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GENERALIZED TYPE 2 DEGENERATE THE EULER-GENOCCHI POLYNOMIALS

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ABSTRACT. Recently, Kim-Kim-Kim introduced generalized degenerate Euler-Genocchi polynomials. From this idea we consider generalized type 2 Euler-Genocchi poplynomials as a degenerate version. In this paper, we study some properties and identities of the generalized type 2 degenerate Euler-Genocchi polynomials. In addition, it was expressed as an equation using the Fermionic integral.

1. INTRODUCTION

Many researchers have studied stirling numbers of the second kind, Euler polynomials, Genocchi polynomials, type 2 Euler polynomials and type 2 Genocchi polynomials of degenerate version and Kim-Kim-Kim studied the generalized degenerate Euler-Genocchi polynomials in [5]. From the generalized degenerate Euler-Genocchi polynomials, we study generalized type 2 degenerate the Euler-Genocchi polynomials. The purpose of this paper is to define the generalized type 2 degenerate Euler-Genocchi polynomials and derived some properties and identities of them. And we also, investigate the generalized type 2 degenerate Euler-Genocchi polynomials as Fermionic integral. Furthermore, we obtain some properties and identities between those polynomials ,the degenrate Stirling number of second kind ,the type 2 degenerate Euler polynomials,the degenrate Genocchi polynomials,the type 2 degenerate Genocchi polynomials, degenerate Euler-Genocchi polynomials and special polynomials.

Let p be a fixed odd prime number. In this paper, $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers,the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm is denoted as $|p|_p = \frac{1}{p}$.

The Fermionic p-adic integral of f on $\mathrm{Z_p}$ was introduced by Kim as

(1)
$$\int_{\mathrm{Z}_p} f(x)d_{\mu_{-1}}(x) = \lim_{N\longrightarrow\infty}\sum_{x=0}^{p^N-1} f(x)\mu_{-1}(x+p^N\mathrm{Z}_p) = \lim_{N\longrightarrow\infty}\sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad (\text{see}[1,9,10,13]).$$

From (1), we note that

(2)
$$\int_{\mathrm{Z}_p} f(x+1)d_{\mu_{-1}} + \int_{\mathrm{Z}_p} f(x)d_{\mu_{-1}} = 2f(0), \quad (\text{see}[1,9,10,13]).$$

For any $\lambda \neq 0 \in \mathbb{R}$, the degenerate exponential functions $e_\lambda^x(t)$ are defined by

(3)
$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \quad (\text{see}[1-12]).$$

where $(x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), (n \geq 1)$.

The Stirling numbers of second kind $S_{2,\lambda}(n,k)$ are defined as

$$x^n=\sum_{l=0}^{n}S_2(n,l)(x)_l,(n\geq 0),\quad (\text{see}[1-12]). \tag{4}$$

From (4),we note that

$$\frac{1}{k!}(e^t-1)^k=\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!},(k\geq 0),\quad (\text{see}[1-12]). \tag{5}$$

The degenerate Stirling numbers of the second kind are defined by

$$(x)_{n,\lambda}=\sum_{k=0}^{n}S_{2,\lambda}(n,k)(x)_{k,\lambda},(n\geq 0),\quad (\text{see}[1-12]). \tag{6}$$

From (6),we note that

$$\frac{1}{k!}(e_\lambda(t)-1)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!}. \tag{7}$$

It is well known that the Euler polynomials $E_n(x)$ are defined by the generating functions to be

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad (\text{see}[2,3,7,10]). \tag{8}$$

When $x=0$, $E_n=E_n(0)$ are called the Euler numbers.

The degenerate Euler polynomials $E_{n,\lambda}(x)$ are defined by

$$\frac{2}{e_\lambda(t)+1}e_\lambda^x(x)=\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!},\quad (\text{see}[2,3,7,10]). \tag{9}$$

The degenerate Genocchi polynomials $G_{n,\lambda}(x)$ are defined by

$$\frac{2t}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}G_{n,\lambda}(x)\frac{t^n}{n!},\quad (\text{see}[3,4,5,8,11]). \tag{10}$$

The degenerate type 2 Euler polynomials $\varepsilon^*_{n,\lambda}(x)$ are given by

$$\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t)=\sum_{n=0}^{\infty}\varepsilon^*_{n,\lambda}(x)\frac{t^n}{n!},\quad (\text{see}[7,14]). \tag{11}$$

When $x=0$ $\varepsilon^*_{n,\lambda}=\varepsilon^*_{n,\lambda}(0)$ are called the degenerate type 2 Euler numbers. The degnerate type 2 Euler polynomials

$$\left(\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}\right)^r e_\lambda^x(t)=\sum_{n=0}^{\infty}\varepsilon^{*(r)}_{n,\lambda}(x)\frac{t^n}{n!},\quad (\text{see}[7,14]). \tag{12}$$

The modified degenerate polyexponential funcion $Ei_{k,\lambda}(x)$ which is given by

$$Ei_{k,\lambda}(x)=\sum_{n=1}^{\infty}\frac{x^n(1)_{n,\lambda}}{(n-1)!n^k},(k\in\mathbb{Z}),\quad (\text{see}[8,11]). \tag{13}$$

The degenerate type 2 poly-Genocchi polynomials $G^*_{n,\lambda}(x)$ are defined by

$$\frac{2Ei_{k,\lambda}(log_\lambda(1+t))}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t)=\sum_{n,\lambda}G^{*(k)}_{n,\lambda}(x)\frac{t^n}{n!},\quad (\text{see}[8,11]). \tag{14}$$

When $x = 0, G^{*(k)}_{n,\lambda} = G^{*(k)}_{n,\lambda}(0)$ are called the type 2 degenerate poly-Genocchi numbers. The type 2 degenerate Genocchi numbers of order r are defined by

$$\left(\frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}\right)^r e_\lambda^x(t) = \sum_{n,\lambda}^{\infty} G^{*(r)}_{n,\lambda}(x)\frac{t^n}{n!}, \quad (\text{see}[8,11]). \tag{15}$$

Note that $G^{*(r)}_{n,\lambda}(x) = G^{*(r)}_{n,\lambda}(0)$ are called the type 2 degenerate Genocchi numbers of order r The generalized Euler-Genocchi polynomials $A^{(r)}_{n,\lambda}(x)$ are given by

$$\frac{2t^r}{e_\lambda(t)+1}e_\lambda^x(t) = \sum_{n=0}^{\infty} A^{(r)}_{n,\lambda}(x)\frac{t^n}{n!}, \quad (\text{see}[5]). \tag{16}$$

When $x = 0, A^{(r)}_{n,\lambda} = A^{(r)}_{n,\lambda}(0)$ are called Euler-Genocchi numbers.

Note that we take $f(x) = e_\lambda^{2x}(t)$, then $f(0) = 1$ and $f(x+1) = e_\lambda^{2(x+1)}(t) = e_\lambda^2(t)e_\lambda^{2x}(t)$.
Thus. by (2) we have

$$\int_{Z_p} e_\lambda^{2x}(t)d_{\mu_{-1}} = \frac{2}{e_\lambda^2(t)+1} \tag{17}$$

From (17), we observe that

$$\begin{aligned} \frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t) &= \int_{Z_p} e_\lambda^{(2y+x+1)}(t)d_{\mu_{-1}}(y) \\ &= \sum_{n=0}^{\infty}\int_{Z_p}(x+2y+1)_{n,\lambda}d_{\mu_{-1}}(y)\frac{t^n}{n!}, \quad (\text{see}[9,10,13]). \end{aligned} \tag{18}$$

Thus, by (2) and (18),we get

$$\varepsilon^*_{n,\lambda}(x) = \int_{Z_p}(x+2y+1)_{n,\lambda}d_{\mu_{-1}}(y), \quad (\text{see}[9,10,13]). \tag{19}$$

2. Generalized of the type2 degenerate Euler-Genocchi polynomials and numbers

For any r$\in \mathbb{Z}$, we consider the generalized type 2 degenerate Genocchi-Euler polynomials given by

$$\frac{2t^r}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t) = \sum_{n=0}^{\infty} A^{*(r)}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{20}$$

When $x = 0, A^{*(r)}_{n,\lambda} = A^{*(r)}_{n,\lambda}(0)$ are called the generalized type 2 Genocchi-Euler numbers.

we observe that sepcial value in (20) Then we note that if $r = 0$,we get

$$\sum_{n=0}^{\infty} A^{*(0)}_{n,\lambda}(x)\frac{t^n}{n!} = \frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t) = \sum_{n=0}^{\infty}\varepsilon^*_{n,\lambda}(x)\frac{t^n}{n!}.$$

That is, $A^{*(0)}_{n,\lambda}(x) = \varepsilon^*_{n,\lambda}(x)$. where $\varepsilon^*_{n,\lambda}(x)$ is the type 2 degenerate Euler polynomials, and if $r = 1$,we get

$$\sum_{n=0}^{\infty} A^{*(1)}_{n,\lambda}(x)\frac{t^n}{n!} = \frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t) = \sum_{n=0}^{\infty} G^*_{n,\lambda}(x)\frac{t^n}{n!}.$$

That is, $A^{*(1)}_{n,\lambda}(x) = G^*_{n,\lambda}(x)$, where $G^{*(0)}_{n,\lambda}$ is the degenerate type 2 poly-Genocchi polynomials with $k = 0$.

By (20),we have

$$\sum_{n=0}^{\infty} A_{n,\lambda}^{*(r)}(x+1)\frac{t^n}{n!} = \frac{2t^r}{e_\lambda(t)+e_\lambda^{-1}(t)} e_\lambda^x(t) e_\lambda(t) \tag{21}$$
$$= \sum_{m=0}^{\infty} A_{m,\lambda}^{*(r)}(x)\frac{t^m}{m!}\sum_{l=0}^{\infty}(1)_{l,\lambda}\frac{t^l}{l!}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}(1)_{n-m,\lambda}A_{m,\lambda}^{*(r)}(x)\frac{n!}{m!(n-m)!}\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}(1)_{n-m,\lambda}A_{m,\lambda}^{*(r)}(x)\frac{t^n}{n!}.$$

Thus by comparing the both sides (21), we have the following theorem.

Theorem 1. *For $n \geq 0$, we have*

$$A_{n,\lambda}^{*(r)}(x+1) = \sum_{m=0}^{n}\binom{n}{m}(1)_{n-m,\lambda}A_{m,\lambda}^{*(r)}(x).$$

By (20), we get

$$\sum_{n=0}^{\infty} A_{n,\lambda}^{*(r)}(x)\frac{t^n}{n!} = \frac{2t^r}{e_\lambda(t)+e_\lambda^{-1}(t)} e_\lambda^x(t) \tag{22}$$
$$= \sum_{m=0}^{\infty} A_{m,\lambda}^{*(r)}\frac{t^m}{m!}\sum_{l=0}^{\infty}(x)_{n,\lambda}\frac{t^l}{l!}$$
$$= \sum_{m=0}^{\infty} A_{m,\lambda}^{*(r)}\frac{t^m}{m!}\sum_{l=0}^{\infty}\sum_{k=0}^{l}S_{2,\lambda}(l,k)(x)_k\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}A_{n-l,\lambda}^{*(r)}S_{2,\lambda}(l,k)(x)_k\frac{t^n}{n!}.$$

By comparing the coefficients of both side (22), we have the following theorem.

Theorem 2. *For $n \geq 0$, we have*

$$A_{n,\lambda}^{*(r)}(x) = \sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}A_{n-l,\lambda}^{*(r)}S_{2,\lambda}(l,k)(x)_k.$$

we observe (18) and (20) , then we have

$$\sum_{n=0}^{\infty} A_{n,\lambda}^{*(r)}(x)\frac{t^n}{n!} = \frac{2t^r}{e_\lambda(t)+e_\lambda^{-1}(t)} e_\lambda^x(t) \tag{23}$$
$$= \int_{\mathbb{Z}_p} e_\lambda^{2y+1}(t)d\mu_{-1}(y)t^r e_\lambda^x(t)$$
$$= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(2y+x+1)_{n,\lambda}d\mu_{-1}(y)\frac{t^{n+r}}{n!}$$
$$= \sum_{n=r}^{\infty}\int_{\mathbb{Z}_p}(2y+x+1)_{n-r,\lambda}d\mu_{-1}(y)(n)_r\frac{t^n}{n!}.$$

Thus by comparing the both sides (23),we have the following theorem.

Theorem 3. *For $n \geq 0$, we have*

$$A_{n,\lambda}^{*(r)}(x) = \begin{cases} \int_{\mathbb{Z}_p}(2y+x+1)_{n-r,\lambda} d\mu_{-1}(y)(n)_r, & \text{if } n \geq r, \\ 0, & \text{if } 0 < n < r. \end{cases}$$

By (20), we have

(24)
$$\begin{aligned} \sum_{n=0}^{\infty} A_{n,\lambda}^{*(r)}(x)\frac{t^n}{n!} &= \frac{2t^r}{e_\lambda(t)+e_\lambda^{-1}(t)} e_\lambda^x(t) \\ &= \int_{\mathbb{Z}_p} e_\lambda^{2y+1}(t) d\mu_{-1}(y) t^r \sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!} \\ &= \sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}(2y+1)_{n,\lambda} d\mu_{-1}(y)\frac{t^m}{m!}\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!} \\ &= \sum_{m=0}^{\infty}\varepsilon_{m,\lambda}^{*}\frac{t^m}{m!}\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}(x)_{n-m,\lambda}\varepsilon_{m,\lambda}^{*}\frac{t^n}{n!}. \end{aligned}$$

Thus by comparing the coefficients of both sides (24), we have following theorem.

Theorem 4. *For $n \geq 0$, we have*

$$A_{n,\lambda}^{*(r)}(x) = \sum_{m=0}^{n}\binom{n}{m}(x)_{n-m,\lambda}\varepsilon_{m,\lambda}^{*}.$$

By (20), we have

(25)
$$e_\lambda^{x+1}(t) = \frac{e_\lambda^2(t)+1}{2t^2}\sum_{n=0}^{\infty}A_{n,\lambda}^{*(r)}(x)\frac{t^n}{n!}$$

From (25), we get

(26)
$$\begin{aligned} \sum_{n=0}^{\infty}(x+1)_{n,\lambda}\frac{t^n}{n!} &= \frac{1}{2}\left(\sum_{m=0}^{\infty}A_{m,\lambda}^{*(r)}(x)\frac{t^{m-r}}{m!}\right)\sum_{l=0}^{\infty}(2)_{l,\lambda}\frac{t^l}{l!} + \frac{1}{2}\sum_{n=0}^{\infty}A_{n,\lambda}^{*(r)}(x)\frac{t^{n-r}}{n!} \\ &= \frac{1}{2}\left(\sum_{m=0}^{\infty}A_{m+r,\lambda}^{*(r)}(x)\frac{m!}{(m+r)!}\frac{t^m}{m!}\right)\sum_{l=0}^{\infty}(2)_{l,\lambda}\frac{t^l}{l!} + \frac{1}{2}\sum_{n=0}^{\infty}A_{n+r,\lambda}^{*(r)}(x)\frac{n!}{(n+r)!}\frac{t^n}{n!} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{m!}{(m+r)!}(2)_{n-m,\lambda}A_{m+r,\lambda}^{*(r)}(x)\frac{t^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty}A_{n+r,\lambda}^{*(r)}(x)\frac{n!}{(n+r)!}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{1}{2}\left(\sum_{m=0}^{n}\binom{n}{m}(2)_{n-m,\lambda}\frac{A_{n+r,\lambda}^{*(r)}(x)}{(m+r)_r} + \frac{A_{n+r,\lambda}^{*(r)}(x)}{(n+r)_r}\right)\frac{t^n}{n!}. \end{aligned}$$

Thus by comparing the coefficients both sides in (26), we have the following theorem.

Theorem 5. *For $n \geq 0$,we have*

$$(x+1)_{n,\lambda} = \left(\sum_{m=0}^{n} \binom{n}{m} (2)_{n-m,\lambda} \frac{A^{*(r)}_{n+r,\lambda}(x)}{(m+r)_r} + \frac{A^{*(r)}_{n+r,\lambda}(x)}{(n+r)_r} \right).$$

For nonzero $\alpha \in \mathbb{C}$ and $r \in \mathbb{Z}$,we consider the generalized type 2 degenerate the Euller-Genocchi polynomials of order α which are given by

$$t^r \left(\frac{2}{e_\lambda(t) + e_\lambda^{-1}(t)} \right)^{\alpha} e_\lambda^x(t) = \sum_{n=0}^{\infty} A^{*(r,\alpha)}_{n,\lambda}(x) \frac{t^n}{n!}. \tag{27}$$

When $x = 0$, $A^{*(r,\alpha)}_{n,\lambda} = A^{*(r,\alpha)}_{n,\lambda}(0)$ are called the generalized type 2 degenerate numbers of order α. By (27), we have

$$\begin{aligned} \sum_{n=0}^{\infty} A^{*(r,\alpha)}_{n,\lambda}(x) \frac{t^n}{n!} &= \sum_{m=0}^{\infty} A^{*(r,\alpha)}_{m,\lambda} \frac{t^m}{m!} \sum_{l=0}^{\infty} (x)_{l,\lambda} \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} A^{*(r,\alpha)}_{m,\lambda}(x)_{n-m,\lambda} \frac{n!}{m!(n-m)!} \frac{t^n}{n!} \\ &= \sum_{n} \sum_{m=0}^{n} \binom{n}{m} A^{*(r,\alpha)}_{n,\lambda}(x)_{n-m,\lambda} \frac{t^n}{n!}. \end{aligned} \tag{28}$$

Thus comparing the coefficients of both sides in (28),we have the following theorem.

Theorem 6. *For $n \geq 0$,we have*

$$A^{*(r,\alpha)}_{n,\lambda}(x) = \binom{n}{m} A^{*(r,\alpha)}_{m,\lambda}(x)_{n-m,\lambda}.$$

Let $\alpha = -m (m \in \mathbb{N})$. Then we have

$$\begin{aligned} \sum_{n=0}^{\infty} A^{*(r,-m)}_{n,\lambda} \frac{t^n}{n!} &= t^r \left(\frac{2}{e_\lambda(t) + e_\lambda^{-1}(t)} \right)^{-m} e_\lambda^x(t) \\ &= \frac{t^r}{2^m} (e_\lambda(t) + e_\lambda^{-1}(t))^m e_\lambda^x(t) \\ &= \frac{t^r}{2^m} (e_\lambda^2 + 1)^m e_\lambda^{x-m}(t) \\ &= \frac{t^r}{2^m} \sum_{k=0}^{m} \binom{m}{k} e_\lambda^{2k+x-m}(t) \\ &= \frac{1}{2^m} \sum_{k=0}^{m} \binom{m}{k} \sum_{l=0}^{\infty} (2k+x-m)_{l,\lambda} \frac{t^{l+r}}{l!} \\ &= \sum_{n=r}^{\infty} \frac{1}{2^m} \sum_{k=0}^{\infty} \binom{m}{k} (2k+x-m)_{n-r}(n)_r \frac{t^n}{n!}. \end{aligned} \tag{29}$$

Thus comparing the coefficientes of both sides in (29),we have the following theorem.

Theorem 7. *For $n \geq 0$,we have*

$$A^{*(r,-m)}_{n,\lambda}(x) = \begin{cases} \frac{1}{2^m} \sum_{k=0}^{m} \binom{m}{k} (2k+x-m)_{n-r}(n)_r, & \textit{if } n \geq r, \\ 0, & \textit{if } 0 < n < r. \end{cases}$$

By (27), we get

$$\sum_{n=0}^{\infty} A_{n,\lambda}^{*(r,\alpha)}(x)\frac{t^n}{n!} = \left(\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}\right)^{\alpha} e_\lambda^x(t) \tag{30}$$
$$= t^r \sum_{m=0}^{\infty} \varepsilon_{m,\lambda}^{*(\alpha)} \frac{t^m}{m!} \sum_{l=0}^{\infty} (x)_{l,\lambda} \frac{t^l}{l!}$$
$$= t^r \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \varepsilon_{m,\lambda}^{*(\alpha)} (x)_{n-m,\lambda} \frac{t^n}{n!}$$
$$= \sum_{n=r}^{\infty} \sum_{m=0}^{n-r} \binom{n-r}{m} (n)_r \varepsilon_{m,\lambda}^{*(\alpha)} (x)_{n-m-r,\lambda} \frac{t^n}{n!}.$$

Thus comparing the coefficients of both sides in (30) we have the following theorem.

Theorem 8. *For $n \geq 0$,we have*

$$A_{n,\lambda}^{*(r,\alpha)}(x) = \begin{cases} \sum_{m=0}^{n-r} \binom{n-r}{m} (n)_r \varepsilon_{m,\lambda}^{*(\alpha)} (x)_{n-m-r,\lambda}, & if\, n \geq r, \\ 0, & if\, 0 < n < r. \end{cases}$$

By (27), we get

$$\sum_{n=0}^{\infty} A_{n,\lambda}^{*(r,\alpha)}(x)\frac{t^n}{n!} = t^{r-\alpha}\left(\frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}\right)^{\alpha} e_\lambda^x(t) \tag{31}$$
$$= t^{r-\alpha} \sum_{m=0}^{\infty} G_{m,\lambda}^{*(\alpha)} \frac{t^m}{m!} \sum_{l=0}^{\infty} (x)_{l,\lambda} \frac{t^l}{l!}$$
$$= \sum_{l=0}^{\infty} (x)_{l,\lambda} \frac{t^l}{l!} \sum_{m=0}^{\infty} G_{m,\lambda}^{*(\alpha)} \frac{t^{m+r-\alpha}}{m!}$$
$$= \sum_{l=0}^{\infty} (x)_{l,\lambda} \frac{t^l}{l!} \sum_{m=r-\alpha}^{\infty} G_{m-r+\alpha,\lambda}^{*(\alpha)} \frac{m!}{(m-r+\alpha)!} \frac{t^m}{m!}$$
$$= \sum_{n=r-\alpha}^{\infty} \sum_{l=0}^{n} \binom{n}{l} (x)_{l,\lambda} G_{n-l-r+\alpha}^{*(\alpha)} (n-l)_{r-\alpha} \frac{t^n}{n!}.$$

Thus comparing the coefficients of both sides (31),we have the following theorem.

Theorem 9. *For $n \geq 0$ and $\alpha, r \in \mathbb{Z}, (r \leq \alpha)$, we have*

$$A_{n,\lambda}^{*(r,\alpha)}(x) = \begin{cases} \sum_{l=0}^{n} \binom{n}{l} (x)_{l,\lambda} G_{n-l-r+\alpha}^{*(\alpha)} (n-l)_{r-\alpha}, & if\, n \geq r-\alpha, \\ 0, & if\, 0 < n < r-\alpha. \end{cases}$$

By (18) and (27), we have

(32)

$$
\begin{aligned}
&\sum_{n=0}^{\infty} A_{n,\lambda}^{*(r,\alpha)}(x)\frac{t^n}{n!}\\
&= t^r\left(\int_{\mathbb{Z}_p} e_{\lambda}^{(2y_1+1)}(t)d\mu_{-1}(y_1)\right)\cdots\left(\int_{\mathbb{Z}_p} e_{\lambda}^{(2y_\alpha+1)}(t)d\mu_{-1}(y_\alpha)\right)e_{\lambda}^{x}(t)\\
&= t^r\left(\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e_{\lambda}^{(2y_1+2y_2+\cdots+2y_\alpha+\alpha)}(t)d\mu_{-1}(y_1)d\mu_{-1}(y_2)\cdots d\mu_{-1}(y_\alpha)\right)\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}\\
&= t^r\sum_{m=0}^{\infty}\left(\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(2y_1+\cdots+2y_\alpha+\alpha)_{m,\lambda}d\mu_{-1}(y_1)d\mu_{-1}(y_2)\cdots d\mu_{-1}(y_\alpha)\frac{t^m}{m!}\right)\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(2y_1+\cdots+2y_\alpha+\alpha)_{m,\lambda}d\mu_{-1}(y_1)\cdots d\mu_{-1}(y_\alpha)\right)(x)_{n-m,\lambda}\binom{n}{m}\frac{t^{n+r}}{n!}\\
&= \sum_{n=r}^{\infty}\sum_{m=0}^{\infty}\left(\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(2y_1+\cdots+2y_\alpha+\alpha)_{m,\lambda}d\mu_{-1}(y_1)\cdots d\mu_{-1}(y_\alpha)\right)(x)_{n-m,\lambda}\binom{n}{m}(n)_r\frac{t^n}{n!}.
\end{aligned}
$$

Thus comparing the coefficients of both sides in (32),we have the following theorem.

Theorem 10. *For $n \geq 0$,we have*

$$
\begin{aligned}
&A_{n,\lambda}^{*(r,\alpha)}(x)\\
&= \begin{cases} \sum_{n=r}^{\infty}\sum_{m=0}^{\infty}\left(\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(2y_1+\cdots+2y_\alpha+\alpha)_{m,\lambda}d\mu_{-1}(y_1)\cdots d\mu_{-1}(y_\alpha)\right)(x)_{n-m,\lambda}\binom{n}{m}(n)_r, & \text{if } n \geq r,\\ 0, & \text{if } 0 \leq n < r. \end{cases}
\end{aligned}
$$

3. Conclusion

Recently,Kim-Kim-Kim introduced degenerate Euler-Genocchi polynomials and obtained interesting results. Motivated by these investigations, we defined the type 2 degenerate Euler-Genocchi polynomials in (18) and obtained some properties in Theorem 3 and Theorem 10 and identities in Theorem 1,Theorem 5-7.We derived interesting relations involving these polynomials with Stirling number of second kind in Theorem 2,degenrate Euler polynomials in Theorem 4, degenerate Genocchi polynomials, degenerate type 2 Euler polynomials in Theorme 8,type 2 poly-Genocchi polynomials in Theorem 9 and Fermionic integral in Theorem 10. In addition, we consider high order α of generalized type 2 degenerate Euler-Genocchi polynomials. Likewise, we optained some properties and identities with the special polynomials and Fermionic integral.

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