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Interval multi-criteria optimization

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ABSTRACT. We consider the method of partial ordering of real intervals with the help of a numerical indicator and describe the applications of the indicator for the interval multi-criteria optimization of interval decision-making problems. The properties of reduced problems are established. Illustrating examples are considered.

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KEYWORDS AND PHRASES. Interval indicator, decision-making problem, multi-criteria optimization.

INTRODUCTION

Many mathematical models used in the natural sciences, engineering, social and humanitarian fields can be considered as interval ones. Initially, the objects of study of interval analysis were problems of computational mathematics, and the main efforts were focused on obtaining two-sided (interval) estimates of solutions. Further, decision-making problems began to be investigated: extreme, game, controlled. Their specificity is related with the need to use the preference relation.

Interval problems can be treated as a parametric family of problems generated by all parameter values in admissible intervals. In the paper [1], the solution of the interval problem is considered to be one «acceptable» solution for the entire family of problems. It is shown that finding an acceptable («universal») solution is reduced to solving a regularized deterministic problem of the same type as the original interval problem.

Another approach employs comparison of intervals. In [2,5,6], the partial order on the set of real intervals is understood in the strong and weak sense. Strong comparison is defined for intervals without common interior points; all other intervals are considered incomparable. A weak comparison allows the intersection of intervals, while the «smaller» interval on the numerical line may be to the right of the «larger» interval.

The problem of partial ordering of intervals can be approached applying the methodology of other mathematical disciplines - probability theory, fuzzy set theory. They use real indicators of binary operations on sets - probabilities of random events, membership functions. By analogy, it is natural to set a partial order relation on the set of real intervals by defining a pairwise comparison of intervals using a numerical indicator of interval inequality [3]. As a result, a formal basis appears for correct mathematical formulations of interval decision-making problems and development of methods for their solution.

The present work is devoted to the applying and further developing of the theory of partial ordering of intervals based on interval inequality indicator introduced in the paper “On reduction of interval decision-making problems” (Advanced Studies in Contemporary Mathematics 33 (2023), No. 1, pp. 33 – 49) [4] for the interval multi-criteria optimization.

The quality of functioning of complex systems in many cases is assessed by several criteria [7]. Criteria may have different physical nature and be incomparable with each other. The needs of practice stimulated the emergence and development of the theory of multi-criteria optimization. Initially, multi-criteria optimization models assumed the uniqueness of the criteria. However, in reality they are often ambiguous. The reason for the ambiguity may be the relative knowledge of the researcher about the phenomenon being modeled or the difficulty in assessing the consequences of the decisions made. A typical example is the modeling of the development of socio-economic processes in conditions of incomplete information. In this regard, there is a need to study multi-criteria problems with interval criteria [8–10].

Thematically, the presentation of the material consists of two related parts. The first part provides the necessary information about the indicator of interval inequality. The second part shows the application of the indicator for the study the interval multi-criteria optimization problems. The reductions of typical problems to similar deterministic problems are described, in which the target conditions and constraints can be meaningfully interpreted in terms of probability. On this base, the interval multi-criteria optimization problems are considered. The approach and obtained results can be applied to the many economic and other models.

1. PRELIMINARIES

1.1. Interval inequality indicator

By [4], we consider the space $\mathbb{IR}$ of regular closed real intervals $\boldsymbol{a} = [\underline{a}, \overline{a}]$, $\underline{a} \le \overline{a}$. The *center* and *radius* of the interval $\boldsymbol{a}$ will be denoted

$$a_0 = 0.5(\underline{a} + \overline{a}), \ \Delta a = 0.5(\overline{a} - \underline{a}).$$

Expressing the ends of the interval in terms of the center and the radius, we obtain an equivalent *symmetrical* representation of the interval

$$\boldsymbol{a} = [a_0 - \Delta a, a_0 + \Delta a].$$

An interval $\boldsymbol{a}$ is called *degenerate* if $\Delta a = 0$ and *non-degenerate* if $\Delta a > 0$.

Following [6], for intervals $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{IR}$ we give the definitions of inequality $\boldsymbol{a} \le \boldsymbol{b}$ in the «strong», «weak» and «central» senses:

$$\begin{gathered} \boldsymbol{a} \le \boldsymbol{b} \Leftrightarrow \big((\forall a \in \boldsymbol{a})(\forall b \in \boldsymbol{b})(a \le b)\big), \\ \boldsymbol{a} \le \boldsymbol{b} \Leftrightarrow \big((\exists a \in \boldsymbol{a})(\exists b \in \boldsymbol{b})(a \le b)\big), \\ \boldsymbol{a} \le \boldsymbol{b} \Leftrightarrow (a_0 \le b_0). \end{gathered} \tag{1.1}$$

According to [4], we consider non-degenerate intersecting intervals $\boldsymbol{a}$, $\boldsymbol{b}$ on the real axis (Fig. 1.1).

Fig. 1.1 The inequality $a \le b$ holds for all points $a \in [\underline{a}, \underline{b}]$, $b \in [\overline{a}, \overline{b}]$

As can be seen from the Fig. 1.1, the points $a \in \boldsymbol{a}$ and $b \in \boldsymbol{b}$ satisfying the inequality $a \le b$ are in the intervals $[\underline{a}, \underline{b}]$ and $[\overline{a}, \overline{b}]$, respectively. The share of the total length of these intervals in the total length of intervals $\boldsymbol{a}$, $\boldsymbol{b}$ is equal to the absolute value of the ratio

$$\frac{(\underline{b}-\underline{a})+(\overline{b}-\overline{a})}{2(\Delta a+\Delta b)}=\frac{b_0-a_0}{\Delta a+\Delta b}. \tag{1.2}$$

Taking into account the above considerations and formula (1.2), we accept the real number

$$R(\boldsymbol{a}\le\boldsymbol{b})=\frac{b_0-a_0}{\Delta a+\Delta b} \tag{1.3}$$

as the *indicator* R *of the interval inequality* $\boldsymbol{a}\le\boldsymbol{b}$.

With an additional agreement, the indicator R can be extended on degenerate intervals. Indeed, we can apply formula (1.3) to intervals $\boldsymbol{a}_\varepsilon=[a-\varepsilon,a+\varepsilon]$, $\boldsymbol{b}_\varepsilon=[b-\varepsilon,b+\varepsilon]$ of small radius $\varepsilon>0$. We get

$$R(\boldsymbol{a}_\varepsilon\le\boldsymbol{b}_\varepsilon)=(b-a)/(2\varepsilon).$$

Hence, in the limit as $\varepsilon\to 0$ we find

$$\begin{aligned}&R(a\le b)=+\infty\Leftrightarrow a<b;\\&R(a\le b)=0\Leftrightarrow a=b;\\&R(a\le b)=-\infty\Leftrightarrow a>b.\end{aligned} \tag{1.4}$$

Correspondences (1.4) determine the indicator (1.3) on degenerate intervals if the interval of its values $(-\infty,+\infty)$ is supplemented with symbols $-\infty,\ +\infty$.

1.2. Indicator properties

By [4], for intervals $\boldsymbol{a},\boldsymbol{b}$ from $\mathbb{IR}$ represented in symmetrical form

$$\boldsymbol{a}=[a_0-\Delta a,a_0+\Delta a],\ \boldsymbol{b}=[b_0-\Delta b,b_0+\Delta b],$$

operations of classical interval arithmetic (addition, multiplication by a real number α) are performed according to the rules

$$\boldsymbol{a}+\boldsymbol{b}=[a_0+b_0-\Delta a-\Delta b,\ a_0+b_0+\Delta a+\Delta b],$$

$$\alpha\boldsymbol{a}=[\alpha a_0-|\alpha|\Delta a,\ \alpha a_0+|\alpha|\Delta a].$$

Using these operations, it is easy to establish [3] the main properties of the indicator which follow from definition (1.3).

- Multiplying the interval inequality by a positive number does not change the inequality indicator; multiplying inequality by a negative number reverses the sign of the indicator.
- Inequality indicator is antisymmetric: $R(\boldsymbol{a} \leq \boldsymbol{b}) = -R(\boldsymbol{b} \leq \boldsymbol{a})$.
- Intervals ***a***, ***b*** with equal centers satisfy opposite inequalities $\boldsymbol{a} \leq \boldsymbol{b}, \boldsymbol{b} \leq \boldsymbol{a}$ with zero indicator.
- When adding interval inequalities $\boldsymbol{a} \leq \boldsymbol{b}$ and $\boldsymbol{c} \leq \boldsymbol{d}$ with equal indicators, an inequality $\boldsymbol{a} + \boldsymbol{c} \leq \boldsymbol{b} + \boldsymbol{d}$ with the same indicator is obtained.
- If pairs of intervals ***a***, ***b*** and ***c***, ***d*** have equal sums of radii then the inequality indicator $\boldsymbol{a} + \boldsymbol{c} \leq \boldsymbol{b} + \boldsymbol{d}$ is equal to the arithmetic mean of the inequality indicators $\boldsymbol{a} \leq \boldsymbol{b}$ and $\boldsymbol{c} \leq \boldsymbol{d}$.
- For intervals $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ with pairwise equal positive sums of radii, the equality

$$R(\boldsymbol{a} \leq \boldsymbol{c}) = R(\boldsymbol{a} \leq \boldsymbol{b}) + R(\boldsymbol{b} \leq \boldsymbol{c})$$

holds true.

1.3. Relation of indicator with probability

According to [4], we consider non-degenerate intervals ***a***, ***b*** from $\mathbb{IR}$ as sets of realization of independent uniformly distributed random variables *a*, *b*. A relation between the probability p of a random event $a \leq b$ and the indicator r of the inequality $\boldsymbol{a} \leq \boldsymbol{b}$ can be established as follows: we distinguish three main cases of the location of the rectangle $\boldsymbol{a} \times \boldsymbol{b}$ relative to the half-plane $x \leq y$ (Fig. 1.2) in the Cartesian coordinate system *x*, *y*. In the case (a), the coordinates of the vertices of the rectangle $\boldsymbol{a} \times \boldsymbol{b}$ satisfy the conditions

$$a_0 - \Delta a \leq b_0 + \Delta b,\ a_0 - \Delta a > b_0 - \Delta b,$$
$$a_0 + \Delta a > b_0 - \Delta b,\ a_0 + \Delta a > b_0 + \Delta b.$$

Then

$$-(\Delta a + \Delta b) \leq b_0 - a_0 < -|\Delta a - \Delta b|$$

or, by (1.3)

$$-1 \le r < -\frac{|\Delta a - \Delta b|}{\Delta a + \Delta b}.$$

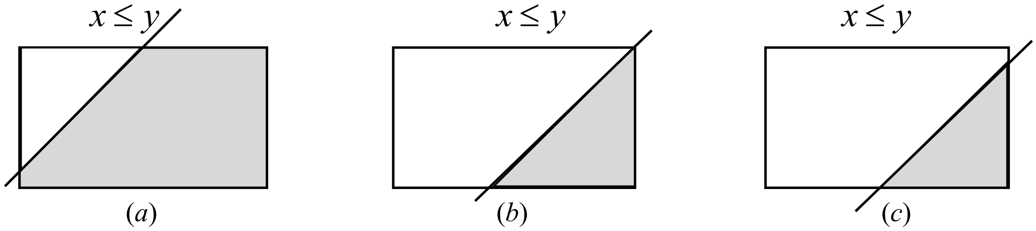

Fig. 1.2 Half plane $x \le y$ of x, y coordinate plane
contains k vertices of rectangle $\boldsymbol{a} \times \boldsymbol{b}$:
(a) $k = 1$, (b) $2 \le k \le 3$, (c) $3 \le k \le 4$

From geometric considerations, the probability p is equal to the ratio of the area of the light triangle in Fig. 1.2 (a) to the area of a rectangle $\boldsymbol{a} \times \boldsymbol{b}$:

$$p = \frac{(b_0 + \Delta b_0 - a_0 + \Delta a)^2}{8\Delta a \Delta b} = \frac{(\Delta a + \Delta b)^2}{8\Delta a \Delta b}(1+r)^2.$$

Fulfilling similar calculations for cases (b) and (c), we obtain [3] the desired relationship between the probability p and indicator r:

$$\begin{aligned} &p = \alpha(1+r)^2, -1 \le r \le -r_1, \\ &p = 0.5(1+\beta r), |r| \le r_1, \\ &p = 1 - \alpha(1-r)^2, r_1 < r \le 1, \end{aligned} \tag{1.5}$$

$$\alpha = \frac{(\Delta a + \Delta b)^2}{8\Delta a \Delta b},\ \beta = 1 + \frac{\Delta a}{\Delta b},\ r_1 = \frac{|\Delta a - \Delta b|}{\Delta a + \Delta b}. \tag{1.6}$$

The last formulas define a continuous strictly increasing function on the segment $-1 \le r \le 1$ (Fig. 1.3). We extend it by continuity to the entire real axis, setting

$$p = 0, r < -1;\ p = 1, r > 1. \tag{1.7}$$

Denote the resulting function by $\pi(r)$. On the interval $-1 < r < 1$, the function $\pi(r)$ has a single-valued inverse function $r = \pi^{-1}(p)$.

Formulas (1.5) − (1.7) give a certain probabilistic interpretation to the relationships in which the indicator of interval inequality participates. For example, if the inequality $R(\boldsymbol{a} \le \boldsymbol{b}) \ge r$ holds for some real r then the probability of an event $a \le b$ for independent random variables a, b uniformly distributed over intervals $\boldsymbol{a}, \boldsymbol{b}$ is not less than $\pi(r)$. For brevity, we agree in such cases to say that *the interval inequality* $\boldsymbol{a} \le \boldsymbol{b}$ *is satisfied with a probability not less than* $\pi(r)$ *and write it in the form* $P(\boldsymbol{a} \le \boldsymbol{b}) \ge \pi(r)$.

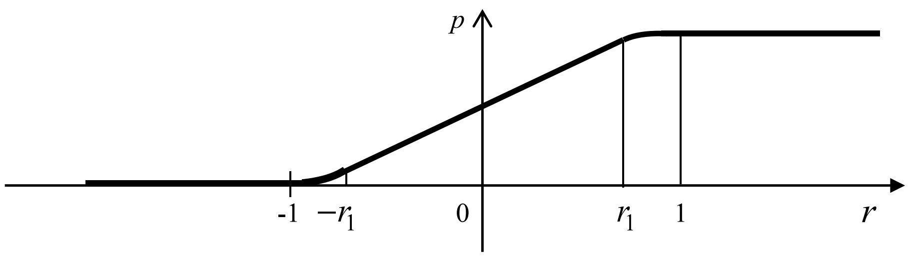

Fig. 1.3 Graph of a function $p = \pi(r)$ given by formulas (1.5) – (1.7)

2. INTERVAL MULTI-CRITERIA OPTIMIZATION

2.1. Statement of the problem

The subject of our attention is the interval problem of multi-criteria optimization

$$\boldsymbol{f}(x) \to \max, \ x \in D, \tag{2.1}$$

where $\boldsymbol{f}$ – given vector function with real interval components $\boldsymbol{f}_1, ..., \boldsymbol{f}_k : \mathrm{R}^n \to \mathrm{IR}$, D - given set in R^n.

The points of the set D are called *admissible points*, functions $\boldsymbol{f}_1, ..., \boldsymbol{f}_k$ - *criteria*, vector function $\boldsymbol{f}$ - *vector criterion*. The problem is to find admissible points of the "maximum" of the vector criterion in the sense specified below.

With the help of the indicator of interval inequality (1.3), we introduce a preference relation on the set *D*. Let problem (2.1) be reduced in one way or another to the problem of single-criterion interval optimization

$$\boldsymbol{F}(x) \to \max,\ x \in D$$

with a given function $\boldsymbol{F} : \mathrm{R}^n \to \mathrm{IR}$. Let us agree to consider from two points $a, b \in D$ a point b *no worse than* a point a if for a given in advance real r it holds the inequality

$$R(\boldsymbol{F}(a) \le \boldsymbol{F}(b)) \ge r. \tag{2.2}$$

The choice of the number r in inequality (2.2) is dictated by two considerations. First, point *b* is no worse than itself, i.e. inequality (2.2) must hold for $a = b$. Then, by virtue of (1.3), we have

$$0 = R(\boldsymbol{F}(b) \le \boldsymbol{F}(b)) \ge r.$$

Secondly, if $r < -1$, then according to formulas (1.5) – (1.7) the probability of fulfillment of the inequality $\boldsymbol{F}(a) \le \boldsymbol{F}(b)$ is equal to zero. Therefore, we further consider $-1 \le r \le 0$.

We present the preference relations and the reductions of the problem (2.1) that follow from them for known methods of reducing a multi-criteria optimization problem to a single-criterion problem.

2.2. Convolution of criteria

Define a vector $\lambda = (\lambda_1, \ldots, \lambda_k)$ with components

$$\lambda_1, \ldots, \lambda_k \ge 0,\ \lambda_1 + \ldots + \lambda_k = 1 \tag{2.3}$$

and function

$$\boldsymbol{F}(x) = \lambda' \boldsymbol{f}(x) = \lambda_1 \boldsymbol{f}_1(x) + \ldots + \lambda_k \boldsymbol{f}_k(x),\ x \in \mathrm{R}^n. \tag{2.4}$$

Let us call the components (2.3) of the vector λ as the *weights*, the function (2.4) - the *convolution* of the vector criterion $\boldsymbol{f}$. In essence, each weight λ_i determines the "contribution" $\lambda_i \boldsymbol{f}_i$ of the criterion $\boldsymbol{f}_i$ to the sum $\boldsymbol{F}$ and implicitly reflects its "importance" among the other criteria.

We consider for a fixed λ one-criterion interval problem of mathematical programming

$$\boldsymbol{F}(x) \to \max, \; x \in D. \tag{2.5}$$

Choose a number $r \in [-1,0]$. An admissible point $x^{(r)}$ that satisfies the inequality

$$R(\boldsymbol{F}(x) \le \boldsymbol{F}(x^{(r)})) \ge r, \; x \in D, \tag{2.6}$$

is called a *solution* of the problem (2.5). If the point $x^{(r)}$ exists then we take it as a *solution* to the multi-criteria problem (2.1). Otherwise, problem (2.1) is assumed to be r-unsolvable.

Assuming

$$\boldsymbol{F}(x) = [F_0(x) - \Delta F(x), F_0(x) + \Delta F(x)]$$

and using formula (1.3), we represent inequality (2.6) in the equivalent form

$$F_0(x^{(r)}) - r\Delta F(x^{(r)}) \ge F_0(x) + r\Delta F(x), \; x \in D. \tag{2.7}$$

For inequality (2.7) to be satisfied, it is sufficient for $x^{(r)}$ to be the global maximum point in the problem of mathematical programming

$$F_0(x) - r\Delta F(x) \to \max, \; x \in D. \tag{2.8}$$

Indeed, let $x^{(r)}$ be a solution of the problem (2.8). Then for $r \in [-1, 0]$ and arbitrary $x \in D$ we have

$$F_0(x^{(r)}) - r\Delta F(x^{(r)}) \ge F_0(x) - r\Delta F(x) \ge F_0(x) + r\Delta F(x)$$

which proves the above statement.

By the Weierstrass theorem, for the existence of a solution $x^{(r)}$ of the problem (2.8), it suffices that the objective function be continuous and that the set D be compact (bounded and closed).

The disadvantage of the criteria convolution method is the absence of recommendations on the choice of weights.

Example 3.1. Consider a multi-criteria interval linear programming problem

$$\boldsymbol{C}x \to \max, \; Ax \le b, \; x \ge 0 \tag{2.9}$$

with given matrices $\boldsymbol{C} \in \mathrm{IR}^{k\times n}$, $A \in \mathrm{R}^{m\times n}$, vector $b \in \mathrm{R}^m$ and unknown vector $x \in \mathrm{R}^n$. Polyhedral set

$$D = \{x \in \mathrm{R}^n : Ax \le b,\ x \ge 0\}$$

is considered not empty. Let us put the weight vector $\lambda \in \mathrm{R}^k$, the convolution of the vector criterion $\boldsymbol{F}(x) = \lambda' \boldsymbol{C} x$ and represent the interval matrix $\boldsymbol{C}$ in a symmetric form $\boldsymbol{C} = [C_0 - \Delta C,\ C_0 + \Delta C]$. Due to the non-negativity of the vectors λ, x we have

$$\boldsymbol{F}(x) = [\lambda' C_0 x - \lambda' \Delta C x,\ \lambda' C_0 x + \lambda' \Delta C x].$$

Problem (2.8) will take the form of an "ordinary" linear programming problem

$$\lambda'(C_0 - r\,\Delta C)x \to \max,\ Ax \le b,\ x \ge 0. \tag{2.10}$$

If a linear function $x \to \lambda'(C_0 - r\,\Delta C)x$ on a polyhedral set D is bounded from above then a solution $x^{(r)}$ of the problem (3.10) exists and simultaneously is a solution of the original problem (2.9).

2.3. Principal criterion

Let the first criterion $\boldsymbol{f}_1$ in the problem (2.1) be chosen as the principal one and for the remaining criteria $\boldsymbol{f}_2,\ldots,\boldsymbol{f}_k$ on the set D acceptable lower numerical estimates $b_2,\ldots,b_k$ be established. Then it is natural to treat problem (2.1) as an interval mathematical programming problem

$$y \to \max,\ y \le \boldsymbol{f}_1(x),\ b_i \le \boldsymbol{f}_i(x),\ i = 2,\ldots,k,\ x \in D. \tag{2.10}$$

Let us formalize the meaning of conditions (2.10). We require that the performance indicators of the interval inequalities (2.10) be no less than the numbers $r_i \in [-1,1]$, $i = 1,\ldots,k$. Using formula (1.3), we obtain

$$y \le f_{10}(x) - r_1 \Delta f_1(x),\ b_i \le f_{i0}(x) - r_i \Delta f_i(x),\ i = 2,\ldots,k. \tag{2.11}$$

Based on conditions (2.10), (2.11), we form a deterministic problem of mathematical programming

$$y = f_{10}(x) - r_1 \Delta f_1(x) \to \max,$$
$$b_i \le f_{i0}(x) - r_i \Delta f_i(x),\ i = 2,...,k,\ x \in D \tag{2.12}$$

with parameters $r_1,...,r_k$.

At points $x \in \mathrm{R}^n$ satisfying constraints (2.12) each interval inequality $b_i \le \boldsymbol{f}_i(x)$ is satisfied with probability $\ge \pi(r_i)$, $i = 2,...,k$ and inequality $y \le \boldsymbol{f}_1(x)$ - with probability $\ge \pi(r_1)$. If all probabilities $\pi(r_i)$ are close to 1 then by formulas (1.10) all the corresponding parameters r_i are close to 1. Then the objective function and the right parts of inequalities (2.12) are close to the left ends of the corresponding intervals and the problem (2.12) can be considered similar in meaning to the problem (2.10). In this case, it is natural to take a solution $x^{(r)}$ of the problem (2.12) as a solution of the problem (2.10).

The disadvantage of the method is the assignment of the principal criterion and the lower estimates of the criteria.

2.4. Directed criteria improvement

Let in the problem of multi-criteria optimization (2.1) all criteria $\boldsymbol{f}_i$ on the set *D* be positive: $\boldsymbol{f}_i(D) \subset (0,\infty),\ i = 1,...,k$. The method of directed criteria improvement consists in changing over from the problem (2.1) to a single-criteria optimization problem

$$\beta \to \max,\ \beta \boldsymbol{c}_i \le \boldsymbol{f}_i(x),\ i = 1,...,k,\ x \in D, \tag{2.13}$$

where $\boldsymbol{c}_1,...,\boldsymbol{c}_k$ are given positive intervals from IR.

We proceed to the reduction of the interval problem (2.13). Let $r = (r_1,...,r_k)$ be some fixed vector from the cube $[-1,0]^k$. A point $x \in D$ is considered *admissible* if

$$R(\beta \boldsymbol{c}_i \le \boldsymbol{f}_i(x)) \ge r_i,\ i = 1,...,k.$$

Taking (1.3) into account, we have in an equivalent form

$$f_{i0}(x) - r_i\ \Delta f_i(x) \ge \beta\ c_{i0} + |\beta| r_i \Delta c_i,\ i = 1,...,k.$$

As a result, the interval problem (2.13) is transformed into a deterministic mathematical programming problem

$$\beta \to \max,$$
$$\beta\, c_{i0} + |\beta| r_i \Delta c_i \le f_{i0}(x) - r_i\, \Delta f_i(x),\ i = 1,\dots,k\,,\ x \in D \tag{2.14}$$

with the vector of parameters $r = (r_1,\dots,r_k) \in [-1,0]^k$ and we take the solution $x^{(r)}$ of the problem (2.14) as the solution of multi-criteria problem (2.1).

The complexity of applying the method of directed criteria improvement lies in the assignment of the vector $\boldsymbol{c}$.

2.5. Ideal point

Suppose we are given a vector $\boldsymbol{b} \in \mathrm{IR}^k$ with components

$$\boldsymbol{b}_i = [\underline{b}_i, \overline{b}_i],\ \underline{b}_i \ge \underline{f}_i(x),\ \overline{b}_i \ge \overline{f}_i(x)),\ x \in D,\ i = 1,\dots,k\,.$$

If equality $\boldsymbol{b} = \boldsymbol{f}(x)$ holds for some $x \in D$ then it is natural to consider vector x as a solution to the multi-criteria problem (2.1). In the general case, the vector $\boldsymbol{b}$ does not lie in the range $\boldsymbol{f}(D)$ of vector criterion (2.1) and is some unattainable "ideal". We define the distance $\rho(\boldsymbol{a},\boldsymbol{b})$ between interval vectors $\boldsymbol{a},\boldsymbol{b} \in \mathrm{IR}^k$ by the formula

$$\rho(\boldsymbol{a},\boldsymbol{b}) = \max\left\{ \max_{1\le i\le k} |\underline{a}_i - \underline{b}_i|,\ \max_{1\le i\le k} |\overline{a}_i - \overline{b}_i| \right\}. \tag{2.15}$$

For the solution of the multi-criteria problem (2.1), we take the solution of the problem of interval mathematical programming

$$\rho(\boldsymbol{b},\boldsymbol{y}) \to \min,\ \boldsymbol{y} = \boldsymbol{f}(x),\ x \in D\,. \tag{2.16}$$

Eliminating the unknown $\boldsymbol{y}$ in conditions (2.16) and taking into account formula (2.15) we arrive at an equivalent deterministic mathematical programming problem

$$\rho \to \min,\ |\underline{b}_i - \underline{f}_i(x)| \le \rho,\ |\overline{b}_i - \overline{f}_i(x)| \le \rho,\ i = 1,\dots,k,\ x \in D\,. \tag{2.17}$$

The main difficulty in using the ideal point method is to construct a vector $\boldsymbol{b}$.

2.6. Fuel supply problem

We demonstrate the above approaches to multi-criteria optimization on an applied problem. Substantially, we are talking about the supply of one type of liquid fuel via the railway transport network from source cities to consumer cities. Tariffs, delivery times and restrictions on the throughput of railway sections between cities are known. It is required to find a transportation plan that guarantees the supply of fuel to consumer cities in the required volumes, takes into account the restrictions on the capacity of railways and ensures a minimum of costs (with possible fluctuations in tariffs) and a minimum of the total number of ton-hours.

To formalize the problem, we represent the transport network in the form of a graph the nodes of which correspond to cities and the arcs corresponding to railways between cities.

Let us introduce the following notation:

$P \subset N$ – a set of network sinks (fuel consuming cities);
$Q = P \setminus N$ – set of sources of the network (fuel producing cities);
(i, j) – oriented arc going from node i to node j;
d_{ij} – arc capacity (i, j);
b_k – the difference between the amount of fuel exported from node k and imported into this node (net flow through node k);
t_{ij} − fuel transportation time from node i to node j;
$\boldsymbol{c}_{ij}$ − interval cost of transporting one ton of fuel from node i to node j;
x_{ij} – the amount of tons of fuel delivered from node i to node j.

In these notations, the fuel supply problem takes the form

$$\boldsymbol{f}_1 = \sum_{i \in N} \sum_{j \in N} \boldsymbol{c}_{ij} x_{ij} \to \min, \; f_2 = \sum_{i \in N} \sum_{j \in N} t_{ij} x_{ij} \to \min, \tag{2.18}$$

$$\sum_{j \in N} x_{kj} - \sum_{i \in N} x_{ik} = b_k, \; k \in N, \; 0 \le x_{ij} \le d_{ij}, \; i, j \in N. \tag{2.19}$$

Criteria (2.18) have the meaning of costs and the total number of ton-hours for fuel transportation, respectively. Conditions (2.19) describe the balances of fuel transportation through the network nodes and restrictions on the throughput of the arcs. By sense, $b_k < 0$ with $k \in P$ and $b_k > 0$ with $k \in Q$.

Problem (2.18), (2.19) was solved for a transport network with sets of nodes $N = \{1,...,13\}$, $Q = \{9,13\}$ (Fig. 2.1). Interval coefficients $\boldsymbol{c}_{ij} = \boldsymbol{c}_{ij0}[0.9, 1.1]$, $i, j = 1,...,13$ take into account 10% fluctuations in fuel transportation tariffs. Values $\boldsymbol{c}_{ij0}$ and coefficients t_{ij} are given in Table 2.1. The balance relations for each network node are compiled according to the formulas (2.19).

When solving the problem, the method of convolution, the principal criterion, the directed improvement of criterion, and the ideal point were used. The parameters of the methods and the results obtained are given in Table 2.2. To select the best of the four solutions, we compose from the normalized minima of the criteria (the last two rows of Table 2.2) the interval vectors

$$\begin{aligned} &\boldsymbol{F}_1 = ([0.81, 0.99], 0.96),\ \boldsymbol{F}_2 = ([0.82, 1], 0.97),\\ &\boldsymbol{F}_3 = ([0.80, 0.98], 1),\ \boldsymbol{F}_4 = ([0.98, 0.99], 0.96). \end{aligned} \tag{2.20}$$

In the metric (2.15), the distances between the vectors (2.20) and the origin of the criteria space IR^2 are 0.99, 1, 1, 0.99 respectively.

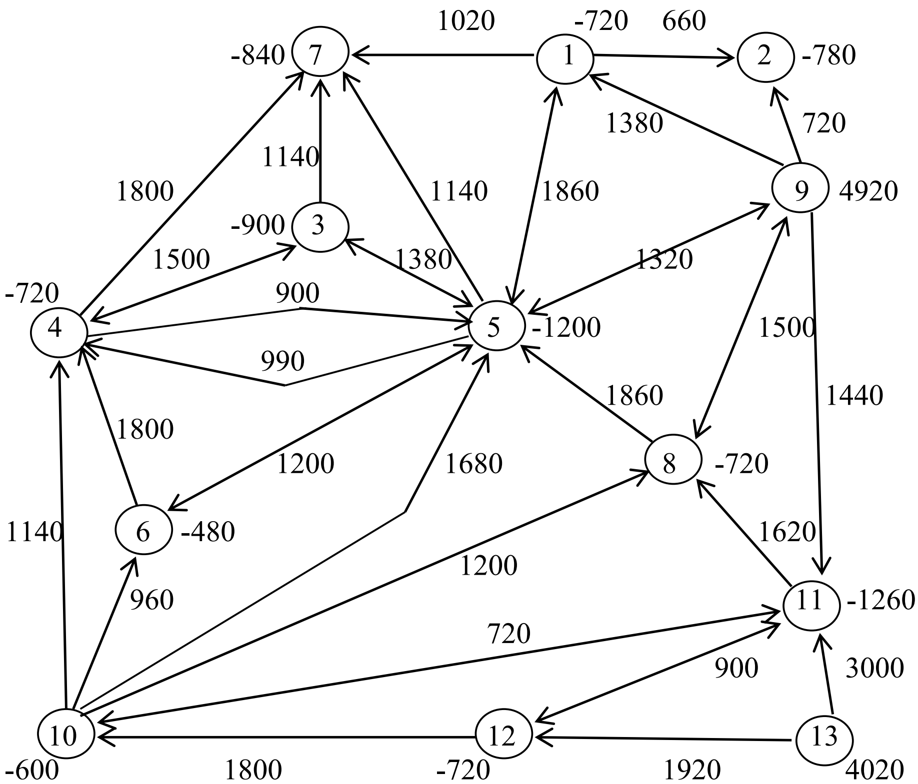

Figure 2.1. Transport network in the problem of fuel supply. Numbered circles denote nodes, one-sided arrows denote oriented arcs, and two-sided arrows denote two oppositely oriented arcs. Net flows are indicated next to the nodes, capacities are indicated next to the arcs.

Table 2.1

Tariffs c_{ij0} (RUB/t) / coefficients t_{ij}

i\j	1	2	3	4	5	6	7	8	9	10	11	12
1		495/2			592/3		1064					
2												
3				731	456		704					
4			731	730	731		1414					
5	592/3		456	731		704	860		1500/5			
6				551	704/3							
7												
8					532/2			1530	1530/5			
9	1264/4	1264/4			1500/5			1530/5				
10				992	860/3	784/4		704/3			860/3	
11								927/3	1456/5	860/3		1127/4
12										897/3	1127/4	
13											704/3	807/4

As one can see, the minimum distance is implemented by the convolution method and the ideal point method. The solution found by the last method is shown in Figure 2.2.

Table 2.2

Results of solving the fuel supply problem

Minima and normalized minima criteria	Name and parameters of the method			
	Convolution of criteria	Principal criterion	Directed criteria improvement	Ideal point
	$\lambda = (0.5, 0.5)$ $r = 0$	$f_2 \to \min, \boldsymbol{f}_1 \leq 17$ $r = 1$	$\boldsymbol{a} = (1,1)$ $r = 1$	$\boldsymbol{b} = (0,0)$
$\boldsymbol{f}_1$, *million rubles*	[13.39, 16.37]	[13.52, 16.53]	[13.20, 16.14]	[13.37, 16.34]
f_2, ton-hour	45000	45630	46920	45000
$\boldsymbol{f}_1$/16.53	[0.81, 0.99]	[0.82, 1.00]	[0.80, 0.98]	[0.98, 0.99]
f_2/46920	0.96	0.97	1	0.96

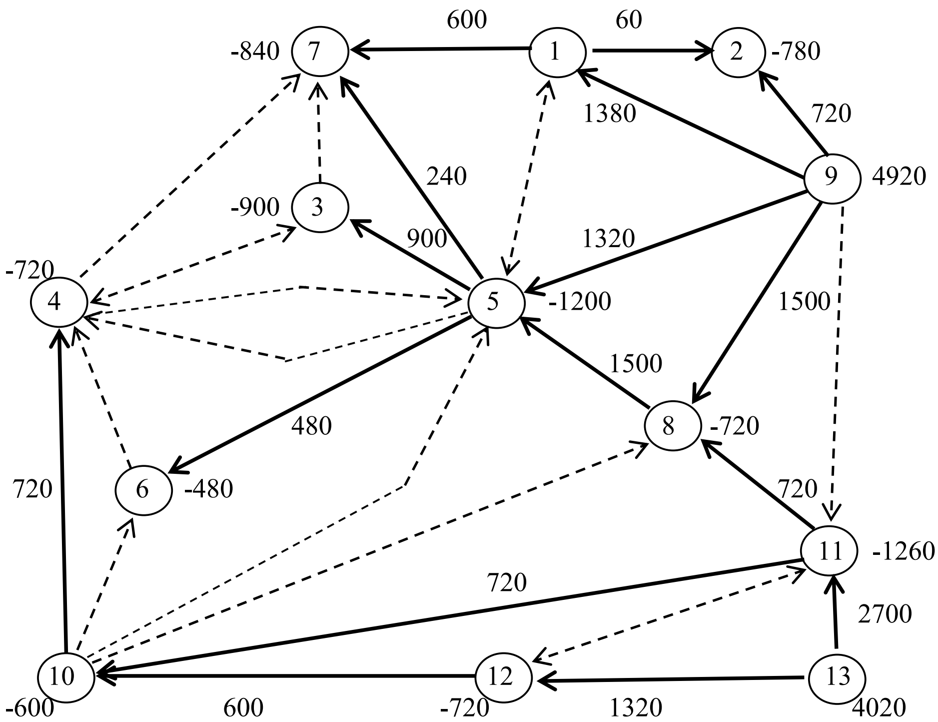

Figure 2.2. Solving the problem of fuel supply by the ideal point method. Bold arrows and numbers next to them show the directions and volumes of fuel transportation

CONCLUSION

In this paper, we considered interval multi-criteria optimization for decision-making models. We introduced the method of partial ordering of real intervals for solving of these optimization problems. With the aid of the indicator of intervals, we converted the initial interval model to a deterministic model of the same type with parameters. Due to this, the solutions of the deterministic model can be meaningfully interpreted in terms of the probability of their implementation, which is important for the final choice of the solution.

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