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CONTINUITY CRITERIA FOR LOCALLY BOUNDED HOMOMORPHISMS OF CERTAIN CONNECTED LIE GROUPS

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Abstract.

We prove that every locally bounded homomorphism of a connected Lie group G whose commutator subgroup G' admits a closed supplementary subgroup Z such that $G = G'Z$ into a linear connected Lie group is continuous if and only if it is continuous on Z.

§ 1. Introduction

As was proved in [1], every locally bounded endomorphism of a central extension of a perfect connected linear Lie group is continuous if and only if it is continuous on the center of the group. In this paper, extending this assertion, we prove that every locally bounded homomorphism of a connected Lie group G whose commutator subgroup G' admits a supplementary subgroup Z such that $G = G'Z$ into a linear connected Lie group is continuous if and only if it is continuous on Z, which refines results of [1–3].

§ 2. Preliminaries

Let us recall some information needed below.

A (not necessarily continuous) homomorphism π of a topological group G into a topological group H is said to be *relatively compact* if there is a

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neighborhood $U = U_{e_G}$ of the identity element e_G in G whose image $\pi(U)$ has compact closure in H.

Obviously, a homomorphism into a locally compact group is relatively compact if and only if it is *locally bounded*, i.e., there is a neighborhood U_e whose image is contained in some element of the filter $\mathfrak{V}$ of neighborhoods of e_V having compact closure.

Let us also recall the notion of discontinuity group of a homomorphism π of a topological group G into a topological group H, see [4] and [5]. Let $\mathfrak{U} = \mathfrak{U}_G$ be the filter of neighborhoods of e_G in G. For every (not necessarily continuous) locally relatively compact homomorphism π of G into H, the set

$$\mathrm{DG}(\pi) = \bigcap_{U \in \mathfrak{U}} \overline{\pi(U)}$$

is called the discontinuity group of π. Here and further, the bar stands for the closure in the corresponding topology (in the case under consideration, the closure is taken in the topology of H). (See Definition 1.1.1 of [4].)

The discontinuity group of a homomorphism has some important properties. Under the above conditions, the set $\mathrm{DG}(\pi)$ is a compact subgroup of the topological group H and a compact normal subgroup of the closed subgroup $\overline{\pi(G)}$ of H. Moreover, the filter basis $\{\overline{\pi(U)} \mid U \in \mathfrak{U}\}$ converges to $\mathrm{DG}(\pi)$, and the homomorphism π is continuous if and only if

$$\mathrm{DG}(\pi) = \{e_H\}.$$

(See Theorem 1.1.2 of [4].)

If G is a connected Lie group, then $\mathrm{DG}(\pi)$ is a compact connected subgroup of H. (See Lemma 1.1.6 of [4].)

Let G be a connected Lie group, let N be a closed normal subgroup of G, and let π be a locally bounded homomorphism of G into a locally compact group H. Let M be the discontinuity group of the restriction $\mathrm{DG}(\pi|_N)$. Then M is a closed normal subgroup of the compact discontinuity group $\mathrm{DG}(\pi)$, and the corresponding quotient group $\mathrm{DG}(\pi)/M$ is isomorphic to the discontinuity group $\mathrm{DG}(\psi)$ of the homomorphism ψ of G obtained as the composition of the homomorphism π and the canonical homomorphism

$$\overline{\pi(G)} \to \overline{\pi(G)}/M.$$

(See Lemma 1.1.7 of [4].)

§ 3. Main result

Theorem 1. *Every locally bounded homomorphism of a connected Lie group G whose commutator subgroup G' admits a closed supplementary subgroup Z such that $G = G'Z$ into a linear connected Lie group is continuous if and only if it is continuous on Z.*

Proof. Obviously, if a homomorphism of a group is continuous, then it is continuous on every subgroup of the group, and thus it suffices to prove the "if" part.

Let G be a connected Lie group G whose commutator subgroup G' admits a closed supplementary subgroup Z such that $G = G'Z$.

Let H be a connected linear Lie group.

Let π be a locally bounded homomorphism of G into H which is continuous on Z.

Since H is linear, we may assume, applying a faithful representation of H, that π is a locally bounded linear representation of G.

As is well known, the discontinuity group of the restriction of π to the commutator subgroup G' is the identity group (see Theorem 1.3.2 of [4]), and hence the restriction of π to the commutator subgroup G' is continuous with respect to the intrinsic Lie topology. (See Theorem 1.1.2 of [4] and the correction in [7].)

Hence the representation π is separately continuous with respect to the subgroups Z and G'.

By the Namioka theorem [5], the representation π has a point of joint continuity, and therefore is continuous. This completes the proof of Theorem 1.

§ 4. Comments

The linearity condition for H can be established using the list in [8] and Harish-Chandra's theorem [9].

The phenomenon relating the continuity of a locally bounded homomorphism to the continuity of its restriction to the center is specific, see the example in [1].

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