_2 Z(\ln\lambda)g_1(M,\nu_2 N) + g_1(\mathcal{V}\nabla_M\nu_1 N,\mu_1 Z) \\ &+ \frac{1}{\lambda^2}g_2((\nabla\gamma_*)(M,\mu_1 Z),\gamma_*(\nu_2 N)). \end{aligned} \tag{4.10}$$

The proof is completed from (4.10). □

Note that, a horizontally conformal submersion $\gamma : (\Sigma_m, \Omega, \xi, \eta, g_1) \to (\Sigma_n, g_2)$ is said to be horizontally homothetic if the gradient of its dilation λ is vertical, i.e., $\mathcal{H}(grad\lambda) = 0$ at regular points. Hence, we have the following.

Corollary 4.6. *Let γ be a conformal quasi hemi-slant Riemannian submersion from a cosymplectic manifold $(\Sigma_m, \Omega, \xi, \eta, g_1)$ onto a Riemannian manifold (Σ_n, g_2), then the horizontal distribution $(\ker\gamma_*)^\perp$ defines totally geodesic foliations on Σ_m if and only if*

(i) γ is a horizontally homothetic map,

(ii) $g_2((\nabla\gamma_*)(M,\nu_1 N),\gamma_*(\mu_2 Z)) - g_2((\nabla\gamma_*)(M,\mu_1 Z),\gamma_*(\nu_2 N)) = \lambda^2 g_1(\mathcal{V}\nabla_M\nu_1 N,\mu_1 Z)$ *for $M, N \in \Gamma(\ker\gamma_*)^\perp$ and $Z \in \Gamma(\ker\gamma_*)$.*

Proof. Since γ defines totally geodesic foliations on Σ_m. Hence, we have (4.10). Suppose that γ is a horizontally homothetic map, so we have from (4.10)

$$0 = \nu_2 N(\ln\lambda)g_1(M,\mu_2 Z) - \mu_2 Z(\ln\lambda)g_1(M,\nu_2 N) \tag{4.11}$$

for $M, N \in \Gamma(\ker\gamma_*)^\perp$ and $Z \in \Gamma(\ker\gamma_*)$. Here, if we take $M = \mu_2 Z$ in (4.11) we get

$$0 = \nu_2 N(\ln\lambda)g_1(\mu_2 Z,\mu_2 Z). \tag{4.12}$$

In (4.12), we get $0 = \nu_2 N(\ln\lambda)$ and it means λ is a constant on μ. Similarly, if we take $M = \nu_2 N$ in (4.11) we get

$$0 = -\mu_2 Z(\ln\lambda)g_1(\nu_2 N,\nu_2 N). \tag{4.13}$$

In (4.13), we get $0 = \mu_2 Z(\ln\lambda)$ and it means λ is a constant on $\mu_2 D_\theta \oplus \Omega(D^\perp)$. Therefore, from (4.12) and (4.13) we say that λ is a constant on horizontal distribution. So, (i) is satisfied. Now, if (i) is satisfied in (4.10), we obtain

$$\begin{aligned}0 = &-\frac{1}{\lambda^2}g_2((\nabla\gamma_*)(M,\nu_1 N),\gamma_*(\mu_2 Z)) + g_1(\mathcal{V}\nabla_M\nu_1 N,\mu_1 Z)\\ &+\frac{1}{\lambda^2}g_2((\nabla\gamma_*)(M,\mu_1 Z),\gamma_*(\nu_2 N)).\end{aligned}\tag{4.14}$$

From (4.14), (ii) is satisfied. The proof is completed. □

A horizontally conformal submersion $\gamma : (\Sigma_m, \Omega, \xi, \eta, g_1) \to (\Sigma_n, g_2)$ is said to be totally geodesic if second fundamental form of the map $(\nabla\gamma_*)(P,Q) = 0$ for $P, Q \in \Gamma(T\Sigma_m)$. Hence, we have the next theorem.

Theorem 4.7. *Let γ be a conformal quasi hemi-slant Riemannian submersion from a cosymplectic manifold $(\Sigma_m, \Omega, \xi, \eta, g_1)$ onto a Riemannian manifold (Σ_n, g_2), then the map γ is totally geodesic if and only if*

(*i*) $\cos^2\theta\mathcal{T}_M f_2 N = \mathcal{H}\nabla_M\mu_2\mu_1 f_2 N + \nu_2\{\mathcal{T}_M\mu_1 f_1 N + \mathcal{H}\nabla_M\mu_2 N\} + \mu_2\{\widehat{\nabla}_M\mu_1 f_1 N + \mathcal{T}_M\mu_2 N\}$,

(*ii*) $0 = \nu_2\{\mathcal{A}_P\mu_1 M + \mathcal{H}\nabla_P\mu_2 M\} + \mu_2\{\mathcal{V}\nabla_P\mu_1 M + \mathcal{A}_P\mu_2 M\}$,

(*iii*) *γ is a horizontally homothetic map,*

for $P, Q \in \Gamma(\ker\gamma_*)^\perp$ and $M, N \in \Gamma(\ker\gamma_*)$.

Proof. Firstly, we examine $(\nabla\gamma_*)(M,N)$ for $M, N \in \Gamma(\ker\gamma_*)$. Because of $\mu_2 f_2 N + \mu_2 f_3 N = \mu_2 N$ we have from (2.4), (2.7) and (3.5)

$$(\nabla\gamma_*)(M,N) = \gamma_*(\Omega\nabla_M(\mu_1 f_1 N + \mu_1 f_2 N + \mu_2 N))$$

for $M, N \in \Gamma(\ker\gamma_*)$. Then, using equations (2.11), (2.12) and (3.9), we have

$$\begin{aligned}(\nabla\gamma_*)(M,N) &= \gamma_*(\Omega\mathcal{T}_M\mu_1 f_1 N + \Omega\widehat{\nabla}_M\mu_1 f_1 N)\\ &\quad+\gamma_*(\nabla_M\mu_1^2 f_2 N + \nabla_M\mu_2\mu_1 f_2 N)\\ &\quad+\gamma_*(\Omega\mathcal{T}_M\mu_2 N + \Omega\mathcal{H}\nabla_M\mu_2 N)\\ &= \gamma_*(\nu_2\mathcal{T}_M\mu_1 f_1 N + \mu_2\widehat{\nabla}_M\mu_1 f_1 N)\\ &\quad-\cos^2\theta\gamma_*(\nabla_M f_2 N) + \gamma_*(\mathcal{H}\nabla_M\mu_2\mu_1 f_2 N)\\ &\quad+\gamma_*(\mu_2\mathcal{T}_M\mu_2 N + \nu_2\mathcal{H}\nabla_M\mu_2 N)\\ &= \gamma_*(\nu_2\{\mathcal{T}_M\mu_1 f_1 N + \mathcal{H}\nabla_M\mu_2 N\}\\ &\quad+\mu_2\{\widehat{\nabla}_M\mu_1 f_1 N + \mathcal{T}_M\mu_2 N\})\\ &\quad-\cos^2\theta\gamma_*(\mathcal{T}_M f_2 N) + \gamma_*(\mathcal{H}\nabla_M\mu_2\mu_1 f_2 N).\end{aligned}\tag{4.15}$$

We obtain (*i*) from (4.15). Second fundamental form of map is symmetric . So, we have $(\nabla\gamma_*)(M,P) = (\nabla\gamma_*)(P,M)$ for $P \in \Gamma(\ker\gamma_*)^\perp$ and $M \in \Gamma(\ker\gamma_*)$. From (2.7), (2.13), (2.14) and (3.4) we obtain

$$\begin{aligned}(\nabla\gamma_*)(P,M) &= \gamma_*(\Omega\nabla_P\mu_1 M + \Omega\nabla_P\mu_2 M)\\ &= \gamma_*(\Omega\mathcal{A}_P\mu_1 M + \Omega\mathcal{V}\nabla_P\mu_1 M)\\ &\quad+\gamma_*(\Omega\mathcal{A}_P\mu_2 M + \Omega\mathcal{H}\nabla_P\mu_2 M)\\ &= \gamma_*(\nu_2\mathcal{A}_P\mu_1 M + \mu_2\mathcal{V}\nabla_P\mu_1 M)\\ &\quad+\gamma_*(\mu_2\mathcal{A}_P\mu_2 M + \nu_2\mathcal{H}\nabla_P\mu_2 M).\end{aligned}\tag{4.16}$$

We obtain (ii) from (4.16). Lastly, from (2.15) we have

$$(\nabla\gamma_*)(P,Q) = P(\ln\lambda)\gamma_*(Q) + Q(\ln\lambda)\gamma_*(P) - g_1(P,Q)\gamma_*(grad(\ln\lambda)) \tag{4.17}$$

for $P, Q \in \Gamma(\ker\gamma_*)^\perp$. For P in (4.17) we obtain

$$0 = Q(\ln\lambda)g_2(\gamma_*(P), \gamma_*(P))$$

$$0 = \lambda^2 Q(\ln\lambda)g_1(P,P). \tag{4.18}$$

In (4.18), we get $Q(\ln\lambda) = 0$. It means λ is a constant on horizontal distribution. So, the map is horizontally homothetic. (iii) is satisfied. Hence the proof is completed. □

5. Examples

The canonical example of a cosymplectic manifold is given by the product $B^{2n}\times\mathbb{R}$ Kaehler manifold $B^{2n}(J, g_1)$ with the $\mathbb{R}$ real line. Now we will introduce a well-known cosymplectic manifold example of $\mathbb{R}^{2n+1}$.

We consider $\mathbb{R}^{2n+1}$ with cartesian coordinats $(u_i, v_i, t)(i = 1, 2, ..., n)$ and its usual contact one-form $\eta = dt$. The Reeb vector field ξ is given by $\frac{\partial}{\partial t}$ and its Riemannian metric g_1 and tensor field Ω are given by

$$g_1 = (dt)^2 + \sum_{i=i}^{n}((du_i)^2 + (dv_i)^2), \quad \Omega = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

this gives a cosymplectic manifold on $\mathbb{R}^{2n+1}$. The vector fields $e_i = \frac{\partial}{\partial v_i}, e_{n+i} = \frac{\partial}{\partial u_i}, \xi$ form a Ω-basis for the cosymplectic structure. On the other hand, it can be shown that $(\mathbb{R}^{2n+1}, \Omega, \xi, \eta, g_1)$ is a cosymplectic manifold.

Example 1. *Using above example, let $\mathbb{R}^9$ have a Cosymplectic structure. Define a map from $\mathbb{R}^9$ to $\mathbb{R}^3$ by,*

$$\gamma(u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, w) = e^2\left(u_1, \frac{\sqrt{3}u_2 + u_3}{2}, v_3\right),$$

where g_2 is Euclidean metric on $\mathbb{R}^3$.
Then, the Jacobian matrix of γ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Since, the rank of above Jacobian matrix is 3, therefore the map γ is a submersion. After computations, we obtain

$$(\ker\gamma_*) = span\{\frac{\partial}{\partial u_4}, \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}, \frac{\partial}{\partial v_4}, \frac{1}{2}\left(\frac{\partial}{\partial u_2} - \sqrt{3}\frac{\partial}{\partial u_3}\right), \frac{\partial}{\partial w}\},$$

$$(\ker\gamma_*)^\perp = span\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_3}, \frac{1}{2}\left(\sqrt{3}\frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_3}\right)\}.$$

Then it follows that,

$$D = span\{\frac{\partial}{\partial u_4}, \frac{\partial}{\partial v_4}\},\ D^{\theta} = span\{\frac{\partial}{\partial v_2}, \frac{1}{2}\left(\frac{\partial}{\partial u_2} - \sqrt{3}\frac{\partial}{\partial u_3}\right)\}\ and\ D^{\perp} = span\{\frac{\partial}{\partial v_1}\}.$$

Thus the map γ is conformal quasi hemi-slant Riemannian submersion with the quasi hemi-slant angle $\theta = \frac{\pi}{3}$ and dilation $\lambda = e^2$.

Example 2. *Let $\mathbb{R}^9$ have a Cosymplectic structure as in above example. Define a map from $\mathbb{R}^9$ to $\mathbb{R}^3$ by,*

$$\gamma(u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, w) = e^7\left(u_2, u_1 \sin\alpha - v_2\cos\alpha, v_3\right),$$

where g_2 is Euclidean metric on $\mathbb{R}^3$.
Then, by direct calculations, we obtain the Jacobian matrix of γ as

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin\alpha & 0 & 0 & 0 & 0 & -\cos\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Since, the rank of above Jacobian matrix is 3, therefore the map γ is a submersion. After computations, we obtain

$$(\ker\gamma_*) = span\{\frac{\partial}{\partial u_3}, \frac{\partial}{\partial u_4}, \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_4}, \cos\alpha\frac{\partial}{\partial u_1} + \sin\alpha\frac{\partial}{\partial v_2}, \frac{\partial}{\partial w}\},$$
$$(\ker\gamma_*)^{\perp} = span\{\frac{\partial}{\partial u_2}, \frac{\partial}{\partial v_3}, \sin\alpha\frac{\partial}{\partial u_1} - \cos\alpha\frac{\partial}{\partial v_2}\}.$$

Then it follows that,

$$D = span\{\frac{\partial}{\partial u_4}, \frac{\partial}{\partial v_4}\},\ D^{\theta} = span\{\frac{\partial}{\partial v_1}, \cos\alpha\frac{\partial}{\partial u_1} + \sin\alpha\frac{\partial}{\partial v_2}\}\ and\ D^{\perp} = span\{\frac{\partial}{\partial u_3}\}.$$

Thus the map γ is conformal quasi hemi-slant Riemannian submersion with the quasi hemi-slant angle $\theta = \alpha$ and dilation $\lambda = e^7$.

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