Proceedings of the Jangjeon Mathematical Society
26 (2023), No. 4, pp. 423 - 441

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2023.26.4.423

UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING TWO SETS WITH Q-SHIFT AND DELAY-DIFFERENTIAL OPERATOR

CHAITHRA C. N., NAVEENKUMAR S. H., AND JAYARAMA H. R.

Abstract. The delay-differential operator is expressed as the sum of a shift and differential operator and a differential operator, i.e., $L_2(f^n(z))+ L_3(f^n(z))$ and is denoted by $\tilde{L}(f^n(z))$. In this article, we mainly investigate the uniqueness of meromorphic function sharing two sets with q-shift by considering a delay-differential operator. Addition to that number of examples are provided for the justification. Our results improves the results due to Meng et al., and Qi.

2010 Mathematics Subject Classification. Primary 30D35.

Keywords and phrases. Uniqueness, Linear delay-differential operator, Sharing value, Meromorphic Functions and Shift operator.

1. Introduction

Throughout the paper we use standard notations of Nevanlinna theory as stated in [1] and by any meromorphic function f we always mean that it is defined on $\mathbb{C}$. Let f and g be two such non constant meromorphic functions. For $a \in \mathbb{C} \cup \{\infty\}$, the following two quantities

$$\delta(a,f) = 1 - \lim_{r\longrightarrow\infty} \sup \frac{N(r,a;f)}{T(r,f)} = \lim_{r\longrightarrow\infty} \inf \frac{m(r,a;f)}{T(r,f)}$$

and

$$\Theta(a;f) = 1 - \lim \sup_{r\longrightarrow\infty} \frac{\overline{N}(r,a;f)}{T(r,f)}$$

are respectively known as Nevanlinna deficiency and ramification index of the value a.

In the beginning of the nineteenth century R. Nevanlinna inaugurated the value distribution theory with his famous Five value and Four value theorems which can be considered as the backbone of the modern uniqueness theory. Illuminated by these two basic results initially the research was performed on the value sharing of meromorphic functions. After five decades, uniqueness theory moved to a new direction led by F. Gross [2], who transformed the traditional value sharing problem to a more general set up namely to the shared set problems. Now we recall the definition of set sharing.

Definition 1.1. *Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$. For a non-constant meromorphic function f, let $E_f(S) = \cup_{a\in S}\{(z,p) \in \mathbb{C}\times\mathbb{N} : f(z) = a$ with multiplicity $p\}$ ($\overline{E}_f(S) = \cup_{a\in S}\{(z,1) \in \mathbb{C}\times\mathbb{N} : f(z) = a)$. Then we say f, g share the set S CM(IM) if $E_f(S) = E_g(S)$ ($\overline{E}_f(S) = \overline{E}_g(S)$).*

Evidently, if S is a singleton, then it coincides with the traditional definition of CM(IM) sharing of values, which are known to the readers.

In 2001, due to a revolutionary approach by Lahiri [3, 4], the notion of weighted sharing of values or sets appeared in the literature and expedited the research work there in. Though now-a-days the definition is widely circulated, we invoke the difinition.

Definition 1.2. [3, 4] *Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k and denote it by (a, k). The IM and CM sharing corresponds to $(a, 0)$ and (a, ∞) respectively.*

Definition 1.3. [3] *Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a non-negative integer or ∞. We denote by $E_f(S, k)$ the set $\cup_{a \in S} E_k(a; f)$. Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.*
If $E_f(S, k) = E_g(S, k)$, then we say that f, g share the set S with weight k and write it as f, g share (S, k).

By $N(r, a; f| < m)$ we mean the counting function of those $a-$points of f whose multiplicities are less than m where each $a-$point is counted according to its multiplicity and by $\overline{N}(r, a; f| \geq m)$ we mean the counting function of those $a-$points of f whose multiplicities are not less than m where each $a-$points is counted ignoring multiplicity. We also denote by $N_2(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f| \geq 2)$.
Usually, $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ for all r outside of a possible exceptional set of finite linear measure. Also $S_1(r, f)$ denotes any quantity satisfying $S_1(r, f) = o(T(r, f))$ for all r on a set of logarithmic density 1, where the logarithmic density of a set F is defined by

$$\lim \sup_{r \longrightarrow \infty} \frac{1}{logr} \int_{[1,r] \cap F} \frac{dt}{t}.$$

Throughout the paper for a positive integer l, S_1, S_1^* and S_2 represents respectively the sets $\{1, \omega, ..., \omega^{l-1}\}$, $\{\alpha_1, \alpha_2, ..., \alpha_l\}$ and $\{\infty\}$, where $\omega = cos\frac{2\pi}{l} + isin\frac{2\pi}{l}$ and $\alpha_i, i = 1, 2, ..., l$ are non zero constants.
Let $a_{t-1}(\neq 0), a_{t-2}, ..., a_0$ and $C(\neq 0)$ be complex numbers. We define

$$P(z) = CzQ(z) = Cz(a_{t-1}z^{t-1} + a_{t-2}z^{t-2} + ... + a_1 z + a_0). \tag{1}$$

For the polynomial $P(z)$ as given in (1), let us define two functions:

$$\chi_0^{t-1} = \begin{cases} 1, & if \quad a_0 \neq 0 \\ 0, & if \quad a_0 = 0. \end{cases}$$

and

$$\mu_0^{t-1} = \begin{cases} 1, & if \quad a_0 = 0, a_1 \neq 0 \\ 0, & otherwise. \end{cases}$$

In view of (1), corresponding to the set S_1^*, let us consider the polynomial $P_*(z)$ as follows:

$$P_*(z) = CzQ_*(z). \tag{2}$$

where $C = \dfrac{1}{(-1)^{l+1}\alpha_1\alpha_2...\alpha_l}$ and $Q_*(z) = \Sigma_{r=0}^{l-1}(-1)^r\Sigma\alpha_1\alpha_2...\alpha_r z^{l-r-1}$,

$\Sigma\alpha_1\alpha_2...\alpha_r$ = sum of the products of the value $\alpha_1, \alpha_2, ..., \alpha_l$ taking r into account. We also denote by m_1 and m_2 as the number of simple and multiple zeros of $Q_*(z)$ respectively.
Next we define linear shift operator, shift-differential operator and differential operator respectively as follows:

$$L_1(f^n(z)) = a_k f^n(z+c_k) + a_{k-1}f^n(z+c_{k-1}) + ... + a_1 f^n(z+c_1) + a_0 f^n(z),$$
$$L_2(f^n(z)) = b_s(f^s)^n(z+c_s) + b_{s-1}(f^{s-1})^n(z+c_{s-1}) + ... + b_1(f^{'})^n(z+c_1),$$
$$L_3(f^n(z)) = d_t(f^t)^n(z) + d_{t-1}(f^{t-1})^n(z) + ... + d_1(f^{'})^n(z),$$

where a_k, b_s and d_t are non zero and k, s, t are natural numbers and all $c_i^{'}(s)$ are non zero. For the sake of convenience we shall call $L_2(f^n(z)) + L_3(f^n(z))$ a delay-differential operator which is denoted by $\tilde{L}(f^n(z))$.

As far a the knowledge of the authors are concerned, Qi-Li-Yang [5] were the first authors who initiated two shared set problems for the derivative of a meromorphic function $f(z)$ with its shift $f(z+c)$ a follows:
Theorem A. [5] Let $f(z)$ be a non constant meromorphic function of finite order, $n \geq 9$ be an integer and a be a non zero complex constant. If $[f^{'}(z)]^n$ and $f^n(z+c)$ share (a, ∞) and (∞, ∞), then $f^{'}(z) = tf(z+c)$, for a constant t that satisfies $t^n = 1$.

Recently employing the notion of weighted sharing, Meng-Liu [6] further investigated Theorem A to obtain the following result.
Theorem B. [6] Let $f(z)$ be a non constant meromorphic function of finite order, $n \geq 10$ be an integer. If $[f^{'}(z)]^n$ and $f^n(z+c)$ share $(1, 2)$ and $(\infty, 0)$, then $f^{'}(z) = tf(z+c)$, for a constant t that satisfies $t^n = 1$.

Considering $f(z) = e^z$ and $\omega = e^{-c}$ satisfying $\omega^n = 1$, it is easy to see that $f^{'}$ and $f(z+c)$ share the sets $(S_1, \infty), (\infty, \infty)$ and $f^{'}(z) = \omega f(z+c)$, for each n. So it is natural to conjecture that in Theorem A and Theorem B the cardinality of n could further be reduced. To this end, we have performed our investigations and have been able to reduce the cardinality of n in Theorem B up to 6. In fact, we have proved our theorem for a more general setting S_1^* rather than to consider only the set S_1.
Theorem 1.1. Let $f(z)$ be a non constant meromorphic function of finite order such that $\tilde{L}(f^n(z))$ and $f^n(z+c)$ share $(S_1^*, 2)$ and $(S_2, 0)$. If $l > 2n(\chi_0^{l-1} + \mu_0^{l-1} + m_1 + 2m_2) + \frac{3n(4n+1)}{2n(l-1)-1}(\chi_0^{l-1} + m_1 + m_2)$, then

$$\prod_{i=1}^{l}(\tilde{L}(f^n(z)) - \alpha_i) \equiv \prod_{i=1}^{l}(f^n(z+c) - \alpha_i).$$

Remark 1.1. From the definitions, we easily can calculate the value of $\chi_0^{l-1}, \mu_0^{l-1}, m_1$ and m_2 for particular set S_1^*. Clearly for the set $S_1, \chi_0^{l-1} = 0; \mu_0^{l-1} = 0; m_1 = 0$ and $m_2 = 1$. Therefore in above theorem for the set S_1 if $l > \frac{3n(4n+1)}{10n-1}$ i.e., if $l \geq 6$ then $\tilde{L}(f^n(z) = tf^n(z+c)$, for a constant t that satisfies $t^l = 1$. For particular choices of coefficients of $\tilde{L}(f^n(z)$ we can easily make $\tilde{L}(f^n(z) = (f^{'})^n$.

Corresponding to q-shift Meng-Liu [6] also investigated the same result like Theorem B as follows:

Theorem C. [6] Let $f(z)$ be a non constant meromorphic function of zero order, $n \geq 10$ be an integer. If $[f'(z)]^n$ and $f^n(qz)$ share $(1, 2)$ and $(\infty, 0)$, then $f'(z) = tf(qz)$, for a constant t that satisfies $t^n = 1$.

In connection to Theorem C below we present our result which improve the same.

Theorem 1.2. Let $f(z)$ be a non constant meromorphic function of zero order such that $\tilde{L}(f^n(z))$ and $f^n(qz)$ share $(S_1^*, 2)$ and $(S_2, 0)$. If $l > 2n(\chi_0^{l-1} + \mu_0^{l-1} + m_1 + 2m_2) + \frac{3n(4n+1)}{2n(l-1)-1}(\chi_0^{l-1} + m_1 + m_2)$, then

$$\prod_{i=1}^{l}(\tilde{L}(f^n(z)) - \alpha_i) \equiv \prod_{i=1}^{l}(f^n(qz) - \alpha_i).$$

In the next theorem we shall show that the lower bound of l can further be reduced at the expense of allowing both the range sets S_1^*, S_2 to be shared CM.

Theorem 1.3. Let $f(z)$ be a non constant meromorphic function of finite order such that $\tilde{L}(f^n(z))$ and $f^n(z+c)$ share (S_1^*, ∞) and (S_2, ∞) with $T(r, f^n) = nN\left(r, \frac{1}{\tilde{L}(f^n(z))}\right) + S(r, f)$ then for $l > 2n[(\chi_0 + m_1 + m_2) + 1]$

$$\prod_{i=1}^{l}(\tilde{L}(f^n(z)) - \alpha_i) \equiv \prod_{i=1}^{l}(f^n(z+c) - \alpha_i).$$

Remark 1.2. In connection of Remark 1.1, for the set S_1 in Theorem 1.3 the result holds for $n \geq 4$.

Theorem 1.4. Let $f(z)$ be a non constant meromorphic function of zero order such that $\tilde{L}(f^n(z))$ and $f^n(qz)$ share (S_1^*, ∞) and (S_2, ∞). If $l > 2n[(\chi_0^{l-1} + m_1 + m_2) + 1]$ then

$$\prod_{i=1}^{l}(\tilde{L}(f^n(z)) - \alpha_i) \equiv \prod_{i=1}^{l}(f^n(qz) - \alpha_i).$$

Recently, corresponding to Theorem A, Qi-Yang [16] obtained the value sharing problem for entire function as follows:

Theorem D. [16] Let f(z) be a transcendental entire function of finite order and let $(a \neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share $(0, \infty)$ and $(a, 0)$, then $f'(z) \equiv f(z+c)$.

In view of Theorem 1.1, [16] we know that $f(z)$ actually becomes a transcendental entire function. Since we are dealing with $\tilde{L}(f^n(z))$ instead of f', it will be reasonable to consider the above theorem for meromorphic function under small function sharing category. In this respect we prove the following theorem.

Theorem 1.5. Let $f(z)$ be transcendental meromorphic function of finite order and let $a(z)(\not\equiv 0) \in S(f)$ be an entire function. If $\tilde{L}(f^n(z))$ and

$f^n(z+c)$ share $(0,\infty),(\infty,\infty)$ and $(a(z),0)$ with $\Theta(0,f^n)+\Theta(\infty,f^n)>0$, then $\tilde{L}(f^n(z))\equiv f^n(z+c)$.

From Theorem 1.5 we can immediatly deduce the following corollary.

Corollary 1.1. Let $f(z)$ be transcendental entire function of finite order and let $a(z)(\not\equiv 0)\in S(f)$. If $\tilde{L}(f^n(z))$ and $f^n(z+c)$ share $(0,\infty)$ and $(a(z),0)$, then $\tilde{L}(f^n(z))\equiv f^n(z+c)$.

The following example shows that in Theorem 1.5 the CM pole sharing can not be replaced by IM.

Example 1.1. Let $f^n(z)=n[(e^{\lambda z}-1)^2+1]$. Choose $e^{\lambda c}=1, \sum_{i=1}^{s} b_i(2\lambda)^i e^{2\lambda c_i}+\sum_{i=1}^{t} d_i(2\lambda)^i=0$ and $\sum_{i=1}^{s} b_i(\lambda)^i e^{\lambda c_i}+\sum_{i=1}^{t} d_i(\lambda)^i=-\frac{1}{2}$. Then $f^n(z+c)=n[(e^{\lambda z}-1)^2+1]$ and $\tilde{L}(f^n(z))=e^{\lambda(z)}$. Clearly $f^n(z+c)$ and $\tilde{L}(f^n(z))$ share $(2,\infty),(\infty,\infty)$ and $(1,0)$ with $\Theta(0,f)+\Theta(\infty,f)>0$. But $\tilde{L}(f^n(z))\neq f^n(z+c)$.

In Theorem 1.5, sharing of the value 0 can be removed at the cost of slightly manipulating the deficiency condition. In this respect, we state the following theorem for transcendental meromorphic function.

Theorem 1.6. Let $f(z)$ be transcendental meromorphic function of finite order and let $a(z)(\not\equiv 0)\in S(f)$ be an entire function. If $\tilde{L}(f^n(z))$ and $f^n(z+c)$ share $(a(z),\infty)$ and (∞,∞) with $\Theta(0,f)>0$, then $\tilde{L}(f^n(z))\equiv f^n(z+c)$.

By an example we now show that $a(z)$ CM sharing can not be replaced by IM in Theorem 1.6.

Example 1.2. Let $f^n(z)=n[\frac{-2e^z-1}{e^{2z}}]$ and $c=\pi i$. Choose $\tilde{L}(f^n(z))=L_3(f^n(z))$ with $2\sum_{i=1}^{t}(-1)^{i+1}d_i=1$ and $\sum_{i=1}^{t}(-2)^i d_i=0$. Then $\tilde{L}(f^n(z))=\frac{n}{e^z}$ and $f^n(z+c)=n[\frac{2e^z-1}{e^{2z}}]$ share $(1,0),(\infty,\infty)$ and $\delta(0;f)=\frac{1}{2}>0$. Clearly $\tilde{L}(f^n(z))\neq f^n(z+c)$.

Our next example shows that $a(z)\not\equiv 0$ in Theorem 1.6 can not be dropped as well as $(a(z),0)$ sharing in Theorem 1.5 can not be removed.

Example 1.3. Let $f^n(z)=ne^{\frac{\pi iz}{c}}$. Choose $\tilde{L}(f^n(z))=(f')n$. then clearly $f^n(z+c)$ and $\tilde{L}(f^n(z))$ share $(0,\infty),(\infty,\infty)$ and $\delta(0;f)>0$. But $\tilde{L}(f^n(z))\neq f^n(z+c)$.

The following two examples show that $\delta(0;f)>0$ in Theorem 1.6 can not be removed.

Example 1.4. In Example 1.2 though $f^n(z+c)$ and $\tilde{L}(f^n(z))$ share $(2,\infty),(\infty,\infty)$ but $\delta(0;f)>0$. Here $\tilde{L}(f^n(z))\neq f^n(z+c)$.

Example 1.5. Let $f^n(z)=n[\frac{e^z+z}{2}]$. and $a(z)=z$ Choose $\tilde{L}(f^n(z))=L_3(f^n(z))$ with $d_1=2c$ and $\sum_{j=2}^{t} d_j=2(e^c-c)$. Then $\tilde{L}(f^n(z))=n(e^c e^z+c)$ and $f^n(z+c)=n[\frac{e^c e^z+z+c}{2}]$ share $(a(z),\infty)$ and (∞,∞) but $\delta(0;f)=0$. Clearly $\tilde{L}(f^n(z))\neq f^n(z+c)$.

Motivation: By looking at all such results, it is interesting to ask what happens to the result when more general meromorphic functions sharing two sets with q-shift and delay differential operator $\tilde{L}(f^n(z))$ are considered?

As an affirmative answer, we have obtained some uniqueness results which are stated in the respective sections of this article.

2. **Lemmas**

In this section, we present some lemmas which will be needed in the sequel.

Lemma 2.1. [7] *Let $f(z)$ be a meromorphic function of finite order ρ and let $c \in \mathbb{C} \setminus \{0\}$ be fixed. Then, for each $\in > 0$, we have*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\rho-1+\varepsilon}) + O(logr).$$

Lemma 2.2. [8] *Let $f(z)$ be a meromorphic function of finite order and $c \in \mathbb{C} \setminus \{0\}$. Then,*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f).$$

Lemma 2.3. [9] *Let $f(z)$ be a non constant meromorphic function of finite order and $c \in \mathbb{C}$. Then,*

$$N\left(r, \frac{1}{f(z+c)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + S(r, f),$$
$$N(r, f(z+c)) \leq N(r, f(z)) + S(r, f),$$
$$\overline{N}\left(r, \frac{1}{f(z+c)}\right) \leq \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r, f)$$

and

$$\overline{N}(r, f(z+c)) \leq \overline{N}(r, f(z)) + S(r, f).$$

Lemma 2.4. [10] *Let $f(z)$ be a meromorphic function of zero order and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = S_1(r, f).$$

Lemma 2.5. *Let $f(z)$ be a non constant of zero order meromorphic function and let $q \in \mathbb{C} \setminus \{0\}$, then*

$$T(r, f(qz)) = (1 + o(1))T(r, f(z))$$

and

$$N(r, f(qz)) = (1 + o(1))N(r, f(z))$$

on a set of lower logarithmic measure 1,
using Lemma 2.4 and Lemma 2.5 and by the help of simple transformation one can easily prove the next lemma.

Lemma 2.6. *Let $f(z)$ be a meromorphic function of zero order and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(z)}{f(qz)}\right) = S_1(r, f).$$

Lemma 2.7. [12] *Let $f(z)$ be a non constant meromorphic function in the complex plane, and let $R(f) = \frac{P(f)}{Q(f)}$, where*

$$P(f) = \sum_{k=0}^{p} a_k(z) f^k$$

and

$$Q(f) = \sum_{j=0}^{q} b_j(z) f^j$$

are two mutually prime polynomials in f. *If the coefficients* $a_k(z)$ *for* $k = 0, 1, ..., p$ *and* $b_j(z)$ *for* $j = 0, 1, ..., q$ *are small functions of* f *with* $a_p(z) \not\equiv 0$ *and* $b_q(z) \not\equiv 0$, *then*

$$T(r, P(f)) = max\{p, q\} T(r, f) + S(r, f).$$

Lemma 2.8. [13] *Suppose that* h *is a non constant meromorphic function satisfying*

$$N(r, h) = N\left(r, \frac{1}{h}\right) = S(r, h).$$

Let $f = a_0 h^p + a_1 h^{p-1} + ... + a_p$ *and* $g = b_0 h^q + b_1 h^{q-1} + ... + b_q$ *be polynomials in* h *with coefficients* $a_0, a_1, ..., a_p; b_0, b_1, ..., b_q$ *being small functions of* h *and* $a_0 b_0 a_p \not\equiv 0$. *If* $q \leq p$, *then* $m\left(r, \frac{g}{f}\right) = S(r, h)$.

Lemma 2.9. *If* $N(r, 0; (f^{(k)})^n | f \neq 0)$ *denotes the counting function of those zeros of* $(f^{(k)})^n$ *which are not the zeros of* f^n, *where a zero of* $(f^{(k)})^n$ *is counted according to its multiplicity then*

$$N(r, 0; (f^{(k)})^n | f \neq 0) \geq n[k\overline{N}(r, \infty; f) + N(r, 0; f| < k) + k\overline{N}(r, 0; f| \geq k)] + S(r, f)$$

Proof: *By the first fundamental theorem and Milloux theorem* [1] *we get*

$$\begin{aligned} N(r, 0; (f^{(k)})^n | f \neq 0) &\leq N\left(r, 0; \left(\frac{f^{(k)}}{f}\right)^n\right) \\ &\leq nN\left(r, \frac{f^{(k)}}{f}\right) + nm\left(r, \frac{f^{(k)}}{f}\right) + O(1) \\ &\leq n[k\overline{N}(r, \infty; f) + N(r, 0; f| < k) + k\overline{N}(r, 0; f| \geq k)] + S(r, f). \end{aligned}$$

This proves the lemma.

Lemma 2.10. *Let* F *be a meromorphic function. Then*

$$\overline{N}(r, \infty; F \geq n(k+1)) \leq \frac{1}{nk}\{\overline{N}(r, 0; F) + \overline{N}(r, \infty; F\} + S(r, F).$$

Since the proof is straight forward, it is omitted.

Lemma 2.11. [15] *Let* F, G *be two meromorphic functions sharing* $(1, 2)$ *and* (∞, k), *where* $0 \leq k \leq \infty$. *Then one of the following cases holds*

$$(i) T(r, F) + T(r, G) \leq 2\{N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \overline{N}_*(r, \infty; F, G)\} + S(r, F) + S(r, G),$$

where $\overline{N}_*(r, \infty; f, g)$ *is the reduced counting function of those poles of* F *whose multiplicities differ from the multiplicities of the corresponding poles of* G,
$(i) F \equiv G$,
$(ii) FG \equiv 1$.

Lemma 2.12. *Let $P_*(f^n)$ and $P_*(g^n)$ be defined in (2), for two non constant meromorphic functions f and g. Then*

$$\overline{N}(r,0;P_*(f^n)) \leq n(\chi_0^{l-1}+m_1+m_2)T(r,f);$$
$$N_2(r,0;P_*(f^n)) \leq n(\chi_0^{l-1}+\mu_0^{l-1}+m_1+2m_2)T(r,f).$$

Similar results occur for $P_*(g^n)$.

Proof: *Rewrite $P_*(f^n)$ and $P_*(g^n)$ as*

$$P_*(f^n) = Cf^n(f^n-\beta_1)...(f^n-\beta_{m_1})(f^n-\beta_{m_1+1})^{n_{m_1+1}}...(f^n-\beta_{m_1+m_2})^{n_{m_1+m_2}}$$

and

(3)

$$P_*(g^n) = Cg^n(g^n-\beta_1)...(g^n-\beta_{m_1})(g^n-\beta_{m_1+1})^{n_{m_1+1}}...(g^n-\beta_{m_1+m_2})^{n_{m_1+m_2}},$$

where $\beta_i^{'}s(i=1,2,...,m_1+m_2)$ are distinct complex constants and l_i is the multiplicity of the factor $(z-\beta_i)$ in $P^*(z)$ for $i=1,2,...,m_1+m_2$ with $l_1=l_2=...=l_{m_1}=1$ and $l_{m_1+1},...,l_{m_1+m_2}\geq 2$.

Here we have to consider two cases:

Case 1: *Suppose none of $\beta_i^{'}s(i=1,2,...,m_1+m_2)$ be zero. Then*

$$\overline{N}(r,0;P_*(f^n)) \leq n\overline{N}(r,0;f)+n\sum_{i=1}^{m_1+m_2}\overline{N}(r,\beta_i;f) \leq n(1+m_1+m_2)T(r,f);$$

$$\begin{aligned}N_2(r,0;P_*(f^n)) &\leq nN(r,0;f)+n\sum_{i=1}^{m_1}N(r,\beta_i;f)+2n\sum_{i=m_1+1}^{m_1+m_2}\overline{N}(r,\beta_i;f)\\ &\leq n(1+m_1+2m_2)T(r,f).\end{aligned}$$

Case 2: *Next let one of $\beta_i^{'}s(i=1,2,...,m_1+m_2)$ be zero.*

Subcase 2.1*: Suppose one among $\beta_i^{'}s(i=1,2,...,m_1)$ be zero. Without loss of generality let us assume that $\beta_1=0$. Then*

$$\overline{N}(r,0;P_*(f^n)) \leq n\overline{N}(r,0;f)+n\sum_{i=2}^{m_1+m_2}\overline{N}(r,\beta_i;f) \leq n(m_1+m_2)T(r,f);$$

$$\begin{aligned}N_2(r,0;P_*(f^n)) &\leq 2n\overline{N}(r,0;f)+n\sum_{i=2}^{m_1}N(r,\beta_i;f)+2n\sum_{i=m_1+1}^{m_1+m_2}\overline{N}(r,\beta_i;f)\\ &\leq n(1+m_1+2m_2)T(r,f).\end{aligned}$$

Subcase 2.2*: Next suppose one among $\beta_i^{'}s(i=m_1+1,m_1+2,...,m_1+m_2)$ be zero. Without loss of generality let us assume that $\beta_{m_1+1}=0$. Then*

$$\begin{aligned}\overline{N}(r,0;P_*(f^n)) &\leq n\overline{N}(r,0;f)+n\sum_{i=1}^{m_1}\overline{N}(r,\beta_i;f)+n\sum_{i=m_1+2}^{m_1+m_2}\overline{N}(r,\beta_i;f)\\ &\leq n(m_1+m_2)T(r,f);\end{aligned}$$

$$\begin{aligned}N_2(r,0;P_*(f^n)) &\leq 2n\overline{N}(r,0;f)+n\sum_{i=1}^{m_1}N(r,\beta_i;f)+2n\sum_{i=m_1+2}^{m_1+m_2}\overline{N}(r,\beta_i;f)\\ &\leq n(m_1+2m_2)T(r,f).\end{aligned}$$

Combining all cases we can write

$$\overline{N}(r,0;P_*(f^n)) \le n(\chi_0^{l-1}+m_1+m_2)T(r,f);$$
$$N_2(r,0;P_*(f^n)) \le n(\chi_0^{l-1}+\mu_0^{l-1}+m_1+2m_2)T(r,f).$$

Similarly we can obtain the same conclusions for the function g.

Lemma 2.13. *Let $P_*(f^n)$ and $P_*(g^n)$ for two non constant meromorphic functions f and g (as defined in (2)) share $(1,2)$ and $(\infty,0)$. If*

$$l > 2n(\chi_0^{l-1}+\mu_0^{l-1}+m_1+2m_2)+\frac{3n(4n+1)}{2n(l-1)+1}(\chi_0^{l-1}+m_1+m_2),$$

then either $P_*(f^n)(z)\equiv P_*(g^n)(z)$ or $P_*(f^n)(z).P_*(g^n)(z)\equiv 1$.
Proof: *Set*

$$\Phi=\frac{P_*(f^n)(P_*(g^n-1)}{P_*(g^n)(P_*(f^n-1))}.$$

Clearly $S(r,\Phi)$ can be replaced by $S(r,f)+S(r,g)$. It is obvious that $\Phi\not\equiv 0$. If $\Phi\equiv 0$ then either $P_*(f^n)=0$ or $P_*(g^n)=1$, which gives f and g are constants, a contradiction.
First suppose that $\Phi\not\equiv 1$. So $P_*(f^n)\not\equiv P_*(g^n)$.
Therefore, using Lemma 2.10 we get

$$\begin{aligned}&\overline{N}(r,0;\Phi)+\overline{N}(r,0;\Phi)\\&\le \overline{N}(r,1;P_*(f^n)|\ge 3n)+\overline{N}(r,1;P_*(f^n))+\overline{N}(r,1;P_*(g^n))\\&\le \frac{1}{2n}\overline{N}(r,0;P_*(f^n))+\overline{N}(r,\infty;P_*(f^n))\\&+\overline{N}(r,0;P_*(f^n))+\overline{N}(r,0;P_*(g^n))+S(r,P_*(f^n))\\&\le \frac{2n+1}{2n}\overline{N}(r,0;P_*(f))+\frac{1}{2n}\overline{N}(r,\infty;f)+n\overline{N}(r,0;P_*(g))+S(r,f).\end{aligned}$$

Now,

$$\Phi-1=\frac{P_*(g^n)-P_*(f^n)}{P_*(g^n)(P_*(f^n-1))}$$

and $\Phi'=\left[\dfrac{P_*(g^n)'}{P_*(g^n)(P_*(g^n-1))}-\dfrac{P_*(f^n)'}{P_*(f^n)(P_*(f^n-1))}\right]\Phi.$

If $\Phi'\equiv 0$ then

$$\left[\frac{P_*(g^n)'}{P_*(g^n)(P_*(g^n-1))}-\frac{P_*(f^n)'}{P_*(f^n)(P_*(f^n-1))}\right]\equiv 0.$$

Integrating we have,

$$\frac{P_*(f^n-1)}{P_*(f^n)}\equiv A\frac{P_*(g^n-1)}{P_*(g^n)},$$

where A is non zero constant. i.e.,

$$1-\frac{1}{P_*(f^n)}\equiv A-\frac{A}{P_*(g^n)}.$$

Since $P_*(f^n)$ and $P_*(g^n)$ share $(\infty,0)$ so $A=1$. Then $P_*(f^n)\equiv P_*(g^n)$ which gives $\Phi\equiv 1$, a contradiction. Therefore $\Phi'\not\equiv 0$. Clearly all poles of $P_*(f^n)$ and $P_*(g^n)$ are multiple poles which are multiple zeros of $\Phi-1$ and *so zeros of Φ' with multiplicity at least $(l-1)$ but not zeros of Φ. Therefore by Lemma 2.9,*

$$\begin{aligned}(l-1)\overline{N}(r,\infty;f^n) &= (l-1)\overline{N}(r,\infty;P_*(f^n)) = (l-1)\overline{N}(r,\infty;P_*(f^n)|\geq l)\\ &\leq N(r,0;\Phi'|\Phi\neq 0) \leq \overline{N}(r,0;\Phi)+\overline{N}(r,\infty;\Phi)+S(r,\Phi).\end{aligned}$$

So,

$$(2nl-2n-1)\overline{N}(r,\infty;f) \leq (2n+1)\overline{N}(r,0;P_*(f))+2n\overline{N}(r,0;P_*(g))+S(r,f).$$

Applying Lemma 2.12 we obtain

$$\begin{aligned}&\overline{N}(r,\infty;f^n)\\ &\leq \frac{n(2n+1)(\chi_0^{l-1}+m_1+m_2)}{2n(l-1)-1}T(r,f)+\frac{(2n^2)(\chi_0^{l-1}+m_1+m_2)}{2n(l-1)-1}T(r,g)\\ &+S(r,f)+S(r,g).\end{aligned}$$

Similarly

$$\begin{aligned}&\overline{N}(r,\infty;g^n)\\ &\leq \frac{n(2n+1)(\chi_0^{l-1}+m_1+m_2)}{2n(l-1)-1}T(r,g)+\frac{2n^2(\chi_0^{l-1}+m_1+m_2)}{2n(l-1)-1}T(r,f)\\ &+S(r,f)+S(r,g).\end{aligned}$$

That is

$$\overline{N}(r,\infty;f^n)+\overline{N}(r,\infty;g^n)$$

$$\leq \frac{n(4n+1)(\chi_0^{l-1}+m_1+m_2)}{2n(l-1)-1}(T(r,f)+T(r,g))+S(r,f)+S(r,g). \tag{4}$$

If possible, we suppose that (i) of Lemma 2.11 holds. Therefore

$$\begin{aligned}&T(r,P_*(f^n))+T(r,P_*(g^n)) \leq 2\{N_2(r,0;P_*(f^n))+N_2(r,0;P_*(g^n))\\ &+\overline{N}(r,\infty;P_*(f^n))+\overline{N}(r,\infty;P_*(g^n))+\overline{N}_*(r,\infty;P_*(f^n),P_*(g^n))\}\\ &+S(r,P_*(f^n))+S(r,P_*(g^n)).\end{aligned}$$

Then using Lemma 2.7, Lemma 2.12 and 4 we have

$$\begin{aligned}&l(T(r,f^n)+T(r,g^n))\\ &\leq \left(2n(\chi_0^{l-1}+\mu_0^{l-1}+m_1+2m_2)+\frac{3n(4n+1)}{2n(l-1)-1}(\chi_0^{l-1}+m_1+m_2)\right)\\ &(T(r,f)+T(r,g))+S(r,f)+S(r,g),\end{aligned}$$

which contradicts our assumption. So by Lemma 2.11 we have

$$P_*(f^n)(z).P_*(g^n)(z)\equiv 1.$$

If $\Phi\equiv 1$, then $P_*(f^n)(z)\equiv P_*(g^n)(z)$.
Hence the lemma is proved.

Lemma 2.14. *Let f and g be two non constant meromorphic functions of finite order. Let $l\geq 2$, and let $\{a_1(z),a_2(z),...,a_l(z)\}\in S(f^n)$ be distinct* *meromorphic periodic functions with period c. If* $m\left(r, \frac{g^n}{f^n-a_k}\right) = S(r,f)$, *for* $k = 1, 2, ..., l$, *then*

$$\sum_{k=1}^{l} m\left(r, \frac{1}{f^n - a_k}\right) \leq m\left(r, \frac{1}{g^n}\right) + S(r,f),$$

where the exceptional set associated with $S(r,f)$ *is of at most finite logarithmic measure.*

Proof: *Set*

$$P(f^n) = \prod_{k=1}^{l} (f^n - a_k).$$

Rewriting we have,

$$\frac{1}{P(f^n)} = \sum_{k=1}^{l} \frac{\alpha_k}{f^n - a_k},$$

where $\alpha_k \in S(f^n)$ *are certain periodic function with period c. Now,*

$$m\left(r, \frac{g^n}{P(f^n)}\right) \leq \sum_{k=1}^{l} m\left(r, \frac{g^n}{f^n - a_k}\right) + S(r, f^n) = S(r,f),$$

and so

$$m\left(r, \frac{1}{P(f^n)}\right) = nm\left(r, \frac{g}{P(f^n)}\right) + nm\left(r, \frac{1}{g}\right) \leq nm\left(r, \frac{1}{g}\right) + S(r,f).$$

By the first fundamental theorem and using the above inequation we get,

$$\begin{aligned} nm\left(r, \frac{1}{g}\right) &\geq nm\left(r, \frac{1}{P(f)}\right) + S(r,f) \\ &= nT(r, P(f)) - nN\left(r, \frac{1}{P(f)}\right) + S(r,f) \\ &\geq nT(r,f) - \sum_{k=1}^{l} nm\left(r, \frac{1}{f - a_k}\right) + S(r,f) \\ &= \sum_{k=1}^{l} nm\left(r, \frac{1}{f - a_k}\right) + S(r,f). \end{aligned}$$

Lemma 2.15. *If* f^n *is a meromorphic function of finite order then* $\tilde{L}(f^n(z))$ *is of finite order and* $m\left(r, \frac{\tilde{L}(f^n(z))}{f^n(z+c)}\right) = S(r,f)$, $m\left(r, \frac{\tilde{L}(f^n(z))}{f^n(z)-\beta_i}\right) = S(r,f)$ *and* $m\left(r, \frac{\tilde{L}(f^n(z))}{f^n(qz)}\right) = S_1(r,f)$,

Proof. Using logarithmic derivative lemma and Lemma 2.2 we have,

(5)

$$m\left(r, \frac{\tilde{L}(f^n(z))}{f^n(z+c)}\right) = m\left(r, \frac{\sum_{j=1}^{s} b_j (f^{(j)})^n (z + c_j) + \sum_{j=1}^{t} d_j (f^{(j)})^n (z)}{f^n(z+c)}\right)$$

$$\leq \sum_{j=1}^{s} nm\left(r, \frac{f^{(j)}(z+c_j)}{f^{(j)}(z)}\right) + \sum_{j=1}^{s} nm\left(r, \frac{f^{(j)}(z)}{f(z)}\right) + \sum_{j=1}^{t} nm\left(r, \frac{f^{(j)}(z)}{f(z)}\right)$$
$$+ n(s+t)m\left(r, \frac{f(z)}{f(z+c)}\right) + O(1)$$
$$= S(r, f).$$

Also

$$m\left(r, \frac{\tilde{L}(f^n(z))}{f^n(z) - \beta_i}\right) = m\left(r, \frac{\sum_{j=1}^{s} b_j (f^{(j)})^n (z + c_{(j)}) + \sum_{j=1}^{t} d_{(j)} (f^{(j)})^n (z)}{f^n(z) + \beta_i}\right)$$
$$\leq \sum_{j=1}^{s} nm\left(r, \frac{f^{(j)}(z+c_j)}{f^{(j)}(z)}\right) + \sum_{j=1}^{t} nm\left(r, \frac{f^{(j)}(z)}{f^{(j)}(z) - \beta_i}\right)$$
$$+ \sum_{j=1}^{s} nm\left(r, \frac{f^{(j)}(z)}{f(z) - \beta_i}\right) + O(1) = S(r, f).$$

Using (5) and Lemma 2.1 we have,

$$T(r, \tilde{L}(f^n(z))) = m(r, \tilde{L}(f^n(z))) + N(r, \tilde{L}(f^n(z))) + O(1)$$
$$\leq m(r, f^n(z+c))$$
$$+ N\left(r, \sum_{j=1}^{s} b_j (f^{(j)})^n (z + c_j) + \sum_{j=1}^{t} d_j (f^{(j)})^n (z)\right) + S(r, f)$$
$$\leq m(r, f^n(z+c)) + \sum_{j=1}^{s} N(r, (f^{(j)})^n (z + c_j))$$
$$+ \sum_{j=1}^{t} N(r, (f^{(j)})^n (z)) + S(r, f)$$
$$\leq nm(r, f(z+c)) + \sum_{j=1}^{s} n(j+1) N(r, f(z + c_j))$$
$$+ \sum_{j=1}^{t} n(j+1) N(r, f(z)) + S(r, f)$$
$$\leq nT(r, f) + \sum_{j=1}^{s} n(j+1) T(r, f) + \sum_{j=1}^{t} n(j+1) T(r, f) + S(r, f)$$
$$= \left(n + \sum_{j=1}^{s} n(j+1) + \sum_{j=1}^{t} n(j+1)\right) T(r, f) + S(r, f)$$
$$= n\left(1 + s + \frac{s(s+1)}{2} + t + \frac{t(t+1)}{2}\right) T(r, f) + S(r, f)$$
$$\frac{1}{n} T(r, \tilde{L}(f^n(z))) = \left[\frac{s^2 + t^2 + 3(s+t) + 2}{2}\right] T(r, f) + S(r, f).$$

As f^n is of finite order so $\tilde{L}(f^n(z))$ and $f^n(z+c)$ are of finite order and $S(r, \tilde{L}(f^n(z)))$ can replaced by $S(r, f)$.

Similarly by using Lemma 2.4, Lemma 2.5 and Lemma 2.6 as and when required we can prove $f^n(qz)$ and $\tilde{L}(f^n(z))$ are zero order when f^n is of zero order and

$$m\left(r, \frac{\tilde{L}(f^n(z))}{f^n(qz)}\right) = S_1(r, f).$$

□

3. Main Results

Proof of Theorem 1.1. Since $E_{f^n(z+c)}(S_1^*, 2) = E_{\tilde{L}(f^n(z))}(S_1^*, 2)$ and $E_{f^n(z+c)}(S_2, 0) = E_{\tilde{L}(f^n(z))}(S_2, 0)$, it follows that $P_*(f^n(z+c)), P_*(\tilde{L}(f^n(z)))$ share $(1, 2)$ and $(\infty, 0)$. So by Lemma 2.13 we have either $P_*(f^n(z+c)) \equiv P_*(\tilde{L}(f^n(z)))$ or $P_*(f^n(z+c)).P_*(\tilde{L}(f^n(z))) \equiv 1$. Suppose that

$$P_*(f^n(z+c)).P_*(\tilde{L}(f^n(z))) \equiv 1. \tag{6}$$

Noting that $P_*(f^n(z+c)), P_*(\tilde{L}(f^n(z)))$ share $(\infty, 0)$, so we can conclude that $P_*(f^n(z+c)), P_*(\tilde{L}(f^n(z)))$ both are entire functions.
So

$$N\left(r, \infty; \frac{P_*(\tilde{L}(f^n(z)))}{P_*(f^n(z+c))}\right) = N(r, 0; P_*(f^n(z+c))).$$

Therefore using Lemma 2.12 and Lemma 2.1, we get

$$N\left(r, \infty; \frac{P_*(\tilde{L}(f^n(z)))}{P_*(f^n(z+c))}\right) \leq n(\chi_0^{l-1}+m_1+m_2)T(r, f(z+c)) \leq nlT(r, f)+S(r, f).$$

Using Lemma 2.2 and Lemma 2.15 we have,

$$\begin{aligned}
m\left(r, \frac{P_*(\tilde{L}(f^n(z)))}{P_*(f^n(z+c))}\right) &= m\left(r, \frac{\tilde{L}(f^n(z))}{f^n(z+c)} \prod_{i=1}^{m_1+m_2} \left(\frac{\tilde{L}(f^n(z)) - \beta_i}{f^n(z+c) - \beta_i}\right)^{l_i}\right) \\
&\leq m\left(r, \frac{\tilde{L}(f^n(z))}{f^n(z+c)}\right) + m\left(r, \prod_{i=1}^{m_1+m_2} \left(\frac{\tilde{L}(f^n(z)) - \beta_i}{f^n(z+c) - \beta_i}\right)^{l_i}\right) \\
&\quad + O(1) \\
&\leq n \sum_{i=1}^{m_1+m_2} l_i m\left(r, \frac{\tilde{L}(f(z)) - \beta_i}{f(z+c) - \beta_i}\right) + S(r, f) \\
&\leq n \sum_{i=1}^{m_1+m_2} l_i m\left(r, \frac{\tilde{L}(f(z))}{f(z) - \beta_i}\right) + n \sum_{i=1}^{m_1+m_2} l_i m\left(r, \frac{1}{f(z) - \beta_i}\right) \\
&\quad + n \sum_{i=1}^{m_1+m_2} l_i m\left(r, \frac{f(z) - \beta_i}{f(z+c) - \beta_i}\right) + S(r, f) \\
&\leq n \sum_{i=1}^{m_1+m_2} l_i m\left(r, \frac{1}{f(z) - \beta_i}\right) + S(r, f) \\
&\leq n(l_1 + l_2 + ... + l_{m_1+m_2})T(r, f) + S(r, f) \\
&\leq n(l-1)T(r, f) + S(r, f).
\end{aligned}$$

By Lemma 2.1, Lemma 2.7 and (6),

$$\begin{aligned}2nlT(r,f) &= 2nlT(r,f(z+c))+S(r,f) = 2nT(r,P_*(f(z+c)))+S(r,f)\\ &\leq nT\left(r,\frac{1}{P_*(f(z+c))^2}\right)+S(r,f)\\ &\leq nT\left(r,\frac{P_*(\tilde{L}(f(z)))}{P_*(f(z+c))}\right)+S(r,f)\\ &\leq n(2l-1)T(r,f)+S(r,f),\end{aligned}$$

which is a contradiction.
Therefore $P_*(f^n(z+c)) \equiv P_*(\tilde{L}(f^n(z)))$, which yields

$$\prod_{i=1}^{l}(\tilde{L}(f^n(z))-\alpha_i) \equiv \prod_{i=1}^{l}(f^n(z+c)-\alpha_i).$$

Proof of Theorem 1.2. By proceeding in a similar way of the proof of Theorem 1.1 we can prove this theorem using Lemma 2.4, Lemma 2.5 and Lemma 2.6 as and when required instead of Lemma 2.1 and Lemma 2.2.
Proof of Theorem 1.3. Since the finite order meromorphic functions $f^n(z+c)$ and $\tilde{L}(f^n(z))$ share $(S_1^*,\infty),(S_2,\infty)$, it follows that $P_*(f^n(z+c)),P_*(\tilde{L}(f^n(z)))$ share $(1,\infty)$ and (∞,∞) which yields

$$N(r,\tilde{L}(f^n(z))) = N(r,f^n(z+c)) \tag{7}$$

and

$$\frac{P_*(\tilde{L}(f^n(z)))-1}{P_*(f^n(z+c))-1} = e^{\gamma(z)}, \tag{8}$$

where $\gamma(z)$ is a polynomial.
Now,

$$T(r,e^{\gamma(z)}) = m(r,e^{\gamma(z)}) = m\left(r,\frac{P_*(\tilde{L}(f^n(z)))-1}{P_*(f^n(z+c))-1}\right).$$

Using the definition of $P_*(z)$ we have,

$$\begin{aligned}T(r,e^{\gamma(z)}) &= m\left(r,\frac{(\tilde{L}(f^n(z))-\alpha_1)(\tilde{L}(f^n(z))-\alpha_2)...(\tilde{L}(f^n(z))-\alpha_l)}{(f^n(z+c)-\alpha_1)(f^n(z+c)-\alpha_2)...(f^n(z+c)-\alpha_l)}\right)\\ &\leq \sum_{j=1}^{l} m\left(r,\frac{\tilde{L}(f^n(z))-\alpha_j}{f^n(z+c)-\alpha_j}\right)+O(1)\\ &\leq \sum_{j=1}^{l} nm\left(r,\frac{\tilde{L}(f(z))}{f(z)-\alpha_j}\right)+\sum_{j=1}^{l} nm\left(r,\frac{1}{f(z)-\alpha_j}\right)\\ &+\sum_{j=1}^{l} nm\left(r,\frac{f(z)-\alpha_j}{f(z+c)-\alpha_j}\right)+O(1).\end{aligned}$$

In view of Lemma 2.2, Lemma 2.14 and Lemma 2.15 and then by the first fundamental theorem and (7) we have,

$$\begin{aligned} T(r,e^{\gamma(z)}) &= \sum_{j=1}^{l} nm\left(r,\frac{1}{f(z)-\alpha_j}\right)+S(r,f)\le nm\left(r,\frac{1}{\tilde{L}(f(z))}\right)+S(r,f)\\ &\le nT(r,\tilde{L}(f(z)))-nN\left(r,\frac{1}{\tilde{L}(f(z))}\right)+S(r,f)\\ &\le nm\left(r,\frac{\tilde{L}(f(z))}{f(z+c)}\right)+nm(r,f(z+c))+nN(r,\tilde{L}(f(z)))\\ &-nN\left(r,\frac{1}{\tilde{L}(f(z))}\right)+S(r,f)\\ &\le nT(r,f(z+c))-nN\left(r,\frac{1}{\tilde{L}(f(z))}\right)+S(r,f)\\ &\le nT(r,f)-nN\left(r,\frac{1}{\tilde{L}(f(z))}\right)+S(r,f). \end{aligned}$$

According to the given condition $T(r,f^n)=nN\left(r,\frac{1}{\tilde{L}(f(z))}\right)+S(r,f)$, so $T(r,e^{\gamma(z)})=S(r,f)$.
Now from (8) we have,

$$P_*(\tilde{L}(f^n(z)))=e^{\gamma(z)}(P_*(f^n(z+c))-1+e^{-\gamma(z)}).$$

Set

$$W(z)=\frac{P_*(f^n(z+c))}{1-e^{-\gamma(z)}}.$$

If $e^{\gamma(z)}\not\equiv 1$, then by applying Nevanlinna's second fundamental theorem to $W(z)$ and using (7) and Lemma 2.12 we obtain,

$$\begin{aligned} &T(r,P_*(f^n(z+c)))\le T(r,W)+S(r,f)\\ &\le \overline{N}(r,0;W)+\overline{N}(r,\infty;W)+\overline{N}(r,0;W-1)+S(r,f)\\ &\le \overline{N}(r,0;P_*(f^n(z+c)))+\overline{N}(r,\infty;P_*(f^n(z+c)))\\ &+\overline{N}(r,0;P_*\tilde{L}(f^n(z)))+S(r,f)\\ &\le (\chi_0^{l-1}+m_1+m_2)(T(r,f^n(z+c))+T(r,\tilde{L}(f^n(z))))\\ &+N(r,\infty;f^n(z+c))+S(r,f)\\ &\le (\chi_0^{l-1}+m_1+m_2)\left(T(r,f^n(z+c))+m(r,f^n(z+c))+m\left(r,\frac{\tilde{L}(f^n(z))}{f^n(z+c)}\right)\right)\\ &+N(r,\infty;f^n(z+c))+N(r,\infty;f^n)+S(r,f). \end{aligned}$$

Using Lemma 2.1 and Lemma 2.15 we get,

$$lT(r,f^n)=n(2\chi_0^{l-1}+2m_1+2m_2+1)T(r,f)+S(r,f).$$

which contradicts $l>n[2(\chi_0^{l-1}+m_1+m_2)+1]$. This gives $e^{\gamma(z)}\equiv 1$, that yields

$$\prod_{i=1}^{l}(\tilde{L}(f^n(z))-\alpha_i)\equiv\prod_{i=1}^{l}(f^n(z+c)-\alpha_i).$$

Proof of Theorem 1.4. Here $\tilde{L}(f^n(z))$ and $f^n(qz)$ are of zero order. Since $f^n(qz)$ and $\tilde{L}(f^n(z))$ share (S_1^*, ∞) and (S_2, ∞), it follows that $P_*(f^n(qz))$ and $P_*(\tilde{L}(f^n(z)))$ share $(1, \infty)$ and (∞, ∞).
Therefore
$$\frac{P_*(\tilde{L}(f^n(z))) - 1}{P_*(f^n(qz)) - 1} = A,$$
where A is a non zero constant.
This gives
$$P_*(\tilde{L}(f^n(z))) = A\left(P_*(f^n(qz)) - 1 + \frac{1}{A}\right).$$
Set $W_1(z) = \frac{P_*(f^n(qz))}{1-\frac{1}{A}}$. If $A \not\equiv 1$, then applying Nevanlinna's second fundamental theorem to $W_1(z)$ and using Lemmas 2.4 and 2.5 and 2.15 as and when required we can calculate the rest of the proof similar to Theorem 1.3.
Proof of Theorem 1.5. Here $\tilde{L}(f^n(z))$ and $f^n(z+c)$ are of finite order. Since $f^n(z+c)$ and $\tilde{L}(f^n(z))$ share $(0, \infty)$ and (∞, ∞), so
$$\frac{\tilde{L}(f^n(z))}{f^n(z+c)} = e^{\delta(z)}, \tag{9}$$
where $\delta(z)$ is a polynomial.
Clearly by Lemma 2.15 we get,
$$T(r, \delta(z)) = S(r, f).$$
When $\delta(z) \equiv 1$ then $\tilde{L}(f^n(z)) \equiv f^n(z+c)$.
When $\delta(z) \not\equiv 1$, using the fact that $\tilde{L}(f^n(z))$ and $f^n(z+c)$ share $(a(z), 0)$ we have,
$$\begin{aligned}\overline{N}\left(r, \frac{1}{\tilde{L}(f^n(z)) - a(z)}\right) &= n\overline{N}\left(r, \frac{1}{f(z+c) - a(z)}\right)\\ &\leq n\overline{N}\left(r, \frac{1}{e^{\delta(z)} - 1}\right) + \overline{N}\left(r, \frac{1}{a(z)}\right)\\ &\leq nT(r, e^{\delta(z)}) + S(r, f) = S(r, f).\end{aligned}$$
Rewriting (9) we get,
$$\tilde{L}(f^n(z)) - a(z) = e^{\delta(z)}(f^n(z+c) - a(z)e^{-\delta(z)}).$$
Clearly $a(z)e^{-\delta(z)} \not\equiv a(z)$. So,
$$\overline{N}\left(r, \frac{1}{f^n(z+c) - a(z)e^{-\delta(z)}}\right) = \overline{N}\left(r, \frac{1}{\tilde{L}(f^n(z)) - a(z)}\right) = S(r, f).$$
Using Lemma 2.1, Lemma 2.3 and second fundamental theorem we obtain,
$$\begin{aligned}&2T(r, f^n) = 2nT(r, f(z+c)) + S(r, f)\\ &\leq n\overline{N}(r, f(z+c)) + n\overline{N}\left(r, \frac{1}{f(z+c)}\right) + n\overline{N}\left(r, \frac{1}{f(z+c) - a(z)}\right)\\ &+ n\overline{N}\left(r, \frac{1}{f(z+c) - a(z)e^{-\delta(z)}}\right) + S(r, f)\\ &\leq n\overline{N}(r, f) + n\overline{N}\left(r, \frac{1}{f}\right) + S(r, f),\end{aligned}$$

which is a contradiction to $\Theta(0; f) + \Theta(\infty; f) > 0$. Hence $\tilde{L}(f^n(z)) \equiv f^n(z+c)$.

Proof of Theorem 1.6. Here $\tilde{L}(f^n(z))$ and $f^n(z+c)$ are of finite order. Since $f^n(z+c)$ and $\tilde{L}(f^n(z))$ share $(a(z), \infty)$ and (∞, ∞), so

$$\frac{\tilde{L}(f^n(z)) - a(z)}{f^n(z+c) - a(z)} = e^{\zeta(z)}, \tag{10}$$

where $\zeta(z)$ is a polynomial. Using logarthmic derivative lemma, Lemmas 2.1 and 2.2 we get,

$$\begin{aligned}
&T(r, e^{\zeta(z)}) = m(r, e^{\zeta(z)})\\
&= m\left(r, \frac{\tilde{L}(f^n(z)) - a(z)}{f^n(z+c) - a(z)}\right)\\
&\leq m\left(r, \frac{\tilde{L}(f^n(z)) - \tilde{L}(a(z-c))}{f^n(z+c) - a(z)}\right) + m\left(r, \frac{\tilde{L}(a(z-c)) - a(z)}{f^n(z+c) - a(z)}\right)\\
&\leq m\left(r, \frac{\tilde{L}(f^n(z)) - \tilde{L}(a(z-c))}{f^n(z) - a(z-c)}\right) + m\left(r, \frac{f(z) - a(z-c)}{f^n(z+c) - a(z)}\right)\\
&+ m\left(r, \frac{1}{f^n(z+c) - a(z)}\right) + S(r, f)\\
&\leq m\left(r, \frac{\sum_{j=1}^{s} b_j((f^{(j)})^n(z+c_j) - a^{(j)}(z-c+c_j)) + \sum_{j=1}^{t} d_j((f^{(j)})^n(z) - a^{(j)}(z-c))}{f^n(z) - a(z-c)}\right)\\
&+ T(r, f^n(z+c)) + S(r, f)\\
&\leq \sum_{j=1}^{s} nm\left(r, \frac{f^{(j)}(z+c_j) - a^{(j)}(z-c+c_j)}{f^{(j)}(z) - a^{(j)}(z-c)}\right) + \sum_{j=1}^{t} nm\left(r, \frac{f^{(j)}(z) - a^{(j)}(z-c)}{f(z) - a(z-c)}\right)\\
&+ \sum_{j=1}^{s} nm\left(r, \frac{f^{(j)}(z) - a^{(j)}(z-c)}{f(z) - a(z-c)}\right) + nT(r, f) + S(r, f)\\
&\leq nT(r, f) + S(r, f).
\end{aligned}$$

So, $S(r, e^{\zeta(z)})$ can be replaced by $S(r, f)$. When $e^{\zeta(z)} \equiv 1$ then $\tilde{L}(f^n(z)) \equiv f^n(z+c)$.

Suppose $e^{\zeta(z)} \not\equiv 1$. Now rewriting (10) we can obtain,

$$\frac{1}{f^n(z+c)} = -\frac{\tilde{L}(f^n(z))}{f^n(z+c)(e^{\zeta(z)} - 1)} + \frac{e^{\zeta(z)}}{a(z)(e^{\zeta(z)} - 1)}.$$

Therefore in view of Lemma 2.15 we have,

$$m\left(r, \frac{1}{f^n(z+c)}\right) \leq 2nm\left(r, \frac{1}{e^{\zeta(z)} - 1}\right) + S(r, f).$$

If $\zeta(z)$ is constant then automatically $m\left(r, \frac{1}{f^n(z+c)}\right) = S(r, f)$. If $\zeta(z)$ is non constant then by Lemma 2.8 we get,

$$m\left(r, \frac{1}{f^n(z+c)}\right) = S(r, e^{\zeta(z)}) = S(r, f).$$

By Lemmas 2.1 and 2.3 we have,

$$\begin{aligned} T(r, f^n) &= T(r, f^n(z+c)) + S(r,f) = T\left(r, \frac{1}{f^n(z+c)}\right) + S(r,f) \\ &\le N\left(r, \frac{1}{f^n(z+c)}\right) + S(r,f) \\ &\le nN\left(r, \frac{1}{f}\right) + S(r,f) \le nT(r,f) + S(r,f). \end{aligned}$$

Therefore,

$$nN\left(r, \frac{1}{f}\right) \le nT(r,f) + S(r,f),$$

which contradicts the fact that $\delta(0,f) > 0$. Hence $\tilde{L}(f^n(z)) \equiv f^n(z+c)$.

References

[1] W. K. Hayman, *Meromorphic functions*(The Clarendon Press, Oxford, 1964.)

[2] F. Gross, *Factorization of meromorphic functions and some open problems*, Proc. Conf. Univ. Kentucky, Leixngton, Kentucky(1976); Lecture Notes in Math., **599**(1977), 51-69, Springer(Berlin).

[3] I. Lahiri, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J., **161**(2001), 193-206 .

[4] I. Lahiri, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theo. Anal. Appl., **46**(2001), 241-253.

[5] X. Qi, N. Li and L. Yang, *Uniqueness of meromorphic functions concerning their differences and solutions of difference PainlevE equations*, Comput. Methods Funct. Theo., **18**(2018), 567-582.

[6] C. Meng and G. Liu, *Uniqueness of meromorphic functions concerning their shifts and derivatives*, J. Appl. Math. Inform. **37**(1-2)(2019), 133-148.

[7] Y. M. Chiang and S. J. Feng, *On the Nevanlinna Characteristic $f(z+n)$ and difference equations in complex plane*, Ramanujan J., **16**(2008), 105-129.

[8] R. G. Halburd and R. J. Korhonen, *Difference analogue of the lemma on the logarithmic derivative with applications to difference equations*, J. Math. Anal. Appl., **314**(2006), 477-487.

[9] R. Halburd, R. Korhonen and K. Tohge, *Holomorphic curves with shift-invariant hyperplane preimages*, Trans. Amer. Math. Soc., **366**(2014), 4267-4298.

[10] D. C. Barnett, R. G. Halburd, R. J. Korhonen and W. Morgan, *Nevanlinna theory for the q-difference operator and meromorphic solutions of q-difference equations*, Proc. Roy. Soc. Edinburgh Sect. A, **137**(2007), 457-474.

[11] J. L. Zhang and R. J. Korhonen, *On the Nevanlinna Characteristic $f(qz)$ and its application*, J. Math. Anal. Appl., **369**(2010), 537-544.

[12] C. C. Yang and H. X. Yi, *Uniqueness of meromorphic functions*, Kluwer Academic, Dordrecht (2003).

[13] P. Li and W. J. Wang, *Entire functions that share a small function with its derivatives*, J. Math. Anal. Appl., **328**(2007), 743-751.

[14] I. Lahiri and S. Dewan, *Value distribution of the product of meromophic function and its derivative*, Kodai Math. J., **26**(2003), 95-100.

[15] A. Banerjee, *Meromorphic functions sharing two sets*, Czech. Math. J., **57**(132)(2007), 1199-1214.

[16] X. Qi and L. Yang, *Uniqueness of meromorphic functions concerning their shifts and derivatives*, Comput. Methods Funct. Theo., **20**(2020), 159-178.

[17] P. Nagaswara and S. Rajeshwari, *Complex delay-differential equations of malmquist type*, Journal of Applied Mathematics and Informatics. **40**(3-4), (2022), 507-513.

[18] S. Rajeshwari, B. Venkateswarlu, and S. H. NaveenKumar, *Uniqueness of certain types of difference-differential polynomials sharing small function*, Journal of Applied Mathematics and Informatics. **39**(5-6), (2021), 839-850.

[19] C. N. Chaithra, S. H. Naveenkumar and H. R. Jayarama, *Uniqueness of meromorphic function sharing two values concerning differential-difference polynomial and its kth derivative*, Mathematics in Engineering, Science and Aerospace, **14**(1), (2023), 129-144.

[20] C. N. Chaithra, S. H. Naveenkumar and H. R. Jayarama, *On the Transcendental solution of the Fermat type q-shift Equation*, Electronic Journal of Mathematical Analysis and Applications, **11**(2), (2023), 1-7.

[21] H. R. Jayarama, S. H. Naveenkumar and C. N. Chaithra, *Uniqueness of L-functions relating to possible Differential polynomial share with some finite weight*, Ganita, **72**(2), (2022), 93-101.

[22] C. N. Chaithra, S. H. Naveenkumar and S. Rajeshwari, *Results on the uniqueness of difference polynomials with diffrence operator of entire functions sharing small function*, Communications in Mathematics and Applications, **14**(2), (2023), 775-790.

[23] H. R. Jayarama, S. H. Naveenkumar, S. Rajeshwari and C. N. Chaithra, *Uniqueness of Transcendental meromorphic functions and certain Differential polynomials*, J. Appl. Math. and Informatics, **41**, (2023), 765-780.

Department of Mathematics, Presidency University, Bengaluru-560 064
E-mail address: `chinnuchaithra15@gmail.com`

Department of Mathematics, Presidency University, Bengaluru-560 064
E-mail address: `naveenkumarnew1991@gmail.com`

Department of Mathematics, Presidency University, Bengaluru-560 064
E-mail address: `jayjayaramhr@gmail.com`