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On the refinements of the Hermite-Hadamard inequality*

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Abstract: The concept of majorization is extended in this paper. Based on this generalized majorization method, a new refinement of Hermite-Hadamard inequality involving infinite series is established. As applications, we obtain a refinement of mean inequality and give new way to estimate the convergence for two kinds of improper integrals for convex functions.

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1 Introduction

Assume that $f(x)$ is a convex function on $[a, b]$, then

$$\frac{f(a)+f(b)}{2} \geq \frac{\int_a^b f(x)\,dx}{b-a} \geq f(\frac{a+b}{2}). \tag{1.1}$$

It is usually called Hermite-Hadamard inequality. As described in [1], the inequality was established by Hermite in 1883 and Hadamard in 1893, respectively. This is an important refinement of Jensen inequality.

In recent years, there are many related results. Integral inequalities involving convex functions are established in [2], by introducing a non-negative function $g(x)$. These results can be seen as generalizations for Hermite-Hadamard inequality as well as its several extensions. Two mappings H and F, which are cited from some previous references, are introduced in [3] to establish several Hadamard-type inequalities for

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M-Lipschizian function. Such inequalities estimate the upper bound for difference between two terms in Hadamard-type inequality. In [4], some Hermite-Hadamard type inequalities involving two log-preinvex functions are proved, which improve several similar results with only one log-preinvex function. Some new integral inequalities of Hermite-Hadamard type for differentiable and h-convex functions are established in [5]. Some quantum estimates for Hadamard-Hadamard type inequality are also given in [6] with the concept of q-calculus. A new generalized quantum integral is introduced in [7] and Hermite-Hadamard inequality is generalized by such integral. As can be seen above, different definitions of convex often lead to different forms of Hermite-Hadamard type inequalities. Surprisingly, it is in [8] pointed out that log-convex functions also satisfy the original form of Hermite-Hadamard inequality for convex functions and the Hermite-Hadamard type inequality chains for log-convex functions is established. A refinement of the both sides of the Hermite-Hadamard inequality is established via the defined sequences in [9].

Especially in the past five years, there're many relevant results about Hermite-Hadamard inequality from different perspective. For connections between Hermite-Hadamard inequality and A-G mean inequality, see [10]-[11]. In [12]-[14], Hermite-Hadamard type inequalities are considered on time scales. Recent results of various kinds of convex functions and convexity for Hermite-Hadamard inequality, see [15]-[16]. Hermite-Hadamard inequality for different integrals or calculus are considered in [17]-[19].

Up to now, there's not much refinements of Hermite-Hadamard inequality using infinite series, especially asymmetric (here, symmetric means the series is symmetric with respect to (a+b)/2) ones. Additionally, the refined Hermite-Hadamard inequality can give estimation about the convergence of improper integrals for convex functions. In this paper, we will establish the refinement of the Hermite-Hadamard inequality with infinite series.

We will use the notation: $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. If the components of the vector x are sorted in descending order $x_{[1]} \geq x_{[2]} \geq \ldots \geq x_{[n]}$, then we denote $\downarrow x = (x_{[1]}, x_{[2]}, \ldots, x_{[n]})$. Similarly, we denote $\uparrow x = (x_{(1)}, x_{(2)}, \ldots, x_{(n)})$, where $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$.

Definition 1.1. Let $x, y \in \mathbb{R}^n$ and satisfy that

$$\sum_{i=1}^{k} x_{(i)} \leq \sum_{i=1}^{k} y_{(i)},\ k = 1, 2, \ldots, n-1 \text{ and } \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.$$

Then we say that y is majorized by x, denoted as $y \prec x$.

First, we give the existing inequality for majorization (see [20], chapter 1 in [21] or chapter 17 in [22]).

Lemma 1.1. *If f is a convex function on $[a,b]$, $x_i, y_i \in [a,b]$, and $y \prec x$, then*

$$\sum_{i=1}^{n} f(x_i) \geq \sum_{i=1}^{n} f(y_i).$$

Similar to the above case, we give the definition with weight as follows:

Definition 1.2. For two sequences $x := \{x_1, \ldots, x_n\}$ and $y := \{y_1, \ldots, y_n\}$ and the weights $p := \{p_1, p_2, \ldots, p_n\}$ $(p_i \geq 0)$, it holds that

$$x_1 \leq x_2 \leq \ldots \leq x_n,\ y_1 \leq y_2 \leq \ldots \leq y_n.$$

and

$$\sum_{i=1}^{k} p_i x_i \leq \sum_{i=1}^{k} p_i y_i,\ k = 1, 2, \ldots, n-1,\ \ \sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i y_i.$$

Then we say that a weighted sequence y is majorized by a weighted sequence x with the same weight, and write it as $y \prec_p x$.

In [23] and chapter 14 of [20], the following inequality for majorization is proved.

Lemma 1.2. *Let f be a convex function on $[a,b]$, two sequences of number $x := \{x_1, \ldots, x_n\}$ and $y := \{y_1, \ldots, y_n\}$, $x_i, y_i \in [a,b]$, $p_i \geq 0$ and $y \prec_p x$. Then we have*

$$\sum_{i=1}^{n} p_i f(x_i) \geq \sum_{i=1}^{n} p_i f(y_i).$$

Further, generalizing Definition 1.1 and Lemma 1.2, we have the following majorization relation for any two sequences:

Definition 1.3. Let two sequences $x := \{x_1, \ldots, x_n\}$ and $y := \{y_1, \ldots, y_n\}$ satisfy that

$$x_1 \leq x_2 \leq \ldots \leq x_n,\ \ y_1 \leq y_2 \leq \ldots \leq y_m$$

and their weights $p_i \geq 0$, $q_j \geq 0$ satisfy that

$$\sum_{i=1}^{k} p_i x_i \leq (\sum_{i=1}^{k} p_i - \sum_{j=1}^{l} q_j) y_{l+1} + \sum_{j=1}^{l} q_j y_j (0 \leq \sum_{i=1}^{k} p_i - \sum_{j=1}^{l} q_j < q_{l+1}\,,\ k = 1, \ldots, n-1)$$

$$\sum_{j=1}^{i} q_j y_j \geq (\sum_{j=1}^{l} q_j - \sum_{i=1}^{k} p_i) x_{k+1} + \sum_{i=1}^{k} p_i x_i (0 \leq \sum_{j=1}^{i} q_j - \sum_{i=1}^{k} p_i < p_{k+1} \, , \, l = 1, \ldots, m-1)$$

$$\sum_{i=1}^{n} p_i x_i = \sum_{j=1}^{m} q_j y_j \, , \, \sum_{i=1}^{n} p_i = \sum_{j=1}^{m} q_j .$$

Then we say that a weighted sequence y is majorized by a weighted sequence x with different weights, and denote it as $y \ {}_q{\prec}_p \ x$.

Lemma 1.3. *Let f be a convex function on $[a, b]$, and $x_i, y_j \in [a, b]$, $p_i \geq 0, q_j \geq 0$, $y \ {}_q{\prec}_p \ x$. Then we have*

$$\sum_{i=1}^{n} p_i f(x_i) \geq \sum_{j=1}^{m} q_j f(y_j). \tag{1.2}$$

Proof. By Definition 1.3, we can reconstruct a sequence $c_r \geq 0$ satisfy that

$$\sum_{i=1}^{k} p_i = \sum_{r=1}^{t_k} c_r (k = 1, \ldots, n)(t_k \in \{1, \ldots, s\}),$$

$$\sum_{j=1}^{l} q_j = \sum_{r=1}^{t_l} c_r (l = 1, \ldots, m)(t_l \in \{1, \ldots, s\}),$$

$$\sum_{r=1}^{s} c_r = \sum_{i=1}^{n} p_i = \sum_{j=1}^{m} q_j .$$

Thus, the two sequences x and y can be reconstructed as $x_r^{'}, y_r^{'}$ with the same weight c_r and satisfy that

$$x_k = x_{t_{k-1}+1}^{'} = \ldots = x_{t_k}^{'} (k = 2, \ldots, n), \ x_1 = x_1^{'} = \ldots = x_{t_1}^{'},$$

$$y_l = y_{t_{l-1}+1}^{'} = \ldots = x_{t_l}^{'} (l = 2, \ldots, m), \ y_1 = y_1^{'} = \ldots = y_{t_1}^{'}.$$

The two new sequences of numbers $x_r^{'}, y_r^{'}$ satisfy the condition of Definition 1.2. Therefore, by Lemma 1.2, we can obtain the inequality

$$\sum_{r=1}^{s} c_r f(x_r^{'}) \geq \sum_{r=1}^{s} c_r f(y_r^{'}),$$

which is equivalent to (1.2). Thus we complete the proof of Lemma 1.3. □

Remark 1: Lemma 1.3 also holds when $n = +\infty$ or $m = +\infty$.

Take $n = +\infty$ for example, from $y \ {}_q{\prec}_p \ x$. we obtain that

$$\exists N : \forall i \geq N, x_i \geq y_m .$$

Then with the conditions in Definition 1.3, we predict that $y\ {}_q{\prec}_{p'}\ x'.$, where

$$x_i^{'} = x_i(i = 1, 2, \ldots, N-1),\ x_N^{'} = \frac{\sum_{i=N}^{\infty} p_i x_i}{\sum_{i=N}^{\infty} p_i},$$

$$p_i^{'} = p_i(i = 1, 2, \ldots, N-1),\ p_N^{'} = \sum_{i=N}^{\infty} p_i.$$

By Lemma 1.3, we obtain that

$$\sum_{i=1}^{N} p_i^{'} f(x_i^{'}) \geq \sum_{j=1}^{m} q_j f(y_j).$$

Let $N \to \infty$ we prove the inequality below

$$\sum_{i=1}^{\infty} p_i f(x_i) \geq \sum_{j=1}^{m} q_j f(y_j).$$

2 Main results

Theorem 2.1. *Assume that f is a convex function on $[a, b]$. Then it holds that*

$$\begin{aligned} \frac{f(a)+f(b)}{2} &\geq \frac{\sqrt{5}-1}{2} \sum_{k=1}^{\infty} \frac{f(b-(b-a)/c^{2k-2})}{c^k} \geq \frac{\int_a^b f(x)dx}{b-a} \\ &\geq (1-t) \sum_{k=1}^{\infty} f(b-(b-u)t^{k-1})t^{k-1} \geq f(\frac{a+b}{2}), \end{aligned}$$

where $c = \frac{\sqrt{5}+1}{2}$ and $t = \frac{b-2u+a}{b-a}(a < u < \frac{a+b}{2})$.

Proof. Step1: For the first inequality, from Jensen inequality we have

$$\begin{aligned} &\frac{\sqrt{5}-1}{2} \sum_{k=1}^{\infty} \frac{f(b-(b-a)/c^{2k-2})}{c^k} \\ &\leq \frac{\sqrt{5}-1}{2} \sum_{k=1}^{\infty} \frac{f(a)((b-a)/c^{2k-2}) + f(b)(b-a-(b-a)/c^{2k-2})}{c^k(b-a)} \\ &= \frac{(\sqrt{5}-1)f(a)}{2} \sum_{k=1}^{\infty} \frac{1}{c^{3k-2}} + \frac{(\sqrt{5}-1)f(b)}{2} \sum_{k=1}^{\infty} (\frac{1}{c^k} - \frac{1}{c^{3k-2}}) \\ &= \frac{f(a)+f(b)}{2}. \end{aligned}$$

Step 2: For the second inequality, basing on the Hermite-Hadamard inequality for any positive integer n, we first have

$$\frac{\int_a^b f(x)dx}{b-a} \leq \frac{1}{2n}\big[f(a)+f(b)+2\sum_{k=1}^{n-1} f(a+\frac{k(b-a)}{n})\big].$$

For some positive integer n, in order to prove

$$\frac{1}{2n}\big[f(a)+f(b)+2\sum_{k=1}^{n-1} f(a+\frac{k(b-a)}{n})\big] \leq \frac{\sqrt{5}-1}{2}\sum_{k=1}^{\infty}\frac{f(b-(b-a)/c^{2k-2})}{c^k}$$

according to Lemma 1.3, we need to prove that $y \ {}_q{\prec}_p \ x$, where

$$x_i = b-(b-a)/c^{2i-2}(i=1,2,\ldots),\ p_i = \frac{\sqrt{5}-1}{2c^i}(i=1,2,\ldots),$$
$$y_j = a+\frac{(j-1)(b-a)}{n}(j=1,2,\ldots,n+1), q_1 = \frac{1}{2n}, q_{n+1} = \frac{1}{2n}, q_j = \frac{1}{n}(j=2,\ldots,n).$$

The sums of two weighted sequences and their weights satisfy the condition of equation in Definition 1.3

$$\sum_{j=1}^{n+1} q_j = \frac{1}{2n}+(n-1)\cdot\frac{1}{n}+\frac{1}{2n} = 1 = \sum_{i=1}^{\infty}\frac{\sqrt{5}-1}{2c^i} = \sum_{i=1}^{\infty} p_i,$$
$$\sum_{j=1}^{n+1} q_j y_j = \frac{1}{2n}(a+b)+\frac{1}{n}\sum_{j=2}^{n}(a+\frac{(j-1)(b-a)}{n})$$
$$= \frac{a+b}{2} = \sum_{i=1}^{\infty}\frac{\sqrt{5}-1}{2c^i}(b-(b-a)/c^{2i-2}) = \sum_{i=1}^{\infty} p_i x_i.$$

For $p \in [0,1]$, define

$$S_1(p) = \sum_{j=1}^{r} q_j y_j + (p-\sum_{j=1}^{r} q_j)y_{r+1}(\sum_{j=1}^{r} q_j < p \leq \sum_{j=1}^{r+1} q_j), S_1(p) = py_1(0 \leq p \leq q_1),$$

$$S_2(p) = \sum_{i=1}^{s} p_i x_i + (p-\sum_{i=1}^{s} p_i)x_{s+1}(\sum_{i=1}^{s} p_i < p \leq \sum_{i=1}^{s+1} p_i), S_2(p) = px_1(0 \leq p \leq p_1).$$

To satisfy the inequality condition of Definition 1.3, the sum should satisfy

$$S_2(p) \leq S_1(p) \ \ p \in [0,1],$$

for some positive integer n. Let $n \to \infty$ in the first sequence, then

$$S_1(p) = p(a+\frac{p(b-a)}{2}) \ \ p \in [0,1].$$

First, consider the following special points

$$p_k = \sum_{i=1}^{k} \frac{\sqrt{5}-1}{2c^i}(k = 1, 2, \ldots).$$

Substituting them into the sum, we obtain that

$$S_1(p_k) = p_k(a + \frac{p_k(b-a)}{2}) = \frac{\sqrt{5}-1(1-\frac{1}{c^k})}{2(c-1)}(a + \frac{\sqrt{5}-1(1-\frac{1}{c^k})}{4(c-1)}(b-a)),$$

$$S_2(p_k) = \sum_{i=1}^{k} \frac{\sqrt{5}-1}{2c^i}\big[b-(b-a)/c^{2i-2}\big] = \frac{\sqrt{5}-1}{2}\big[\frac{1-\frac{1}{c^k}}{c-1}b - \frac{c^2-\frac{1}{c^{3k-2}}}{c^3-1}(b-a)\big].$$

From the following inequality

$$\frac{(1-\frac{1}{c^k})^2}{2(c-1)}+\frac{c^2-\frac{1}{c^{3k-2}}}{c^3-1} \geq \frac{1-\frac{1}{c^k}}{c-1} \Rightarrow \frac{1-\frac{1}{c^k}}{c-1}a+[\frac{(1-\frac{1}{c^k})^2}{2(c-1)}+\frac{c^2-\frac{1}{c^{3k-2}}}{c^3-1}](b-a) \geq \frac{1-\frac{1}{c^k}}{c-1}b$$

$$\Rightarrow \frac{1-\frac{1}{c^k}}{c-1}(a + \frac{\sqrt{5}-1(1-\frac{1}{c^k})}{4(c-1)}(b-a)) \geq \frac{1-\frac{1}{c^k}}{c-1}b - \frac{c^2-\frac{1}{c^{3k-2}}}{c^3-1}(b-a),$$

we have

$$S_2(p_k) \leq S_1(p_k)(k = 1, 2, \ldots),\ S_2(0) = S_1(0).$$

With the characteristic of two functions of sums

$$[S_1(p)]_p' \leq [S_1(p_{k+1})]_p' = a + p_{k+1}(b-a) = b - \frac{1}{c^{k+1}}(b-a)(p \in (p_k, p_{k+1})),$$

$$[S_2(p)]_p' = b-(b-a)/c^{2k}(p \in (p_k, p_{k+1})),$$

we can affirm that the derivative functions have relationship below

$$[S_1(p)]_p' \geq [S_1(0)]_p' = a = [S_2(p)]_p'(p \in (0, p_1)),$$

$$[S_2(p)]_p' \geq [S_1(p)]_p'(p \in (p_k, p_{k+1}), k = 1, 2, \ldots).$$

From above we get

$$S_2(p) \leq S_1(p)(p \in [0, 1]).$$

Thus, the first weighted sequence is majorized by the second weighted sequence, the second inequality of Theorem 2.1 is proved.

Step 3: For the third inequality, the proof is similar to that of the second inequality above. In order to prove

$$\frac{1}{2n}[f(a)+f(b)+2\sum_{k=1} n-1f(a+\frac{k(b-a)}{n})] \geq (1-t)\sum_{k=1}^{\infty} f(b-(b-u)t^{k-1})t^{k-1},$$

for some positive integer n , according to Lemma 1.3, we need to prove that $y_q \prec_p z$, where

$z_i = b-(b-u)t^{i-1}(i=1,2,\ldots)$, $p_i=(1-t)t^{i-1}(i=1,2,\ldots)$,
$y_j = a+\dfrac{(j-1)(b-a)}{n}(j=1,2,\ldots,n+1), q_1=\dfrac{1}{2n}, q_{n+1}=\dfrac{1}{2n}, q_j=\dfrac{1}{n}(j=2,\ldots,n)$.

The sums of two weighted sequences and their weights satisfy the condition of equation in Definition 1.3

$$\sum_{j=1}^{n+1} q_j = \frac{1}{2n}+(n-1)\cdot\frac{1}{n}+\frac{1}{2n} = 1 = \sum_{i=1}^{\infty}(1-t)t^{k-1} = \sum_{i=1}^{\infty} p_i,$$

$$\begin{aligned}\sum_{j=1}^{n+1} q_j y_j &= \frac{1}{2n}(a+b)+\frac{1}{n}\sum_{j=2}^{n}(a+\frac{(j-1)(b-a)}{n})\\ &= \frac{a+b}{2} = \sum_{i=1}^{\infty}(1-t)t^{i-1}(b-(b-u)t^{i-1}) = \sum_{j=1}^{\infty} p_i z_i.\end{aligned}$$

For $p\in[0,1]$, define

$$S_1(p) = \sum_{j=1}^{r} q_j y_j + (p-\sum_{j=1}^{r} q_j) y_{r+1} (\sum_{j=1}^{r} q_j < p \le \sum_{j=1}^{r+1} q_j),\ S_1(p) = p y_1 (0\le p\le q_1),$$

$$S_3(p) = \sum_{i=1}^{s} p_i z_i + (p-\sum_{i=1}^{s} p_i) z_{s+1} (\sum_{i=1}^{s} p_i < p \le \sum_{i=1}^{s+1} p_i),\ S_3(p) = p z_1 (0\le p\le p_1).$$

To satisfy the inequality condition of Definition 1.3, the sum should satisfy

$$S_3(p)\ge S_1(p) \ \ p\in[0,1],$$

for some positive integer n. Let $n\to\infty$ in the first sequence, then

$$S_1(p) = p(a+\frac{p(b-a)}{2}) \ \ p\in[0,1].$$

First, consider the following special points

$$p_k = \sum_{i=1}^{k}(1-t)t^{i-1} = (1-t^k)(k=1,2,\ldots).$$

Substituting them into the sum, we obtain that

$$\begin{aligned}S_1(p_k) &= p_k(a+\frac{p_k(b-a)}{2}) = (1-t^k)(a+\frac{(1-t^k)(b-a)}{2}),\\ S_3(p_k) &= (1-t)\sum_{i=1}^{k}(b-(b-u)t^{i-1})t^{i-1} = (1-t^k)b-\frac{(b-u)(1-t^{2k})}{1+t}.\end{aligned}$$

From the following inequality

$$a+\frac{(2-t^k+t^{2k})(b-a)}{2}\le b\Rightarrow a+\frac{(1-t^k)(b-a)}{2}\le b-\frac{(1+t^k)(b-a)}{2}$$
$$\Rightarrow a+\frac{(1-t^k)(b-a)}{2}\le b-\frac{(b-u)(1+t^k)}{1+t},$$

we have

$$S_3(p_k)\ge S_1(p_k)(k=1,2,\ldots), S_3(0)=S_1(0).$$

With the characteristic of two functions of sums and Jensen inequality for the first convex function of the sum $S_1(p)$,

$$\begin{aligned}S_3(p)&=\frac{(p-p_k)S_3(p_k)+(p_{k+1}-p)S_3(p_{k+1})}{p_{k+1}-p_k}\\&\ge\frac{(p-p_k)S_1(p_k)+(p_{k+1}-p)S_1(p_{k+1})}{p_{k+1}-p_k}\ge S_1(p)(p\in(p_k,p_{k+1})),\\S_3(p)&=\frac{(p_1-p)S_3(p_1)}{p_1}\ge\frac{(p_1-p)S_1(p_1)}{p_1}\ge S_1(p)(p\in(0,p_1)).\end{aligned}$$

From above we get

$$S_3(p)\ge S_1(p)\ \ p\in[0,1].$$

Thus, the third weighted sequence is majorized by the first weighted sequence, the third inequality of Theorem 2.1 is proved.

Step 4: For the fourth inequality, from Jensen inequality we have

$$\begin{aligned}&(1-t)\sum_{k=1}^{\infty}f(b-(b-u)t^{k-1})t^{k-1}\\&\ge f(\frac{\sum_{k=1}^{\infty}(b-(b-u)t^{k-1})(1-t)t^{k-1}}{\sum_{k=1}^{\infty}(1-t)t^{k-1}})\\&=f(\frac{a+b}{2}).\end{aligned}$$

Thus, we complete the proof of Theorem 2.1. □

Remark 2. Let $n\to a$ in Theorem 2.1, then $(1-t)\sum\limits_{k=1}^{\infty}f(b-(b-u)t^{k-1})t^{k-1}=\frac{\int_a^b f(x)dx}{b-a}$. Let $u\to\frac{a+b}{2}$ in Theorem 2.1, then $(1-t)\sum\limits_{k=1}^{\infty}f(b-(b-u)t^{k-1})t^{k-1}=f(\frac{a+b}{2})$. We wonder if $(1-t)\sum\limits_{k=1}^{\infty}f(b-(b-u)t^{k-1})t^{k-1}$ is monotone decreasing with u.

3 Application

Corollary 3.1. *Let $b > a > 0$, then*

$$\frac{a+b}{2} \geq \prod_{k=1}^{\infty}(b-(b-u)t^{k-1})^{(1-t)t^{k-1}} \geq \frac{b\ln b - a\ln a}{b-a} - 1$$
$$\geq \prod_{k=1}^{\infty}(b-(b-u)/c^{2k-2})^{\frac{\sqrt{5}-1}{2c^k}} \geq \sqrt{ab}.$$

where $c = \frac{\sqrt{5}+1}{2}$ and $t = \frac{b-2u+a}{b-a}(a < u < \frac{a+b}{2})$.

In fact, set $f(x) = -\ln x$ in Theorem 2.1, we get Corollary 3.1.

Assume that is a convex function on $[a, b)$ and $\lim\limits_{x\to b} f(x) \to +\infty$. Consider the convergence of the improper integral $\int_a^b f(x)dx$. It's invalid to use Hermite-Hadamard inequality $\int_a^b f(x)dx \leq (b-a)(f(a)+f(b))/2$ to estimate. However, the conclusion below may make a more accurate estimation.

Corollary 3.2. *Assume that f is a convex function on $[a, b)$ and $\lim\limits_{x\to b} f(x)\to+\infty$. If $\sum_{k=1}^{\infty} \frac{f(b-(b-a)/c^{2k-2})}{c^k} < \infty$, where $c = \frac{\sqrt{5}+1}{2}$, then $\int_a^b f(x)dx < \infty$.*

Proof. For $\forall t \in (a, b)$, with Theorem 2.1 we have

$$\frac{\int_a^t f(x)dx}{t-a} \leq \frac{\sqrt{5}-1}{2}\sum_{k=1}^{\infty}\frac{f(t-(t-a)/c^{2k-2})}{c^k}$$
$$\Rightarrow \int_a^t f(x)dx \leq \frac{\sqrt{5}-1}{2}(b-a)\sum_{k=1}^{\infty}\frac{f(t-(t-a)/c^{2k-2})}{c^k}.$$

Let $t \to b$ in the inequality, we get Corollary 3.2. □

Let $g(a+b-x) = f(x)$ in Corollary 3.2, we can obtain the following conclusions.

Corollary 3.3. *Assume that g is a convex function on $(a, b]$ and $\lim\limits_{x\to a} g(x)\to+\infty$. If $\sum_{k=1}^{\infty} \frac{g(a+(b-a)/c^{2k-2})}{c^k} < \infty$, where $c = \frac{\sqrt{5}+1}{2}$, then $\int_a^b g(x)dx < \infty$.*

Corollary 3.3 can also prove similar conclusion for another kind of improper integral. Assume that f is a convex and strictly monotone decreasing function on $[a, +\infty)$ and $\lim\limits_{x\to+\infty} f(x) = 0$. Consider the convergence of the improper integral $\int_a^{+\infty} f(x)dx$. It's invalid to use Hermite-Hadamard inequality $\int_a^b f(x)dx \leq (b-a)(f(a)+f(b))/2$ to estimate. The following theorem can make a more accurate estimation.

Corollary 3.4. *Assume that f is a convex and strictly monotone decreasing function on $[a,+\infty)$ and $\lim\limits_{x\to+\infty} f(x)=0$. If $\sum_{k=1}^{\infty}\frac{f^{-1}(f(a)/c^{2k-2})}{c^k}<\infty$, where $c=\frac{\sqrt{5}+1}{2}$, then $\int_a^{+\infty} f(x)dx<\infty$.*

Proof. Since f is a convex and strictly monotone decreasing function on $[a,+\infty)$ and $\lim\limits_{x\to+\infty} f(x)=0$, f^{-1} is a convex function on $(0,f(a)]$. By Corollary 3.3, it can be obtained that

$$\sum_{k=1}^{\infty}\frac{f^{-1}(f(a)/c^{2k-2})}{c^k}<\infty\Rightarrow\int_0^{f(a)} f^{-1}(y)dy<\infty.$$

Noticing that

$$\int_0^{f(a)} f^{-1}(y)dy=af(a)+\int_a^{+\infty} f(x)dx,$$

Corollary 3.4 is proved. □

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