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MEROMORPHIC CONTINUATIONS OF MULTIPLE q-HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. The purpose of this article is to compute meromorphic continuations of several multiple q-hypergeometric functions simply by using the formula for meromorphic continuation of ${}_2\phi_1$. This leads to q-analogues of analytic continuation formulas by Appell and Kampé de Fériet, Exton and Srivastava. All these formulas are proved in detail. For each of the multiple functions, we write down the canonical system of q-difference equations, and by the meromorphic continuation formulas, we automatically get other solutions to these systems in the neighbourhood of infinity. We recall that for q-hypergeometric functions, branching only occurs at the 'principal' regular singular points 0 and ∞.

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1. INTRODUCTION

We refer to our standard work [2], where all q-analysis definitions are made. The convergence regions for our multiple q-hypergeometric functions are often replaced by the corresponding q-addition regions [3], these regions are always larger than the convergence regions for the corresponding multiple q-hypergeometric functions. Some examples are a q-deformed rhombus in dimension 2 and a q-deformed octahedron in dimension 3. In q-calculus, differential equations are replaced by q-difference equations, which often have similar solutions. The canonical systems of q-difference equations for multiple q-hypergeometric functions are not well-known, although it has the same form as the canonical system of differential equations for multiple hypergeometric functions. This system was first studied by Hjalmar Mallin [8], in the footsteps of Weierstrass. Meromorphic continuation, which occurs quite often in q-calculus, means analytic continuation when branching singularities are replaced by an infinite number of poles and zeros. We write down and prove meromorphic continuations of several q-hypergeometric functions; only two corresponding analytic continuation for the Appell functions F_2, which could not be found in the literature, is given without proof.

The paper is organized as follows: In section 1 we define all multiple q-functions together with some simple lemmas, which are used in the proofs. We also state the meromorphic continuation formula for the q-hypergeometric function ${}_2\phi_1$. In sections 2, 3, 4, 5 we write down the systems of q-difference equations for Φ_1, Φ_2, Φ_3, Φ_4 and prove the corresponding meromorphic continuation formulas.

In section 6 we write down the system of q-difference equations for the Srivastava triple q-hypergeometric function $\mathrm{H_C}$ and prove the corresponding meromorphic continuation formula. In section 7 we write down the system of q-difference equations for $\Phi_{\mathrm{D}}^{(3)}$ and prove the corresponding meromorphic continuation formula. In section 8 we prove the corresponding meromorphic continuation formula for $\Phi_{\mathrm{C}}^{(n)}$. The formulas for Φ_1, Φ_2, Φ_3, Φ_4, and $\Phi_{\mathrm{C}}^{(n)}$ are proved by using the meromorphic continuation formula for ${}_2\phi_1$. The formulas for $\mathrm{H_C}$ and $\Phi_{\mathrm{D}}^{(3)}$ are proved by using the meromorphic continuation formula for Φ_1.

We now repeat some notation from [2]. Let $\delta > 0$ be an arbitrary small number. We will always use the following branch of the logarithm: $-\pi+\delta < \mathrm{Im}\,(\log q) \leq \pi + \delta$. This defines a simply connected space in the complex plane.

The power function is defined by

$$q^a \equiv e^{a\log(q)}. \tag{1}$$

A q-analogue of a complex number is also a complex number.

Definition 1. The q-analogue of a complex number a is defined as follows:

$$\{a\}_q \equiv \frac{1-q^a}{1-q},\ q \in \mathbb{C}\backslash\{0,1\}. \tag{2}$$

The q-shifted factorial is defined by

$$\langle a;q\rangle_n \equiv \prod_{m=0}^{n-1}(1-q^{a+m}). \tag{3}$$

The following notation is often used when we have long exponents.

$$\mathrm{QE}(x) \equiv q^x. \tag{4}$$

The q-derivative is defined by

$$(\mathrm{D}_q\varphi)(x) \equiv \begin{cases} \frac{\varphi(x)-\varphi(qx)}{(1-q)x}, & \text{when } q \in \mathbb{C}\backslash\{1\},\ x \neq 0; \\ \frac{d\varphi}{dx}(x), & \text{when } q = 1; \\ \frac{d\varphi}{dx}(0), & \text{when } x = 0. \end{cases} \tag{5}$$

Definition 2. The following operator will also be useful.

$$\theta_{q,j} \equiv x_j \mathrm{D}_{q,x_j}. \tag{6}$$

Definition 3. [2]. The q-analogues of the Appell functions are

(7)

$$\Phi_1(\alpha;\beta,\beta';\gamma|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle\alpha;q\rangle_{m_1+m_2}\langle\beta;q\rangle_{m_1}\langle\beta';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle\gamma;q\rangle_{m_1+m_2}} x_1^{m_1}x_2^{m_2},$$

$\max(|x_1|,|x_2|) < 1$.

(8)

$$\Phi_2(\alpha;\beta,\beta';\gamma,\gamma'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle\alpha;q\rangle_{m_1+m_2}\langle\beta;q\rangle_{m_1}\langle\beta';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle\gamma;q\rangle_{m_1}\langle\gamma';q\rangle_{m_2}} x_1^{m_1}x_2^{m_2},$$

$|x_1| \oplus_q |x_2| < 1$.

(9)

$$\Phi_3(\alpha,\alpha';\beta,\beta';\gamma|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty}\frac{\langle\alpha;q\rangle_{m_1}\langle\alpha';q\rangle_{m_2}\langle\beta;q\rangle_{m_1}\langle\beta';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle\gamma;q\rangle_{m_1+m_2}}x_1^{m_1}x_2^{m_2}, \max(|x_1|,|x_2|)<1.$$

(10)

$$\Phi_4(\alpha;\beta;\gamma,\gamma'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty}\frac{\langle\alpha;q\rangle_{m_1+m_2}\langle\beta;q\rangle_{m_1+m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle\gamma;q\rangle_{m_1}\langle\gamma';q\rangle_{m_2}}x_1^{m_1}x_2^{m_2},$$

$|\sqrt{x_1}|\oplus_q|\sqrt{x_2}|<1.$

Definition 4. Two q-Lauricella functions are

$$\Phi_{\mathrm{C}}^{(n)}(\alpha,\beta;\vec{\gamma}|q;\vec{x}) \equiv \sum_{\vec{m}}\frac{\langle\alpha,\beta;q\rangle_m\vec{x}^{\vec{m}}}{\langle\vec{\gamma},\vec{1};q\rangle_{\vec{m}}},\ |\sqrt{x_1}|\oplus_q\ldots\oplus_q|\sqrt{x_n}|<1, \tag{11}$$

$$\Phi_{\mathrm{D}}^{(n)}(\alpha,\vec{\beta};\gamma|q;\vec{x}) \equiv \sum_{\vec{m}}\frac{\langle\alpha;q\rangle_m\langle\vec{\beta};q\rangle_{\vec{m}}\vec{x}^{\vec{m}}}{\langle\gamma;q\rangle_m\langle\vec{1};q\rangle_{\vec{m}}},\ \max(|x_1|,\ldots,|x_n|)<1. \tag{12}$$

Definition 5. [3] The triple q-$\mathrm{H_C}$ function is defined by

$$\mathrm{H_C}(\alpha,\beta_1,\beta_2;\gamma|q;x_1,x_2,x_3) \equiv \sum_{m,n,p=0}^{\infty}\frac{\langle\alpha;q\rangle_{m+p}\langle\beta_1;q\rangle_{m+n}\langle\beta_2;q\rangle_{n+p}}{\langle 1;q\rangle_m\langle 1;q\rangle_n\langle 1;q\rangle_p\langle\gamma;q\rangle_{m+n+p}}x_1^m x_2^n x_3^p. \tag{13}$$

For didactic reasons, we present some formulas for q-shifted factorials.

Lemma 1.1.

(14)

$$\frac{\langle b-c+1-m;q\rangle_n}{\langle c-b;q\rangle_m} = \langle b-c+1;q\rangle_{n-m}(-1)^m\mathrm{QE}\left(-\binom{m}{2}-m(c-b)\right),$$

$$\frac{\langle a-b;q\rangle_m}{\langle b-a+1-m;q\rangle_n} = \langle a-b;q\rangle_{m-n}(-1)^n\mathrm{QE}\left(-\binom{n}{2}-n(b-a+1-m)\right).$$

Proof. Use [2, 6.14-15]. □

Definition 6. [2, p.212]. The multiplication operator $\mathbb{C}[[\vec{x}]]\to\mathbb{C}[[\vec{x}]]$, i.e. multiplication with $x_i,\ 0\le i\le n$, is denoted by $\mathbf{x}_i$. We have skipped the I for the unit operator to the far right in all q-difference equations.

Theorem 1.2. *Mimachi* [9], *Watson* [12], [7, 4.3.2].

Meromorphic continuation of ${}_2\phi_1$ *to the region* $|z|>1$.

(15)

$$\begin{aligned}&{}_2\phi_1\left[\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\middle|q;z\right]\\ &=\Gamma_q\left[\begin{matrix}\gamma,\beta-\alpha\\ \beta,\gamma-\alpha\end{matrix}\right]\frac{(zq^\alpha,z^{-1}q^{1-\alpha};q)_\infty}{(z,\frac{q}{z};q)_\infty}\,{}_2\phi_1\left[\begin{matrix}\alpha,\alpha-\gamma+1\\ \alpha-\beta+1\end{matrix}\middle|q;q^{\gamma+1-\alpha-\beta}z^{-1}\right]\\ &+\Gamma_q\left[\begin{matrix}\gamma,\alpha-\beta\\ \alpha,\gamma-\beta\end{matrix}\right]\frac{(zq^\beta,z^{-1}q^{1-\beta};q)_\infty}{(z,\frac{q}{z};q)_\infty}\,{}_2\phi_1\left[\begin{matrix}\beta,\beta-\gamma+1\\ \beta-\alpha+1\end{matrix}\middle|q;q^{\gamma+1-\alpha-\beta}z^{-1}\right],\end{aligned}$$

where $|\arg(-z)|<\pi,\ \alpha,\ \beta,\ \gamma,\ \alpha-\beta\neq\mathbb{Z}$.

Corollary 1.3. *The q-analogue of the multivalued function $(-z)^{-\alpha}$ is the meromorphic function*

$$\frac{(zq^{\alpha}, z^{-1}q^{1-\alpha}; q)_\infty}{(z, \frac{q}{z}; q)_\infty}. \tag{16}$$

2. The q-Appell function Φ_1

The q-difference equation for Φ_1 can be written in the following canonical form, [2, 11.60].

$$\begin{cases} -\mathbf{x}\{\theta_{q,1}+\beta\}_q\{\theta_{q,1}+\theta_{q,2}+\alpha\}_q+\{\theta_{q,1}\}_q\{\theta_{q,1}+\theta_{q,2}+\gamma-1\}_q=0, \\ -\mathbf{y}\{\theta_{q,2}+\beta'\}_q\{\theta_{q,1}+\theta_{q,2}+\alpha\}_q+\{\theta_{q,2}\}_q\{\theta_{q,1}+\theta_{q,2}+\gamma-1\}_q=0. \end{cases} \tag{17}$$

Theorem 2.1. *A q-analogue of* [10, (2.3), p.103], [11, (59), p.292]. *Meromorphic continuation of the first q-Appell function to the region $|y|>1$.*

$$\begin{aligned} &\Phi_1(\alpha,\beta,\beta';\gamma|q;x,y) \\ &=\Gamma_q\left[\begin{matrix}\gamma,\beta'-\alpha\\ \beta',\gamma-\alpha\end{matrix}\right]\sum_{m,n=0}^{\infty}\frac{\langle\alpha;q\rangle_{m+n}\langle\beta;q\rangle_m\langle 1-\gamma+\alpha;q\rangle_n(yq^{\alpha+m},y^{-1}q^{1-\alpha-m};q)_\infty}{\langle 1;q\rangle_m\langle 1;q\rangle_n\langle\alpha-\beta'+1;q\rangle_{m+n}(y,\frac{q}{y};q)_\infty} \\ &(-1)^m x^m y^{-n}\mathrm{QE}\left(\binom{m}{2}+m(\alpha-\beta'+1)+n(\gamma+1-\alpha-\beta')\right) \\ &+\Gamma_q\left[\begin{matrix}\gamma,\alpha-\beta'\\ \alpha,\gamma-\beta'\end{matrix}\right]\frac{(yq^{\beta'},y^{-1}q^{1-\beta'};q)_\infty}{(y,\frac{q}{y};q)_\infty} \\ &\sum_{m,n=0}^{\infty}\frac{\langle\beta;q\rangle_m\langle\beta';q\rangle_n\langle\alpha-\beta';q\rangle_{m-n}\langle\beta'-\gamma+1;q\rangle_{n-m}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}(-1)^{m+n} \\ &x^m y^{-n}\mathrm{QE}\left(-\binom{m}{2}-\binom{n}{2}+m(\beta'-\gamma)+n(\gamma+m-2\beta')\right). \end{aligned} \tag{18}$$

Proof.

$$\begin{aligned} &\text{LHS}\overset{\text{by}(15)}{=}\sum_{m=0}^{\infty}\frac{\langle\alpha,\beta;q\rangle_m}{\langle 1,\gamma;q\rangle_m}\Bigg[\Gamma_q\left[\begin{matrix}\gamma+m,\beta'-\alpha-m\\ \beta',\gamma-\alpha\end{matrix}\right]\frac{(yq^{\alpha+m},y^{-1}q^{1-\alpha-m};q)_\infty}{(y,\frac{q}{y};q)_\infty} \\ &{}_2\phi_1\left[\begin{matrix}\alpha+m,\alpha-\gamma+1\\ \alpha-\beta'+m+1\end{matrix}\middle|q;y^{-1}q^{\gamma+1-\alpha-\beta'}\right] \\ &+\Gamma_q\left[\begin{matrix}\gamma+m,\alpha+m-\beta'\\ \alpha+m,\gamma-\beta'+m\end{matrix}\right]\frac{(yq^{\beta'},y^{-1}q^{1-\beta'};q)_\infty}{(y,\frac{q}{y};q)_\infty} \\ &{}_2\phi_1\left[\begin{matrix}\beta',\beta'-m-\gamma+1\\ \beta'-\alpha-m+1\end{matrix}\middle|q;y^{-1}q^{\gamma+1-\alpha-\beta'}\right]\Bigg]x^m\overset{\text{by}2\times[2,\,1.46],\,(14)}{=}\text{RHS}. \end{aligned} \tag{19}$$

□

The second double sum in (18) is obviously a q-analogue of the Horn function G_2.

3. The q-Appell function Φ_2

The second Appell function has the analytic continuation

$$
\begin{aligned}
&\mathrm{F}_2(\alpha;\beta,\beta';\gamma,\gamma'|x,y) = \Gamma\left[\begin{array}{c}\gamma',\beta'-\alpha\\ \beta',\gamma'-\alpha\end{array}\right](-y)^{-\alpha}\sum_{m,n=0}^{\infty}\\
&\frac{(\alpha)_{m+n}(\beta,1-\gamma'+\alpha)_m(\alpha-\gamma'+m+1)_n}{(1-\beta'+\alpha)_{m+n}(1,\gamma)_m n!}x^m y^{-n}(-y)^{-m}\\
&+\Gamma\left[\begin{array}{c}\gamma',\alpha-\beta'\\ \alpha,\gamma'-\beta'\end{array}\right](-y)^{-\beta'}\\
&\sum_{m,n=0}^{\infty}\frac{(\beta)_m(\beta',1-\gamma'+\beta')_n(\alpha-\beta')_{m-n}}{(1,\gamma)_m n!}(-y)^{-n}x^m.
\end{aligned}\tag{20}
$$

Now consider the q-analogue Φ_2. The q-difference equation for Φ_2 can be written in the following canonical form, [2, 11.61].

$$
\begin{cases}-\mathbf{x}\{\theta_{q,1}+\beta\}_q\{\theta_{q,1}+\theta_{q,2}+\alpha\}_q+\{\theta_{q,1}\}_q\{\theta_{q,1}+\gamma-1\}_q=0,\\ -\mathbf{y}\{\theta_{q,2}+\beta'\}_q\{\theta_{q,1}+\theta_{q,2}+\alpha\}_q+\{\theta_{q,2}\}_q\{\theta_{q,2}+\gamma'-1\}_q=0.\end{cases}\tag{21}
$$

Theorem 3.1. *Meromorphic continuation of the second q-Appell function to the region $|y|>1$.*

(22)

$$
\begin{aligned}
&\Phi_2(\alpha;\beta,\beta';\gamma,\gamma'|q;x,y)=\Gamma_q\left[\begin{array}{c}\gamma',\beta'-\alpha\\ \beta',\gamma'-\alpha\end{array}\right]\sum_{m,n=0}^{\infty}\\
&\frac{\langle\alpha;q\rangle_{m+n}\langle\beta,1-\gamma'+\alpha;q\rangle_m\langle\alpha-\gamma'+m+1;q\rangle_n(yq^{\alpha+m},y^{-1}q^{1-\alpha-m};q)_\infty}{\langle 1-\beta'+\alpha;q\rangle_{m+n}\langle 1,\gamma;q\rangle_m\langle 1;q\rangle_n(y,\frac{q}{y};q)_\infty}\\
&x^m y^{-n}\mathrm{QE}\left(n(\gamma'+1-\alpha-m-\beta')\right)+\Gamma_q\left[\begin{array}{c}\gamma',\alpha-\beta'\\ \alpha,\gamma'-\beta'\end{array}\right]\frac{(yq^{\beta'},y^{-1}q^{1-\beta'};q)_\infty}{(y,\frac{q}{y};q)_\infty}\\
&\sum_{m,n=0}^{\infty}\frac{\langle\beta;q\rangle_m\langle\beta',1-\gamma'+\beta';q\rangle_n\langle\alpha-\beta';q\rangle_{m-n}}{\langle 1,\gamma;q\rangle_m\langle 1;q\rangle_n}(-1)^n\\
&x^m y^{-n}\mathrm{QE}\left(-\binom{n}{2}+n(\gamma'-2\beta')\right).
\end{aligned}
$$

Proof.

(23)

$$
\begin{aligned}
&\mathrm{LHS}\overset{\text{by}(15)}{=}\sum_{m=0}^{\infty}\frac{\langle\alpha,\beta;q\rangle_m}{\langle 1,\gamma;q\rangle_m}\Bigg[\Gamma_q\left[\begin{array}{c}\gamma',\beta'-\alpha-m\\ \beta',\gamma'-\alpha-m\end{array}\right]\frac{(yq^{\alpha+m},y^{-1}q^{1-\alpha-m};q)_\infty}{(y,\frac{q}{y};q)_\infty}\\
&{}_2\phi_1\left[\begin{array}{c}\alpha+m,\alpha+m-\gamma'+1\\ \alpha-\beta'+m+1\end{array}\middle|q;y^{-1}q^{\gamma'+1-\alpha-m-\beta'}\right]\\
&+\Gamma_q\left[\begin{array}{c}\gamma',\alpha+m-\beta'\\ \alpha+m,\gamma'-\beta'\end{array}\right]\frac{(yq^{\beta'},y^{-1}q^{1-\beta'};q)_\infty}{(y,\frac{q}{y};q)_\infty}\\
&{}_2\phi_1\left[\begin{array}{c}\beta',\beta'-\gamma'+1\\ \beta'-\alpha-m+1\end{array}\middle|q;y^{-1}q^{\gamma'+1-\alpha-m-\beta'}\right]\Bigg]x^m\overset{\text{by}2\times[2,1.46],(14)}{=}\mathrm{RHS}.
\end{aligned}
$$

□

The second double sum in (22) is obviously of the same form as the original (20).

4. The q-Appell function Φ_3

The third Appell function has the analytic continuation [11, (67), p. 296].

$$\begin{aligned}
&\mathrm{F}_3(\alpha,\alpha';\beta,\beta';\gamma|x,y)=\Gamma\begin{bmatrix}\gamma,\beta'-\alpha'\\ \beta',\gamma-\alpha'\end{bmatrix}(-y)^{-\alpha'}\\
&\mathrm{H}_2\left[\alpha'-\gamma+1,\alpha',\alpha,\beta;\alpha'-\beta'+1\middle|\frac{1}{y},-x\right]+\Gamma\begin{bmatrix}\gamma,\alpha'-\beta'\\ \alpha',\gamma-\beta'\end{bmatrix}\\
&(-y)^{-\beta'}\mathrm{H}_2\left[\beta'-\gamma+1,\beta',\alpha,\beta;\beta'-\alpha'+1\middle|\frac{1}{y},-x\right],
\end{aligned}\tag{24}$$

where H_2 is a Horn function. Now consider the q-analogue Φ_3.

The q-difference equation for Φ_3 can be written in the following canonical form, [2, 11.62].

$$\begin{cases}-\mathbf{x}\{\theta_{q,1}+\alpha\}_q\{\theta_{q,1}+\beta\}_q+\{\theta_{q,1}\}_q\{\theta_{q,1}+\theta_{q,2}+\gamma-1\}_q=0,\\ -\mathbf{y}\{\theta_{q,2}+\alpha'\}_q\{\theta_{q,2}+\beta'\}_q+\{\theta_{q,2}\}_q\{\theta_{q,1}+\theta_{q,2}+\gamma-1\}_q=0.\end{cases}\tag{25}$$

Theorem 4.1. *Meromorphic continuation of the third q-Appell function to the region $|y|>1$.*

$$\begin{aligned}
&\Phi_3(\alpha,\alpha';\beta,\beta';\gamma|q;x,y)=\Gamma_q\begin{bmatrix}\gamma,\beta'-\alpha'\\ \beta',\gamma-\alpha'\end{bmatrix}\sum_{m,n=0}^{\infty}\\
&\frac{\langle\alpha,\beta;q\rangle_m\langle\alpha';q\rangle_n(yq^{\alpha'},y^{-1}q^{1-\alpha'};q)_\infty}{\langle 1;q\rangle_m\langle 1,\alpha'-\beta'+1;q\rangle_n\langle\gamma-\alpha';q\rangle_{m-n}(y,\frac{q}{y};q)_\infty}\\
&(-1)^n x^m y^{-n}\mathrm{QE}\left(\binom{n}{2}+n(1-\beta')\right)\\
&+\Gamma_q\begin{bmatrix}\gamma,\alpha'-\beta'\\ \alpha',\gamma-\beta'\end{bmatrix}\frac{(yq^{\beta'},y^{-1}q^{1-\beta'};q)_\infty}{(y,\frac{q}{y};q)_\infty}\\
&\sum_{m,n=0}^{\infty}\frac{\langle\alpha,\beta;q\rangle_m\langle\beta';q\rangle_n}{\langle 1;q\rangle_m\langle 1,\beta'-\alpha'+1;q\rangle_n\langle\gamma-\beta';q\rangle_{m-n}}(-1)^n\\
&x^m y^{-n}\mathrm{QE}\left(\binom{n}{2}+n(1-\alpha')\right).
\end{aligned}\tag{26}$$

Proof.

(27)

$$\text{LHS} \overset{\text{by}(15)}{=} \sum_{m=0}^{\infty} \frac{\langle \alpha, \beta; q\rangle_m}{\langle 1, \gamma; q\rangle_m} \left[\Gamma_q \left[\begin{array}{c} \gamma+m, \beta'-\alpha' \\ \beta', \gamma-\alpha'+m \end{array} \right] \frac{(yq^{\alpha'}, y^{-1}q^{1-\alpha'}; q)_\infty}{(y, \frac{q}{y}; q)_\infty} \right.$$

$$ {}_2\phi_1 \left[\begin{array}{c} \alpha', \alpha'-\gamma-m+1 \\ \alpha'-\beta'+1 \end{array} \middle| q; y^{-1}q^{\gamma+m-\alpha'-\beta'} \right]$$

$$+\Gamma_q \left[\begin{array}{c} \gamma+m, \alpha'-\beta' \\ \alpha', \gamma-\beta'+m \end{array} \right] \frac{(yq^{\beta'}, y^{-1}q^{1-\beta'}; q)_\infty}{(y, \frac{q}{y}; q)_\infty}$$

$$\left. {}_2\phi_1 \left[\begin{array}{c} \beta', \beta'-m-\gamma+1 \\ \beta'-\alpha'+1 \end{array} \middle| q; y^{-1}q^{\gamma+m-\alpha'-\beta'} \right] \right] x^m \overset{\text{by}2\times[2,\ 1.46,\ 6.15]}{=} \text{RHS}.$$

□

Since Φ_3 is symmetric in the variables, the meromorphic continuation has the same form as the original and is itself symmetric.

5. The q-Appell function Φ_4

The q-difference equation for Φ_4 can be written in the following canonical form, [2, 11.63].

(28)

$$\begin{cases} -\mathbf{x}\{\theta_{q,1}+\theta_{q,2}+\alpha\}_q\{\theta_{q,1}+\theta_{q,2}+\beta\}_q+\{\theta_{q,1}\}_q\{\theta_{q,1}+\gamma-1\}_q=0, \\ -\mathbf{y}\{\theta_{q,1}+\theta_{q,2}+\alpha\}_q\{\theta_{q,1}+\theta_{q,2}+\beta\}_q+\{\theta_{q,2}\}_q\{\theta_{q,2}+\gamma'-1\}_q=0. \end{cases}$$

Theorem 5.1. *A q-analogue of* [1, (37), p.26], [11, (69), p. 297]. *Meromorphic continuation of the fourth q-Appell function to the region* $|y|>1$.

(29)

$$\Phi_4(\alpha;\beta;\gamma,\gamma'|q;x,y)$$

$$= \Gamma_q \left[\begin{array}{c} \gamma', \beta-\alpha \\ \beta, \gamma'-\alpha \end{array} \right] \sum_{m,n=0}^{\infty} \frac{\langle \alpha, 1-\gamma'+\alpha; q\rangle_{m+n} (yq^{\alpha+m}, y^{-1}q^{1-\alpha-m}; q)_\infty}{\langle 1, \gamma; q\rangle_m \langle 1, \alpha-\beta+1; q\rangle_n (y, \frac{q}{y}; q)_\infty} (-1)^m$$

$$x^m y^{-n} \text{QE}\left(-\binom{m}{2} + m(\gamma'-\alpha-1) + n(\gamma'+1-\alpha-\beta-2m) \right)$$

$$+ \Gamma_q \left[\begin{array}{c} \gamma', \alpha-\beta \\ \alpha, \gamma'-\beta \end{array} \right] \sum_{m,n=0}^{\infty} \frac{\langle \beta, 1-\gamma'+\beta; q\rangle_{m+n} (yq^{\beta+m}, y^{-1}q^{1-\beta-m}; q)_\infty}{\langle 1, \gamma; q\rangle_m \langle 1, \beta-\alpha+1; q\rangle_n (y, \frac{q}{y}; q)_\infty} (-1)^m$$

$$x^m y^{-n} \text{QE}\left(-\binom{m}{2} + m(\gamma'-\beta-1) + n(\gamma'+1-\alpha-\beta-2m) \right).$$

Proof.

(30)

$$\text{LHS} \overset{\text{by}(15)}{=} \sum_{m=0}^{\infty} \frac{\langle \alpha, \beta; q\rangle_m}{\langle 1, \gamma; q\rangle_m} \left[\Gamma_q \left[\begin{array}{c} \gamma', \beta-\alpha \\ \beta+m, \gamma'-\alpha-m \end{array} \right] \frac{(yq^{\alpha+m}, y^{-1}q^{1-\alpha-m}; q)_\infty}{(y, \frac{q}{y}; q)_\infty} \right.$$

$$ {}_2\phi_1 \left[\begin{array}{c} \alpha+m, \alpha+m-\gamma'+1 \\ \alpha-\beta+1 \end{array} \middle| q; y^{-1}q^{\gamma'+1-\alpha-\beta-2m} \right]$$

$$+\Gamma_q \left[\begin{array}{c} \gamma', \alpha-\beta \\ \alpha+m, \gamma'-\beta-m \end{array} \right] \frac{(yq^{\beta+m}, y^{-1}q^{1-\beta-m}; q)_\infty}{(y, \frac{q}{y}; q)_\infty}$$

$$\left. {}_2\phi_1 \left[\begin{array}{c} \beta+m, \beta+m-\gamma'+1 \\ \beta-\alpha+1 \end{array} \middle| q; y^{-1}q^{\gamma'+1-\alpha-\beta-2m} \right] \right] x^m \overset{\text{by}2\times[2,\ 1.46]}{=} \text{RHS}.$$

□

Again, since Φ_4 is symmetric in the variables, the meromorphic continuation has almost the same form as the original and is itself symmetric.

6. The q-function $\mathrm{H_C}$

The q-difference equation for $\mathrm{H_C}$ can be written in the following canonical form, a q-analogue of [10, (3.1), p.105].

(31)

$$\begin{cases} -\mathbf{x}\{\theta_{q,1}+\theta_{q,3}+\alpha\}_q\{\theta_{q,1}+\theta_{q,2}+\beta_1\}_q+\{\theta_{q,1}\}_q\{\theta_{q,1}+\theta_{q,2}+\theta_{q,3}+\gamma-1\}_q=0, \\ -\mathbf{y}\{\theta_{q,1}+\theta_{q,2}+\beta_1\}_q\{\theta_{q,2}+\theta_{q,3}+\beta_2\}_q+\{\theta_{q,2}\}_q\{\theta_{q,1}+\theta_{q,2}+\theta_{q,3}+\gamma-1\}_q=0, \\ -\mathbf{z}\{\theta_{q,1}+\theta_{q,3}+\alpha\}_q\{\theta_{q,2}+\theta_{q,3}+\beta_2\}_q+\{\theta_{q,3}\}_q\{\theta_{q,1}+\theta_{q,2}+\theta_{q,3}+\gamma-1\}_q=0. \end{cases}$$

Theorem 6.1. *A q-analogue of* [10, (3.4), p.105], [11, (62), p.293]. *The meromorphic continuation of the q-$\mathrm{H_C}$ function to the region $|y|>1$ is*

(32)

$$\mathrm{H_C}(\alpha,\beta_1,\beta_2;\gamma|q;x,y,z) = \Gamma_q \left[\begin{array}{c} \gamma, \beta_2-\beta_1 \\ \gamma-\beta_1, \beta_2 \end{array} \right] \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{(yq^{\beta_1+m}, y^{-1}q^{1-\beta_1-m}; q)_\infty}{(y, \frac{q}{y}; q)_\infty}$$

$$\frac{\langle \alpha; q\rangle_{m+k}\langle \beta_1; q\rangle_{m+n}\langle 1-\gamma+\beta_1; q\rangle_{n-k}}{\langle \vec{1}; q\rangle_{\vec{m}}\langle \beta_1-\beta_2+1; q\rangle_{m+n-k}}(-1)^k x^m y^{-n} z^k$$

$$\mathrm{QE}\left(\binom{m}{2} + m(\beta_1-k-\beta_2+1) + n(\gamma+1-\beta_1-\beta_2) + k(\beta_2-\gamma) \right)$$

$$+\Gamma_q \left[\begin{array}{c} \gamma, \beta_1-\beta_2 \\ \beta_1, \gamma-\beta_2 \end{array} \right] \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{(yq^{\beta_2+k}, y^{-1}q^{1-\beta_2-k}; q)_\infty}{(z, \frac{q}{z}; q)_\infty} x^m y^{-n} z^k$$

$$\frac{\langle \alpha; q\rangle_{m+k}\langle \beta_2; q\rangle_{n+k}\langle \beta_2-\gamma+1; q\rangle_{n-m}}{\langle \vec{1}; q\rangle_{\vec{m}}\langle 1+\beta_1-\beta_2; q\rangle_{k-m+n}}$$

$$\mathrm{QE}\left(\binom{k}{2} + k(1+\beta_2-\beta_1) + m(\beta_1-\gamma-k) + n(\gamma-\beta_1-\beta_2+1) \right).$$

Proof.

(33)
$$\begin{aligned}
&\mathrm{LHS} \overset{\text{by(18)}}{=} \sum_k \frac{\langle\alpha,\beta_2;q\rangle_k}{\langle 1,\gamma;q\rangle_k}\Bigg[\Gamma_q\left[\begin{array}{c}\gamma+k,\beta_2+k-\beta_1\\ \beta_2+k,\gamma+k-\beta_1\end{array}\right]\sum_{m,n=0}^{\infty}\\
&\frac{\langle\beta_1;q\rangle_{m+n}\langle\alpha+k;q\rangle_m\langle 1+\beta_1-\gamma-k;q\rangle_n(yq^{\beta_1+m},y^{-1}q^{1-\beta_1-m};q)_\infty}{\langle 1;q\rangle_m\langle 1;q\rangle_n\langle\beta_1-\beta_2+1-k;q\rangle_{m+n}(y,\frac{q}{y};q)_\infty}\\
&(-x)^m y^{-n}\mathrm{QE}\left(\binom{m}{2}+m(\beta_1-k-\beta_2+1)+n(\gamma+1-\beta_1-\beta_2)\right)\\
&+\Gamma_q\left[\begin{array}{c}\gamma+k,\beta_1-k-\beta_2\\ \beta_1,\gamma-\beta_2\end{array}\right]\frac{(yq^{\beta_2+k},y^{-1}q^{1-\beta_2-k};q)_\infty}{(y,\frac{q}{y};q)_\infty}\sum_{m,n=0}^{\infty}\\
&\frac{\langle\beta_2+k;q\rangle_n\langle\alpha+k;q\rangle_m\langle 1+\beta_2-\gamma;q\rangle_{n-m}\langle\beta_1-\beta_2-k;q\rangle_{m-n}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}(-x)^m\\
&(-y)^{-n}\mathrm{QE}\left(-\binom{m}{2}-\binom{n}{2}+m(\beta_2-\gamma)+n(\gamma-k+m-2\beta_2)\right)\Bigg]\\
&\times z^k \overset{\text{by}2\times[2,\ 1.46,\ 6.14],\ (14)}{=} \mathrm{RHS}.
\end{aligned}$$

□

We conclude that the q-difference system (31) has solutions in accordance with (32) in the neighbourhood of $(0,\infty,0)$.

7. The q-Lauricella function $\Phi_{\mathrm{D}}^{(3)}$

The q-difference equation for $\Phi_{\mathrm{D}}^{(3)}$ can be written in the following canonical form, a q-analogue of [10, (2.6), p.104].

(34)
$$\begin{cases}-\mathbf{x}\{\theta_{q,1}+\theta_{q,2}+\theta_{q,3}+\alpha\}_q\{\theta_{q,1}+\beta_1\}_q+\{\theta_{q,1}\}_q\{\theta_{q,1}+\theta_{q,2}+\theta_{q,3}+\gamma-1\}_q=0,\\ -\mathbf{y}\{\theta_{q,1}+\theta_{q,2}+\theta_{q,3}+\alpha\}_q\{\theta_{q,2}+\beta_2\}_q+\{\theta_{q,2}\}_q\{\theta_{q,1}+\theta_{q,2}+\theta_{q,3}+\gamma-1\}_q=0,\\ -\mathbf{z}\{\theta_{q,1}+\theta_{q,2}+\theta_{q,3}+\alpha\}_q\{\theta_{q,3}+\beta_3\}_q+\{\theta_{q,3}\}_q\{\theta_{q,1}+\theta_{q,2}+\theta_{q,3}+\gamma-1\}_q=0.\end{cases}$$

Theorem 7.1. *A q-analogue of* [10, (2.5), p.104], [6, (6.7.1.2), p.201]. *Put*

(35) $$|m|\equiv k+m+n,\ \vec{m}\equiv(k,m,n),\ \vec{x}\equiv(x,y,z),\ \vec{\beta}\equiv(\beta_1,\beta_2,\beta_3).$$

Then the meromorphic continuation of the q-Lauricella function $\Phi_{\mathrm{D}}^{(3)}$ *to the neighbourhood of the point* $(0,0,\infty)$ *is*

(36)

$$\Phi_{\mathrm{D}}^{(3)}(\alpha,\vec{\beta};\gamma|q;\vec{x})=\Gamma_q\left[\begin{array}{c}\gamma,\beta_3-\alpha\\ \beta_3,\gamma-\alpha\end{array}\right]\sum_{\vec{m}=\vec{0}}^{\vec{\infty}}\frac{(zq^{\alpha+m+k},z^{-1}q^{1-\alpha-k-m};q)_\infty}{(z,\frac{q}{z};q)_\infty}$$
$$\frac{\langle\alpha;q\rangle_{|m|}\langle\beta_1;q\rangle_k\langle\beta_2;q\rangle_m\langle 1-\gamma+\alpha;q\rangle_n}{\langle\vec{1};q\rangle_{\vec{m}}\langle\alpha-\beta_3+1;q\rangle_{|m|}}x^k y^m z^{-n}(-1)^{k+m}$$
$$\mathrm{QE}\left(\binom{k}{2}+\binom{m}{2}+k(1+\alpha-\beta_3)+m(1+\alpha+k-\beta_3)+n(\gamma+1-\alpha-\beta_3)\right)$$
$$+\Gamma_q\left[\begin{array}{c}\gamma,\alpha-\beta_3\\ \alpha,\gamma-\beta_3\end{array}\right]\frac{(zq^{\beta_3},z^{-1}q^{1-\beta_3};q)_\infty}{(z,\frac{q}{z};q)_\infty}$$
$$\sum_{\vec{m}=\vec{0}}^{\vec{\infty}}\frac{\langle\alpha-\beta_3;q\rangle_{m-n+k}\langle\vec{\beta};q\rangle_{\vec{m}}}{\langle\vec{1};q\rangle_{\vec{m}}\langle\gamma-\beta_3;q\rangle_{m-n+k}}x^k y^m z^{-n}\mathrm{QE}\left(n(1-\beta_3)\right).$$

Proof.

(37)

$$\mathrm{LHS}\overset{\text{by(18)}}{=}\sum_{k=0}^{\infty}\frac{\langle\alpha,\beta_1;q\rangle_k}{\langle 1,\gamma;q\rangle_k}\left[\Gamma_q\left[\begin{array}{c}\gamma+k,\beta_3-\alpha-k\\ \beta_3,\gamma-\alpha\end{array}\right]\sum_{m,n=0}^{\infty}(zq^{\alpha+m+k};q)_\infty\right.$$
$$\frac{(z^{-1}q^{1-\alpha-k-m};q)_\infty}{(z,\frac{q}{z};q)_\infty}\frac{\langle\alpha+k;q\rangle_{m+n}\langle\beta_2;q\rangle_m\langle 1-\gamma+\alpha;q\rangle_n(-1)^m}{\langle 1;q\rangle_m\langle 1;q\rangle_n\langle\alpha+k-\beta_3+1;q\rangle_{m+n}}$$
$$y^m z^{-n}\mathrm{QE}\left(\binom{m}{2}+m(1+\alpha+k-\beta_3)+n(\gamma+1-\alpha-\beta_3)\right)$$
$$+\Gamma_q\left[\begin{array}{c}\gamma+k,\alpha+k-\beta_3\\ \alpha+k,\gamma+k-\beta_3\end{array}\right]\frac{(zq^{\beta_3},z^{-1}q^{1-\beta_3};q)_\infty}{(z,\frac{q}{z};q)_\infty}$$
$$\sum_{m,n=0}^{\infty}\frac{\langle\alpha+k-\beta_3;q\rangle_{m-n}\langle\beta_3-\gamma-k+1;q\rangle_{n-m}\langle\beta_2;q\rangle_m\langle\beta_3;q\rangle_n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}(-1)^{m+n}$$
$$\left.y^m z^{-n}\mathrm{QE}\left(-\binom{m}{2}-\binom{n}{2}+m(\beta_3-\gamma-k)+n(\gamma+k+m-2\beta_3)\right)\right]$$
$$x^k\overset{\text{by}2\times[2,\ 1.46,\ 6.14],\ (14)}{=}\mathrm{RHS}.$$

□

The meromorphic continuation of the q-Lauricella function $\Phi_{\mathrm{D}}^{(n)}$ to the neighbourhood of the point $(0,\ldots,0,\infty)$ is obtained in a similar way.

8. The q-Lauricella function $\Phi_{\mathrm{C}}^{(n)}$

Theorem 8.1. *A q-analogue of* [6, (6.5.3), p.198]. *Put*

$$\vec{x}^{\star}\equiv(x_1,\ldots,x_{n-1}),\ \vec{m}^{\star}\equiv(m_1,\ldots,m_{n-1})$$

(38)
$$m^{\star}\equiv\sum_{i=1}^{n-1}m_i,\ \vec{c}\equiv(c_1,\ldots,c_n),\ \vec{c}^{\star}\equiv(c_1,\ldots,c_{n-1}).$$

The vector $\vec{c}^\star$ is only used in the proof below. Then the meromorphic continuation of the q-Lauricella function $\Phi_{\mathrm{C}}^{(n)}$ to the neighbourhood of the point $(0,\ldots,0,\infty)$ is

$$\begin{aligned}
&\Phi_{\mathrm{C}}^{(n)}(a,b;\vec{c}|q;\vec{x}) = \Gamma_q\left[\begin{matrix} c_n, b-a \\ b, c_n-a\end{matrix}\right] \\
&\sum_{\vec{m}} \frac{(-1)^{m^\star}\langle a, 1+a-c_n;q\rangle_{\vec{m}}(x_n q^{a+m^\star}, x_n^{-1}q^{1-a-m^\star};q)_\infty}{\langle 1,c_1;q\rangle_{m_1}\ldots\langle 1,c_{n-1};q\rangle_{m_{n-1}}\langle 1,a-b+1;q\rangle_{m_n}(x_n,\frac{q}{x_n};q)_\infty}\vec{x}^{\star\vec{m}^\star} \\
&x_n^{-m_n}\mathrm{QE}\left(-\binom{m^\star}{2}+m^\star(c_n-a-1)+m_n(c_n+1-a-b-2m^\star)\right) \\
&+\Gamma_q\left[\begin{matrix} c_n, a-b \\ a, c_n-b\end{matrix}\right] \\
&\sum_{\vec{m}} \frac{(-1)^{m^\star}\langle b, 1+b-c_n;q\rangle_{\vec{m}}(x_n q^{b+m^\star}, x_n^{-1}q^{1-b-m^\star};q)_\infty}{\langle 1,c_1;q\rangle_{m_1}\ldots\langle 1,c_{n-1};q\rangle_{m_{n-1}}\langle 1,b-a+1;q\rangle_{m_n}(x_n,\frac{q}{x_n};q)_\infty}\vec{x}^{\star}\vec{m}^\star \\
&x_n^{-m_n}\mathrm{QE}\left(-\binom{m^\star}{2}+m^\star(c_n-b-1)+m_n(c_n+1-a-b-2m^\star)\right).
\end{aligned} \tag{39}$$

Proof.

(40)

$$\begin{aligned}
&\mathrm{LHS} \overset{\text{by(15)}}{=} \sum_{\vec{m^\star}} \frac{\langle a,b;q\rangle_{m^\star}}{\langle \vec{1^\star},\vec{c^\star};q\rangle_{\vec{m^\star}}}\Bigg[\Gamma_q\left[\begin{matrix} c_n, b-a \\ b+m^\star, c_n-a-m^\star\end{matrix}\right]\frac{(x_nq^{a+m^\star},x_n^{-1}q^{1-a-m^\star};q)_\infty}{(x_n,\frac{q}{x_n};q)_\infty} \\
&{}_2\phi_1\left[\begin{matrix} a+m^\star, a+m^\star+1-c_n \\ a-b+1\end{matrix}\middle| q; x_n^{-1}q^{c_n+1-a-b-2m^\star}\right] \\
&+\Gamma_q\left[\begin{matrix} c_n, a-b \\ a+m^\star, c_n-b-m^\star\end{matrix}\right]\frac{(x_nq^{b+m^\star},x_n^{-1}q^{1-b-m^\star};q)_\infty}{(x_n,\frac{q}{x_n};q)_\infty} \\
&{}_2\phi_1\left[\begin{matrix} b+m^\star, b+m^\star-c_n+1 \\ b-a+1\end{matrix}\middle| q; x_n^{-1}q^{c_n+1-a-b-2m^\star}\right]\Bigg]\vec{x}^{\star\vec{m}^\star} \overset{\text{by}2\times[2,\ 1.46]}{=} \mathrm{RHS}.
\end{aligned}$$

□

9. Conclusion

We have pointed out the importance of symmetric multiple functions for prettier meromorphic continuation formulas. Starting from the meromorphic continuation formula for ${}_2\phi_1$ (15), the meromorphic continuations for Φ_1, Φ_2, Φ_4, $\mathrm{H_C}$, $\Phi_{\mathrm{C}}^{(n)}$ and $\Phi_{\mathrm{D}}^{(3)}$ contain nested sums of similar type. At first sight, this seems confusing, since the power series do not have the same exponents for the variables as in the ordinary cases. The reason is that one of these powers is included in the continuation formula (15), and cannot be seen explicitly. If no sign changes occur in the last step of the proofs, like in formula (29), the extra minus sign corresponds to the summation index, whose power seems to be missing in the formula. This could be important when checking similar meromorphic continuation formulas in the future. This rule could be generalized when the number of variables increases. For further discussions on this theme, see [5].

10. Discussion

Of course, there are many other forms of these q-difference equations and meromorphic continuations, which can be obtained by symmetry of the variables. The more variables, the more symmetries. An equivalent form of these q-difference equations is presented in the recent paper [4]. The meromorphic continuations to the neighbourhood of the point $(0,\dots,0,1)$ etc. can be obtained in a similar way by using Euler integrals instead. These computations are much simpler than the ones in this article. Finally, we have pointed out some new paths to continue this research.

References

[1] P. Appell, J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques.* Paris 1926.

[2] T. Ernst, *A comprehensive treatment of q-calculus.* Birkhäuser 2012

[3] T. Ernst, *On the q-analogues of Srivastava's triple hypergeometric functions* Axioms 2(2), 85-99, 2013

[4] T. Ernst, *A new q-hypergeometric symbolic calculus in the spirit of Horn, Borngässer, Debiard and Gaveau* Axioms 11(2), 2022

[5] T. Ernst, A new semantics for special functions. Submitted.

[6] H. Exton, *Multiple hypergeometric functions and applications.* Mathematics and its Applications. Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York-London-Sydney, 1976.

[7] G. Gasper and M.Rahman, *Basic hypergeometric series.* Cambridge 1990.

[8] Hj. Mellin, *Über den Zusammenhang zwischen den linearen Differential- und Differenzengleichungen* Acta Math. **25**, (1902), 139–164.

[9] K. Mimachi, *Connection problem in holonomic q-difference system associated with a Jackson integral of Jordan-Pochhammer type* Nagoya Math. J. 116 1989, 149–161.

[10] H.M. Srivastava, *A note on certain hypergeometric differential equations* Mat. Vesn., N. 1972 Ser. 9(24), 101–107

[11] H.M. Srivastava, P.W. Karlsson, *Multiple Gaussian Hypergeometric Series* Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, (1985)

[12] G.N. Watson, *The continuations of functions defined by generalised hypergeometric series* (x) Cambr. Phil. Soc. Trans. 1910 21, 281–299

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