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A LARGE CLASS IN KÖTHE-TOEPLITZ DUALS OF GENERALIZED CESÀRO DIFFERENCE SEQUENCE SPACES WITH FIXED POINT PROPERTY FOR AFFINE ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. In 1983, Orhan introduced Cesàro Difference Sequence Spaces. Later, various authors generalized Cesàro Difference Sequence Spaces and investigated their Köthe-Toeplitz Duals. In this study, we first explain how these spaces were born and discuss the fixed point property for them. Then, we recall that Goebel and Kuczumow showed that there exists a very large class of closed, bounded, convex subsets in Banach space of absolutely summable scalar sequences, ℓ^1 with fixed point property for nonexpansive mappings. In 2004, Kaczor and Prus investigated if similar result could be done for asymptotically nonexpansive mappings and they saw that there exists a large class of closed, bounded, convex subsets in ℓ^1 with fixed point property for affine asymptotically nonexpansive mappings. Everest, in his Ph.D. thesis, written under supervision of Chris Lennard, also obtained large classes of closed, bounded and convex subsets in ℓ^1 with fixed point property for affine asymptotically nonexpansive mappings. Analogous to these facts, we consider a different Banach space and its dual. For each $m \in \mathbb{N}$, we take a generalized Cesàro Difference Sequence Space and consider its Köthe-Toeplitz Dual $D_m := \left\{a = (a_n)_n \subset \mathbb{R} \;\middle|\; (n^m a_n)_n \in \ell^1\right\}$. We show that there exists a large class of closed, bounded and convex subsets E^m of D_m such that every affine asymptotically nonexpansive mapping $T : E^m \to E^m$ has a fixed point in E^m. The case $m = 1$ was earlier done by the first author and İlgar. Thus, we mainly extend their results and Everest's.

1. Introduction and Preliminaries

There is a strong relation between reflexivity and fixed point property for non-expansive mappings. It is an open question whether or not every non-reflexive fails the fixed point property for non-expansive mappings but it was shown by Lin [22] that a non-reflexive Banach space failing to have the fixed point property for non-expansive mappings can be renormed to have the fixed point property for non-expansive mappings. Lin showed this fact by setting an equivalent norm on Banach space of absolutely summable scalar sequences, ℓ^1. Because of sharing many common properties, it is natural to ask if c_0, Banach space of scalar sequences converging to 0, can be renormed to have the fixed point property for non-expansive mappings as another well known classical non-reflexive Banach space. Maria and Hernandes Lineares [23] obtained the first example for the class of nonreflexive Banach spaces which can be renormed to have the fixed point property for affine nonexpansive mappins and their space was the Banach space of Lebesgue integrable functions on [0,1], $L_1[0, 1]$. It can be said that all these works are inspired

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by the work of Goebel and Kuczumow [16]. Goebel and Kuczumow showed that there exists very large class of non-weakly compact, closed, bounded and convex subsets of ℓ^1 respect to weak* topology of ℓ^1 with fixed point property for non-expansive mappings. Later, Kaczor and Prus [18] investigated if similar result could be done for asymptotically nonexpansive mappings and they saw that there exists a large class of closed, bounded, convex subsets in ℓ^1 with fixed point property for affine asymptotically non-expansive mappings. Moreover, Everest, in his Ph.D. thesis [14], written under supervision of Chris Lennard, considered large classes in ℓ^1 with fixed point property for affine asymptotically non-expansive mappings by generalizing Kaczor and Prus' work.

In this study, we work on Kaczor and Prus analogy for a Banach space contained in ℓ^1, which is a smaller space than ℓ^1. The space we consider is a Köthe-Toeplitz Dual of a generalized Cesàro Difference Sequence Space. We show that there exists a very large class of closed, bounded and convex subsets of the space with the fixed point property for asymptotically non-expansive mappings under affinity condition.

We recall that the Cesàro sequence spaces

$$\mathrm{ces}_p = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty \right\}$$

and

$$\mathrm{ces}_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| \; < \infty \right\}$$

were introduced by Shiue [30] in 1970, where $1 \leq p < \infty$. It has been shown that $\ell^p \subset \mathrm{ces}_p$ for $1 < p \leq \infty$. Moreover, it has been shown that Cesàro sequence spaces ces_p for $1 < p < \infty$ are seperable reflexive Banach spaces. Furthermore, it was also proved by CuiHudzik [9], Cui-Hudzik-Li [10] and Cui-Meng-Pluciennik [11] that Cesàro sequence spaces ces_p for $1 < p < \infty$ have the fixed point property. They prove this result using different methods. One method is to calculate Garcia-Falset coefficient. It is known that if Garcia-Falset coefficient is less than 2 for a Banach space, then it has the fixed point property for nonexpansive mappings [15]. Using this fact, since they calculate this coeeficient for ces_p as $2^{1/p}$ similary to what it is for ℓ^p, they point the result for the Cesàro sequence spaces. Another fact is that they see that the space has normal structure for $1 < p < \infty$. Then using the fact via Kirk [20] that reflexive Banach spaces with normal structure has the fixed point property, they easily deduce that the space has the fixed point property for $1 < p < \infty$. Their results on Cesàro sequence spaces as a survey can be seen in [8].

Later, in 1981, Kızmaz [19] introduced difference sequence spaces for ℓ^∞, c and c_0 where they are the Banach spaces of bounded, convergent and null sequences $x = (x_n)_n$, respectively. As it is seen below, his definitions for these spaces were given using difference operator applied to the sequence x, $\triangle\, x = (x_k - x_{k+1})_k$.

$$\ell^\infty(\triangle) = \left\{ x = (x_n)_n \subset \mathbb{R} \mid \triangle\, x \in \ell^\infty \right\},$$

$$\mathrm{c}(\triangle) = \left\{ x = (x_n)_n \subset \mathbb{R} \mid \triangle\, x \in \mathrm{c} \right\},$$

$$\mathrm{c}_0(\triangle) = \left\{ x = (x_n)_n \subset \mathbb{R} \mid \triangle\, x \in \mathrm{c}_0 \right\}.$$

Kızmaz investigated Köthe-Toeplitz Duals and some properties of these spaces.

Furthermore, Cesàro sequence spaces X^p of non-absolute type were defined by Ng and Lee [27] in 1977 as follows:

$$X^p = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|^p \right)^{1/p} < \infty \right\}$$

and

$$X^\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right| < \infty \right\},$$

where $1 \leq p < \infty$. They prove that X^p is linearly isomorphic and isometric to ℓ^p for $1 \leq p \leq \infty$. Thus, one would easily deduce that they have similar properties in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail.

Later, in 1983, Orhan [28] introduced Cesàro Difference Sequence Spaces by the following definitions:

$$C_p = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle\, x_k \right|^p \right)^{1/p} < \infty \right\}$$

and

$$C_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \triangle\, x_k \right| < \infty \right\},$$

where $1 \leq p < \infty$. He noted that their norms are given as below for any $x = (x_n)_n$:

$$\|x\|_p^* = |x_1| + \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle\, x_k \right|^p \right)^{1/p} \quad \text{and} \quad \|x\|_\infty^* = |x_1| + \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \triangle\, x_k \right|$$

respectively.

Orhan showed that there exists a linear bounded operator $S : C_p \to C_p$ for $1 \leq p \leq \infty$ such that Köthe-Toeplitz $\beta-$Duals of these spaces are given respectively as follows:

$$S(C_p)^\beta = \{ a = (a_n)_n \subset \mathbb{R} \mid (na_n)_n \in \ell^q \}, \text{ where } 1 < p < \infty \text{ and } q = \frac{p}{p-1},$$

$$S(C_1)^\beta = \{ a = (a_n)_n \subset \mathbb{R} \mid (na_n)_n \in \ell^\infty \} \text{ and}$$
$$S(C_\infty)^\beta = \left\{ a = (a_n)_n \subset \mathbb{R} \;\middle|\; (na_n)_n \in \ell^1 \right\}.$$

It might be better to use the notation $X^p(\triangle)$ instead of C_p for $1 \leq p \leq \infty$ since we also recalled the difference sequence spaces and used similar type of notation.

We note that Orhan also proved that $X^p \subset X^p(\triangle)$ for $1 \leq p \leq \infty$ strictly. Also, one can clearly see that $X^p(\triangle)$ is linearly isomorphic and isometric to ℓ^p for $1 \leq p \leq \infty$. Thus, one would easily deduce that they have similar properties in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail.

Note also that Köthe-Toeplitz Dual for $p = \infty$ case in Orhan's study and ℓ^∞ case in Kızmaz study coincide.

Furthermore, Et and Çolak [12] generalized the spaces introduced in Kızmaz's work [19] in the following way for $m \in \mathbb{N}$.

$$\ell^{\infty}\left(\Delta^{m}\right)=\left\{x=\left(x_{n}\right)_{n} \subset \mathbb{R} \mid \Delta^{m} x \in \ell^{\infty}\right\},$$

$$\mathrm{c}\left(\Delta^{m}\right)=\left\{x=\left(x_{n}\right)_{n} \subset \mathbb{R} \mid \Delta^{m} x \in \mathrm{c}\right\},$$

$$\mathrm{c}_{0}\left(\Delta^{m}\right)=\left\{x=\left(x_{n}\right)_{n} \subset \mathbb{R} \mid \Delta^{m} x \in \mathrm{c}_{0}\right\}$$

where $\Delta\, x=(\Delta\, x_k)=(x_k-x_{k+1})_k$, $\Delta^0 x=(x_k)_k$, $\Delta^m x=(\Delta^m x_k)=(\Delta^{m-1}x_k-\Delta^{m-1}x_{k+1})_k$ and $\Delta^m x_k=\sum_{i=0}^{m}(-1)^i\binom{m}{i}x_{k+i}$.

Also, Et [13] and Tripathy et. al. [31] generalized the space introduced by Orhan in the following way for $m \in \mathbb{N}$.

$$X^{p}\left(\Delta^{m}\right)=\left\{x=\left(x_{n}\right)_{n} \subset \mathbb{R} \;\middle|\; \left(\sum_{n=1}^{\infty}\left|\frac{1}{n} \sum_{k=1}^{n} \Delta^{m} x_{k}\right|^{p}\right)^{1/p}<\infty\right\}$$

and

$$X^{\infty}\left(\Delta^{m}\right)=\left\{x=\left(x_{n}\right)_{n} \subset \mathbb{R} \;\middle|\; \sup_{n}\left|\frac{1}{n} \sum_{k=1}^{n} \Delta^{m} x_{k}\right|<\infty\right\},$$

Then, it is seen that that Köthe-Toeplitz Dual for $p=\infty$ case in Et's study [13] and ℓ^{∞} case in Et and Çolak study [12] coincide such that Köthe-Toeplitz Dual was given as below for any $m \in \mathbb{N}$.

$$D_{m}:=\left\{a=\left(a_{n}\right)_{n} \subset \mathbb{R} \mid \left(n^{m} a_{n}\right)_{n} \in \ell^{1}\right\}$$

$$=\left\{a=\left(a_{k}\right)_{k} \subset \mathbb{R}\colon \|a\|=\sum_{k=1}^{\infty} k^{m}\left|a_{k}\right|<\infty\right\}.$$

Note that $D_m \subset \ell^1$ for any $m \in \mathbb{N}$. Moreover, there are other types of generalized difference sequence spaces (see, for instances, [3], [4], [5], [7], [21]).

One can see that corresponding function space for these duals can be given as below:

$$U_{m}:=\left\{\text{Lebesgue measurable functions } f \text{ on } \mathrm{I}=[0,1]: \ \|f\|=\int_{0}^{1} t^{m}|f(t)|\, dt<\infty\right\}.$$

Note that $L_1[0,1] \subset U_m$ and D_m is the space when counting measure is used for U_m.

As we have already stated, in this study, we consider Kaczor and Prus [18] analogy for a Köthe-Toeplitz Dual of a generalized Cesàro Difference Sequence Space. We show that for any $m \in \mathbb{N}$ there exists a large class of closed, bounded and convex subsets of Köthe-Toeplitz Dual for $X^{\infty}(\Delta^m)$ with fixed point property for affine asymptotically nonexpansive mappings. There are recent important researchers related to these Cesàro Difference Sequences by correlating the difference sequence via Cesàro and weighted means such as [17],[24] and [29] that we find interesting. Moreover, the books ([1], [2], [6]) contain several interesting and useful topics concerning the theory of summability those may be considered further in the spirit of this paper.

Now we provide some preliminaries before giving our main results.

Definition 1.1. *Let* $(\mathrm{X},\|\cdot\|)$ *be a Banach space and* C *is a non-empty closed, bounded, convex subset.*

1. If $T:C\to\mathrm{C}$ *is a mapping such that for all* $\lambda\in[0,1]$ *and for all* $\mathrm{x},\mathrm{y}\in\mathrm{C}$*,* $\mathrm{T}\left((1-\lambda)\,\mathrm{x}+\lambda\ \mathrm{y}\right)=(1-\lambda)\,\mathrm{T(x)}+\lambda\ \mathrm{T(y)}$ *then* T *is said to be an affine mapping.*

2. If $\mathrm{T}:\mathrm{C}\to\mathrm{C}$ *is a mapping such that* $\|\mathrm{T(x)}-\mathrm{T(y)}\|\leq\|\mathrm{x}-\mathrm{y}\|$, for all $\mathrm{x},\mathrm{y}\in\mathrm{C}$ *then* T *is said to be a nonexpansive mapping.*

Also, if for every nonexpansive mapping $\mathrm{T}:\mathrm{C}\to\mathrm{C}$*, there exists* $\mathrm{z}\in\mathrm{C}$ *with* $\mathrm{T(z)}=\mathrm{z}$*, then* C *is said to have the fixed point property for nonexpansive mappings [fpp(ne)].*

3. If $\mathrm{T}:\mathrm{C}\to\mathrm{C}$ *is a mapping such that there exists a sequence of scalars* $(\mathrm{k_n})_{\mathrm{n}\in\mathrm{N}}$ *decreasing to 1 and* $\|\mathrm{T}^n(\mathrm{x})-\mathrm{T}^n(\mathrm{y})\|\leq\mathrm{k_n}\|\mathrm{x}-\mathrm{y}\|$, for all $\mathrm{x},\mathrm{y}\in\mathrm{C}$ *and for all* $\mathrm{n}\in\mathbb{N}$ *then* T *is said to be an asymptotically nonexpansive mapping.*

Also, if for every asymptotically nonexpansive mapping $\mathrm{T}:\mathrm{C}\to\mathrm{C}$*, there exists* $\mathrm{z}\in\mathrm{C}$ *with* $\mathrm{T(z)}=\mathrm{z}$*, then* C *is said to have the fixed point property for asymptotically nonexpansive mappings [fpp(ane)].*

In 1979, Goebel and Kuczumow [16] showed there exists a large class of closed, bounded and convex subsets of ℓ^1 using a key lemma they obtained. Their lemma says that if $\{\mathrm{x_n}\}$ is a sequence in ℓ^1 converging to x in weak-star topology, then for any $\mathrm{y}\in\ell^1$,

$$\mathrm{r\,(y)}=\mathrm{r\,(x)}+\|\mathrm{y}-\mathrm{x}\|_1 \quad \text{where} \quad \mathrm{r\,(y)}=\limsup_{\mathrm{n}}\|\mathrm{x_n}-\mathrm{y}\|_1 \quad .$$

Since Köthe-Toeplitz Dual for $X^\infty\left(\triangle^m\right)$ is contained in ℓ^1 for any $m\in\mathbb{N}$ and in fact it is isometrically isomorphic to ℓ^1, for any $m\in\mathbb{N}$, Goebel and Kuczumow's lemma above (Lemma 3 in [16]) applies in Köthe-Toeplitz Dual for $X^\infty\left(\triangle^m\right)$. We will call this fact ♡.

2. Main Result

In this section, for each $m\in\mathbb{N}$, we consider Kaczor and Prus [18] analogy for a Köthe-Toeplitz Dual $D_m=\{a=(a_n)_n\subset\mathbb{R}\mid(n^m a_n)_n\in\ell^1\}$ of a generalized Cesàro Difference Sequence Space $X^\infty\left(\triangle^m\right)$. We show that for any $m\in\mathbb{N}$, there exists a large class of closed, bounded and convex subsets of Köthe-Toeplitz Dual for $X^\infty\left(\triangle^m\right)$ with fixed point property for affine asymptotically nonexpansive mappings. We note that case $m=1$ was done by Nezir and İlgar in [25]. Thus, this paper generalizes it and here we present the general case for any $m\in\mathbb{N}$. We also note that we are influenced by Ph.D. thesis of Everest [14], written under supervision of Chris Lennard and we adapt some ideas from his thesis.

Example 2.1. *Fix* $m\in\mathbb{N}$ *and* $b\in(0,1)$*. Define a sequence* $(f_n)_{n\in\mathbb{N}}$ *by setting* $f_1:=b\ e_1$*, and* $f_n:=\frac{1}{n^m}e_n$ *for all integers* $n\geq2$ *where the sequence* $(e_n)_{n\in\mathbb{N}}$ *is the canonical basis of both* c_0 *and* ℓ^1*. Then, a closed, bounded, convex subset* $E^{(m)}={E_b}^{(m)}$ *of the Köthe-Toeplitz Dual for* $X^\infty\left(\triangle^m\right)$ *for arbitrary* $m\in\mathbb{N}$ *can be defined as below.*

$$E^{(m)}:=\left\{\sum_{n=1}^{\infty}t_n\ f_n:\quad\forall n\in\mathbb{N},\,t_n\geq0\quad and\quad\sum_{n=1}^{\infty}t_n=1\right\}\quad .$$

The following theorem uses this large class of closed, bounded and convex subsets of D_m for each $m\in\mathbb{N}$.

Theorem 2.2. *For any* $m\in\mathbb{N}$ *and* $b\in(0,1)$*, the set* $E^{(m)}\subset D_m$ *defined as in the example above has the fixed point property for affine asymptotically* $\|.\|$*-nonexpansive mappings.*

Proof. Fix $m \in \mathbb{N}$ and $b \in (0,1)$. Let $T{:}E^{(m)}{\to}E^{(m)}$ be an affine asymptotically nonexpansive mapping. Then, since T is affine, by Lemma 1.1.2 in the Ph.D. thesis of Everest [14] written under supervision of Lennard, there exists a sequence $\left(x^{(n)}\right)_{n\in\mathbb{N}}{\in}E^{(m)}$ such that $\left\|Tx^{(n)}{-}x^{(n)}\right\| \underset{n}{\to}0$. Here we may recall from the previous section that our space D_m is an analogue to ℓ^1 and both itself and its predual have exactly similar results to ℓ^1 and c_0, respectly, in terms of fixed point theory. For example, we may remind to the reader that on bounded subsets, weak star topology on ℓ^1 is equivalent to the coardinatewise convergence, and c_0 is separable. Due to isometry and isomorphism, similar facts apply to D_m for each $m \in \mathbb{N}$ and for its dual respectively. Then, we can say that we apply Banach-Alaoglu theorem and get the unit closed ball in D_m is weak*-sequentially compact. Moreover, we can define the weak* closure of the set $E^{(m)}$ as it is seen below.

$$W^{(m)}{:=}\overline{E^{(m)}}^{w^*}=\left\{\sum_{\mathrm{n=1}}^{\infty}t_n\ f_n:\quad each\quad t_n\geq 0\quad and\quad \sum_{n=1}^{\infty}\ t_n{\leq}1\right\}$$

Then, without loss of generality, passing to a subsequence if necessary, there exists $x{\in}W^{(m)}$ such that $x^{(n)}$ converges to x in weak* topology. Then, by Goebel Kuczumow analogous fact ♡ given as the final sentence of the introduction and preliminaries section, we can define a function $s{:}D_m{\to}[0{,}\infty)$ by

$$s\left(y\right)=\limsup_n\left\|x^{(n)}{-}y\right\|\qquad,\ \ \forall y{\in}D_m$$

and so

$$s\left(y\right)=s\left(x\right)+\left\|x-y\right\|\quad,\ \ \forall y{\in}D_m.$$

Since T is asyymptotically nonexpansive mapping, there exists a decreasing sequence $(\mathrm{k_n})_{\mathrm{n\in N}}$ in $[1,\infty)$ converging to 1 such that $\forall \mathrm{x},y{\in}\mathrm{E}^{(\mathrm{m})}$ and $\forall \mathrm{n} \in \mathbb{N}$,

$$\left\|T^n\mathrm{x}{-}T^n\mathrm{y}\right\|\leq\mathrm{k_n}\left\|\mathrm{x}-\mathrm{y}\right\|.$$

Case 1: $x{\in}E^{(m)}$.

Fix $\mathrm{q} \in \mathbb{N}$ and note that $T^0 = I$, where I is the identity mapping. Then, we have $s(T^qx) = s(x){+}\left\|T^qx{-}x\right\|$ and $\forall \mathrm{n} \in \mathbb{N}$,

$$\begin{aligned} s\left(T^qx\right) &= \limsup_n\left\|T^qx{-}x^{(n)}\right\| \\ &\leq \limsup_n\left\|T^qx{-}T^q\left(x^{(n)}\right)\right\|+\limsup_n\left\|x^{(n)}{-}T^q\left(x^{(n)}\right)\right\| \qquad (2.1)\\ &\leq \mathrm{k_q}\limsup_n\left\|x{-}x^{(n)}\right\|+\limsup_n\left\|x^{(n)}{-}T^q\left(x^{(n)}\right)\right\| \\ &\leq \mathrm{k_q}\limsup_n\left\|x{-}x^{(n)}\right\|+\limsup_n\sum_{j=1}^{q}\left\|T^{j-1}\left(x^{(n)}\right)-T^j\left(x^{(n)}\right)\right\| \\ &\leq \mathrm{k_q}\limsup_n\left\|x{-}x^{(n)}\right\|+\limsup_n\sum_{j=1}^{q}\mathrm{k_{j-1}}\left\|x^{(n)}{-}T\left(x^{(n)}\right)\right\| \\ &= \mathrm{k_q}s(x). \end{aligned}$$

Therefore, $\left\|T^qx{-}x\right\|\leq\left(\mathrm{k_q}-1\right)s(x)$ and so by taking limit as $\mathrm{q}{\to}\infty$, we have $\lim_q\left\|T^qx{-}x\right\|=0$ but then since $\lim_q\left\|TT^qx{-}\mathrm{T}x\right\|\leq\lim_q\mathrm{k_1}\left\|T^qx{-}x\right\|=0$,

$\lim_q \|T^{q+1}x - \mathrm{T}x\| = 0$ and so $T^q x$ converges to x and Tx. Thus, by the uniqueness of limits $Tx{=}x$.

Therefore, $s\,(Tx) = s\,(x) + \|Tx - x\| \le s(x)$ and so $\|Tx - x\| = 0$. Thus, $Tx{=}x$.

Case 2: $x \in W \backslash E^{(m)}$.

Then, x is of the form $\sum_{n=1}^{\infty} \gamma_n f_n$ suchthat $\sum_{n=1}^{\infty} \gamma_n < 1$ and $\gamma_n \ge 0$, $\forall n \in \mathbb{N}$.
Define $\delta := 1 - \sum_{n=1}^{\infty} \gamma_n$ and next define

$$\mathrm{h} := (\gamma_1 + \delta) f_1 + \sum_{n=2}^{\infty} \gamma_n f_n.$$

Then, $\|\mathrm{h} - x\|_1 = \|b\delta e_1\|_1 = b\delta$.
Now fix $y \in E^{(m)}$ of the form $\sum_{n=1}^{\infty} t_n f_n$ such that $\sum_{n=1}^{\infty} t_n = 1$ with $t_n \ge 0$, $\forall n \in \mathbb{N}$.
Then,

$$\begin{aligned}
\|\mathrm{y} - x\| &= \left\| \sum_{k=1}^{\infty} t_k f_k - \sum_{k=1}^{\infty} \gamma_k f_k \right\| = \mathrm{b}\,|t_1 - \gamma_1| + \sum_{k=2}^{\infty} |t_k - \gamma_k| \\
&= b\,|t_1 - \gamma_1| + b \sum_{k=2}^{\infty} |t_k - \gamma_k| + (1-b) \sum_{k=2}^{\infty} |t_k - \gamma_k| \\
&\ge \mathrm{b} \left| \sum_{k=1}^{\infty} t_k - \gamma_k \right| + (1-b) \sum_{k=2}^{\infty} |t_k - \gamma_k| \\
&= b \left| \sum_{k=1}^{\infty} t_k - \sum_{k=1}^{\infty} \gamma_k \right| + (1-b) \sum_{k=2}^{\infty} |t_k - \gamma_k| \\
&= b\,|1 - (1-\delta)| + (1-b) \sum_{k=2}^{\infty} |t_k - \gamma_k|\,.
\end{aligned}$$

Hence,

$$\|\mathrm{y} - x\| \ge b\delta + (1-b) \sum_{k=2}^{\infty} |t_k - \gamma_k| \ge \|\mathrm{h} - x\|\ .$$

Next, we have the following.

$$\begin{aligned}
s\,(h) = s\,(x) + \|\mathrm{h} - x\| &\le s\,(x) + \|T^q \mathrm{h} - x\| = s\,(T^q \mathrm{h}) \\
&= \limsup_n \left\| T^q h - x^{(n)} \right\| \text{ then similarly to the inequality } 2.1 \\
&\le \limsup_n \left\| T^q h - T^q \left(x^{(n)} \right) \right\| + \limsup_n \left\| x^{(n)} - T^q \left(x^{(n)} \right) \right\| \\
&\le \mathrm{k_q} \limsup_n \left\| h - x^{(n)} \right\| + \limsup_n \left\| x^{(n)} - T^q \left(x^{(n)} \right) \right\| \\
&\le \mathrm{k_q} \limsup_n \left\| h - x^{(n)} \right\| + \limsup_n \sum_{j=1}^{q} \left\| T^{j-1} \left(x^{(n)} \right) - T^j \left(x^{(n)} \right) \right\| \\
&\le \mathrm{k_q} \limsup_n \left\| h - x^{(n)} \right\| + \limsup_n \sum_{j=1}^{q} \mathrm{k_{j-1}} \left\| x^{(n)} - T \left(x^{(n)} \right) \right\| \\
&= \mathrm{k_q} s\,(h)\,.
\end{aligned}$$

Hence, $s(h) \leq s(T^q \mathrm{h}) \leq \mathrm{k_q} s(h)$ and so taking limit as $\mathrm{q} \to \infty$, we have $\lim_q \; s(T^q \mathrm{h}) = s(h)$ since $\lim_q \mathrm{k_q} = 1$. That is, $\lim_q \; s(x) + \|T^q \mathrm{h} - x\| = s(x) + \|\mathrm{h} - x\|$ which means

$$\lim_q \; \|T^q \mathrm{h} - x\| = \|\mathrm{h} - x\| . \tag{2.2}$$

Moreover, for any $y \in E^{(m)}$,

$$\begin{aligned}
\|\mathrm{y} - h\| &= \left\| \sum_{k=1}^{\infty} t_k f_k - (\gamma_1 + \delta) f_1 - \sum_{k=2}^{\infty} \gamma_k f_k \right\| \\
&= \left\| \sum_{k=2}^{\infty} (t_k - \gamma_k) f_k - (\gamma_1 + \delta - t_1) f_1 \right\| \\
&= \sum_{k=2}^{\infty} |t_k - \gamma_k| + b |\gamma_1 + \delta - t_1| \\
&= \sum_{k=2}^{\infty} |t_k - \gamma_k| + b \left| \gamma_1 + 1 - \sum_{k=1}^{\infty} \gamma_k - 1 + \sum_{k=2}^{\infty} t_k \right| \\
&\leq \sum_{k=2}^{\infty} |t_k - \gamma_k| + \mathrm{b} \sum_{k=2}^{\infty} |t_k - \gamma_k| \\
&= (1 + \mathrm{b}) \sum_{k=2}^{\infty} |t_k - \gamma_k| \\
&= \frac{1+b}{1-b} (1 - \mathrm{b}) \sum_{k=2}^{\infty} |t_k - \gamma_k| \\
&= \frac{1+b}{1-b} \left[b\delta - b\delta + (1 - \mathrm{b}) \sum_{k=2}^{\infty} |t_k - \gamma_k| \right] \\
&= \frac{1+b}{1-b} \left[b (1 - (1 - \delta)) - b\delta + (1 - \mathrm{b}) \sum_{k=2}^{\infty} |t_k - \gamma_k| \right] \\
&= \frac{1+b}{1-b} \left[b (1 - (1 - \delta)) + (1 - \mathrm{b}) \sum_{k=2}^{\infty} |t_k - \gamma_k| - b\delta \right] \\
&= \frac{1+b}{1-b} \left[b \left(\sum_{k=1}^{\infty} t_k - \sum_{k=1}^{\infty} \gamma_k \right) + (1 - \mathrm{b}) \sum_{k=2}^{\infty} |t_k - \gamma_k| - b\delta \right] \\
&\leq \frac{1+b}{1-b} \left[b \sum_{k=1}^{\infty} |t_k - \gamma_k| + (1 - \mathrm{b}) \sum_{k=2}^{\infty} |t_k - \gamma_k| - b\delta \right] .
\end{aligned}$$

Hence,

$$\|\mathrm{y} - h\| \leq \frac{1+b}{1-b} \left[b |t_1 - \gamma_1| \; + \sum_{k=2}^{\infty} |t_k - \gamma_k| - b\delta \right] = \frac{1+b}{1-b} \left[\|\mathrm{y} - x\| \; - \|\mathrm{h} - x\| \right] .$$

Now, fix $\varepsilon > 0$ and recall that $b \in (0,1)$. Then, we can choose $\mu(\varepsilon) := \frac{1-b}{1+b}\varepsilon \in (0,\infty)$ such that for any $y = \sum_{k=1}^{\infty} t_k f_k \in E^{(m)}$,

$$|\|y-x\| - \|h-x\|| \leq \|y-x\| - \|h-x\| < \mu.$$

Then, $\|y-h\| < \frac{1+b}{1-b}\mu = \varepsilon$.

Hence, for every $\varepsilon > 0$, there exists $\mu = \mu(\varepsilon)$ such that if $|\|y-x\| - \|h-x\|| < \mu$ then $\|y-h\| < \varepsilon$ so this implies for any sequence $(z_n)_n$ in $E^{(m)}$ with $\lim_n \|z_n - x\| = \|h-x\|$ implies $\lim_n \|z_n - h\| = 0$. But then since in 2.2 we obtained $\lim_q \|T^q h - x\| = \|h-x\|$, we have $\lim_q \|T^q h - h\| = 0$.

Furthermore,

$$\begin{aligned}\|h-Th\| &\leq \lim_q \|T^q h - h\| + \lim_q \|T^q h - Th\| \\ &\leq k_1 \lim_q \left\|T^{q-1} h - h\right\| = 0\end{aligned}$$

Hence, $Th = h$ and so $E^{(m)}$ has fpp(ane) as desired. □

As we recall the initial note in this section, our result extends the result in [25]. Indeed, by taking $m = 1$ firstly in Example 2.1 and next in Theorem 2.2 we obtain clearly the following corollary coinciding with the result in [25].

Corollary 2.3. *Let $b \in (0,1)$ and define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b\, e_1$, and $f_n := \frac{1}{n} e_n$ for all integers $n \geq 2$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Next, take the closed, bounded, convex subset $E = E_b$ of the Köthe-Toeplitz Dual for $X^{\infty}(\triangle)$ defined as follows:*

$$E := \left\{ \sum_{n=1}^{\infty} t_n\ f_n : \ \forall n \in \mathbb{N}, t_n \geq 0 \ \ and \ \ \sum_{n=1}^{\infty} t_n = 1 \right\} .$$

Then the set $E \subset D_1$ has the fixed point property for affine asymptotically $\|.\|$-nonexpansive mappings.

As we mentioned in the introduction and preliminaries section, in 1979, Goebel and Kuczumow showed in [28] that there exists a large class of closed, bounded and convex subsets in ℓ^1 with the fixed point property for nonexpansive mappings and so clearly the same class of sets has the fixed point property for affine nonexpansive mappings. In his PhD Thesis [23], Everest also showed the existence of larger classes of closed, bounded and convex subsets with the fixed point property for nonexpansive mappings than those of Goebel and Kuczumow. Similarly to the previous idea, we would have the classes of Everest's when we consider the fixed point property for the affine nonexpansive mappings. Considering the space D_1 which is alike ℓ^1 and given its definition in the introduction and preliminaries section, from Corollary 2.3, we get a large class of closed, bounded and convex subsets with the fixed point property for affine nonexpansive mappings. Indeed, we can present this result by the following corollary and its proof is very easy since every nonexpansive mapping is also an asymptotically nonexpansive mapping.

Corollary 2.4. *Let $b \in (0,1)$ and define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b\, e_1$, and $f_n := \frac{1}{n} e_n$ for all integers $n \geq 2$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both*

c_0 and ℓ^1. Next, take the closed, bounded, convex subset $E = E_b$ of the Köthe-Toeplitz Dual for $X^\infty(\triangle)$ defined as follows:

$$E := \left\{ \sum_{n=1}^{\infty} t_n \ f_n : \quad \forall n \in \mathbb{N}, t_n \geq 0 \quad and \quad \sum_{n=1}^{\infty} t_n = 1 \right\} \quad .$$

Then the set $E \subset D_1$ has the fixed point property for affine $\|.\|$-nonexpansive mappings.

In fact, using Theorem 2.2, we can extend the result above by the similar fact that every nonexpansive mapping is also asymptotically nonexpansive. That is, for any $m \in \mathbb{N}$, we can find a large class of closed, bounded and convex subsets in the Köthe-Toeplitz Dual D_m of a generalized Cesàro Difference Sequence Space $X^\infty(\triangle^m)$ with fixed point property for affine nonexpansive mappings. We present this result by the following corollary and the proof for this is also easy since every nonexpansive mapping is also an asymptoically nonexpansive mapping.

Corollary 2.5. *Fix $m \in \mathbb{N}$ and let $b \in (0,1)$ and define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b \ e_1$, and $f_n := \frac{1}{n^m} e_n$ for all integers $n \geq 2$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Next, take the closed, bounded, convex subset $E^m = E_b^m$ of the Köthe-Toeplitz Dual for $X^\infty(\triangle^m)$ defined as follows:*

$$E^m := \left\{ \sum_{n=1}^{\infty} t_n \ f_n : \quad \forall n \in \mathbb{N}, t_n \geq 0 \quad and \quad \sum_{n=1}^{\infty} t_n = 1 \right\} \quad .$$

Then the set $E^m \subset D_m$ has the fixed point property for affine $\|.\|$-nonexpansive mappings.

Here, we need to note that in the study [26], the first author and Mustafa were able to remove the affinity condition from the previous result, Corollary 2.5. But in their study, they could not get an analogous result for asymptotically nonexpansive mappings. However, by adding an affinity condition, we did have the desired result as in Theorem 2.2 which was, in fact, our main result.

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