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EXTENSIONS OF UNITARY ONE-DIMENSIONAL REPRESENTATIONS FROM SUBGROUPS

A. I. Shtern

Abstract. We obtain a criterion for the existence of an extension of a unitary character (a unitary one-dimensional complex representation) of a subgroup of a (not necessarily Abelian) group to the whole group.

§ 1. Introduction

The extension of unitary characters (one-dimensional complex unitary representations) of closed subgroups of locally compact groups to unitary characters of the ambient group was discussed many times (see, e.g., [1, 2]).

As is well known, for locally compact Abelian groups, the answer is positive (see below).

For non-Abelian locally compact groups, some conditions are needed. Indeed, if G is a semisimple compact Lie group with nontrivial center, then every nontrivial unitary character of the center cannot be extended to a character of the whole group since this group has the unitary character identically equal to one only.

In this paper, we obtain a criterion for a continuous unitary character on a closed subgroup of a group to have an extension to a continuous unitary character on the whole group and indicate some consequences.

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§ 2. Preliminaries

Recall a result mentioned above and needed below.

Theorem 1. *Let H be a closed subgroup of an Abelian locally compact group G. Every continuous unitary character ψ of H admits an extension χ over G that is a continuous unitary character of G.*

For the proof, see Corollary (24.12) in [3].

§ 3. Main Result

Theorem 2. *Let H be a closed subgroup of a locally compact group G. A continuous unitary character ψ of H admits an extension χ over G that is a continuous unitary character of G if and only if ψ is trivial on the intersection $H \cap \overline{G'}$ of the subgroup H with the closure $\overline{G'}$ of the commutator subgroup G' of G.*

Proof. Every continuous unitary character on a group is trivial (identically equal to 1) on the commutator subgroup G' of G, and hence on the closure $\overline{G'}$ of G' by continuity.

Thus, if a continuous unitary character ψ of H admits an extension to a unitary character χ on G, then χ is trivial on G', and hence on the closure $\overline{G'}$ of G'.

Therefore, the original unitary character ψ of H (satisfying the relation $\psi(h) = \chi(h)$ for all $h \in H$) is trivial on the intersection $H \cap \overline{G'}$. This proves the "only if" part of the assertion.

Conversely, let the unitary character ψ be trivial on the intersection $H \cap \overline{G'}$ of the subgroup H with the closure $\overline{G'}$ of the commutator subgroup G' of G.

Then ψ defines a continuous unitary character of the image of H in the Abelian locally compact quotient group $G/\overline{G'}$, which is naturally isomorphic to the Abelian subgroup $H/H \cap \overline{G'}$ of $G/\overline{G'}$.

By Theorem 1, this unitary character admits a continuous extension χ, which is a continuous unitary character of $G/\overline{G'}$, and hence can be viewed as a continuous unitary character of G, which is obviously an extension of ψ to G.

This completes the proof of the theorem.

This theorem has the following corollary.

Theorem 3. *The condition that every continuous unitary character of every closed subgroup of a connected locally compact group admits an extension to a* *continuous unitary character of the group itself holds if and only if the group is commutative.*

Proof. The "if" part is Theorem 1.

By Theorem 1 (or Theorem 2) of [2], if a connected locally compact group G satisfies the extension condition in the theorem, then G is a direct product of a vector group V and a connected compact group K.

The commutativity of the connected compact group K holds since the commutator subgroup K' of the connected compact group K is semisimple (see [4], Definition 9.5 and Corollary 9.6).

Indeed, if the commutator subgroup K' of the compact part K is nontrivial, then K' has a nonidentity Abelian subgroup $H \subset K'$, and thus H has a nontrivial continuous unitary character.

This character cannot be extended to a continuous unitary character of the whole compact group since every character of K is trivial on $(K')'$, and hence on $K' = (K')'$.

This means that the extension property under consideration implies that K is commutative.

§ 4. Discussion

Our Theorem 2 perfectly explains the situation with nontrivial centers of compact semisimple Lie groups: in this situation, the centers belong to the commutator subgroup.

A finite group has the above extension property for every character of every subgroup if and only if the commutator subgroup of the group has no nontrivial unitary characters. The proof of this fact readily follows from the above consideration.

The corresponding extension problem for not necessarily unitary characters of connected locally compact groups will be discussed elsewhere.

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Moscow Center for Fundamental and Applied Mathematics, Moscow, 119991 Russia
Department of Mechanics and Mathematics,
Moscow State University,
Moscow, 119991 Russia
Federal State Institution
“Scientific Research Institute for System Analysis of the Russian Academy of Sciences” (FSI SRISA RAS),
Moscow, 117312 Russia
E-mail: aishtern@mtu-net.ru, rroww@mail.ru