Advanced Studies in Contemporary Mathematics
33 (2023), No. 4, pp. 319 – 329

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2023.33.4.319

HOP HUB NUMBER OF A GRAPH

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Abstract

Hub number is a graph parameter introduced by modelling a transportation problem for rapid transit in any system. In this paper, we coin a new hub parameter called hup hub number of graphs and we determine the hop hub number of some standard graphs. Also upper and lower bounds for the hop hub number are obtained. In addition, the relationship between hop hub number and other graph parameters is discussed.

2000 Mathematics Subject Classification: 05C69.
Keywords: Hop hub number, Hub number, Domination number, Connected hub number.

1 Introduction

In this paper, we consider a simple graph $G = (V, E)$, that is nonempty, finite, having no loops, no multiple and directed edges. Let n and m be the number of its vertices and edges, respectively. The symbols $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of G, respectively. For graph theoretic terminology, we refer to [7].
A double star $S_{n,m}$ where $n \geq m \geq 0$, is the graph consisting of the union of two stars $K_{1,n}$ and $K_{1,m}$, together with an edge joining their central vertices. A coloring of a graph is an assignment of colors to its vertices, so that no two adjacent vertices have the same color. The chromatic number $\chi(G)$ of a graph G is the minimum number of colors required to assign to the vertices of G in such a way that no two adjacent vertices of G receive the same color.

We define $\lfloor x \rfloor$ to be the greatest integer not exceeding x and $\lceil x \rceil$ to be the smallest integer not smaller than x. For any subset S of $V(G)$, the induced subgraph $\langle S \rangle$ is the maximal subgraph of G with vertex set S. A set $S \subseteq V(G)$ is called a dominating set of G if each vertex of $V - S$ is adjacent to at least one vertex of S. The domination number of a graph G denoted as $\gamma(G)$ is the minimum cardinality of a dominating set in G [9]. A set $S \subseteq V$ of a graph G is a hop dominating set of G if for every $v \in V - S$, there exists $u \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hop dominating set of G is called the hop domination number and is denoted by $\gamma_h(G)$ [2].

M. Walsh [26] introduced the theory of hub numbers in the year 2006. Suppose that $S \subseteq V(G)$ and let $x, y \in V(G)$. An S-path between x and y is a path where all intermediate vertices are from S. (This includes the degenerate cases where the path consists of the single edge xy or a single vertex x if $x = y$, call such an H-path trivial). A set $S \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G) - S$, there is an S-path in G between x and y. The smallest size of a hub set in G is called the hub number of G, and is denoted by $h(G)$. S. S. Mahde and A. S. Sand [20] introduced the concept of hop hub-integrity of graphs, which is defined as follows. The hop hub-integrity of a graph is denoted as $H_hI(G) = \{|S| + m(G-S), S \textit{ is a hop hub set}\}$, where $m(G - S)$ is the order of a maximum component of $G - S$.

The concept of hub number quantifies the connectivity of vertices in graphs. Due to which it has wide application in the field of networks. For this reason, several hub parameters have been explored and studied extensively [6, 12, 13, 14, 15, 16, 19, 21, 23]. Motivated by this, in this article we try to term a new hub parameter called hop hub number of graphs as follows.

Definition 1.1. *A hub set S is a hop hub set of G if for every $v \in V - S$, there exists $u \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hop hub set of G is called the hop hub number and is denoted by $h_h(G)$.*

The following results will be useful in the proof of our results.

Theorem 1.2. *[26] For any cycle graph C_n, $h(C_n) = n - 3$.*

Theorem 1.3. *[3] For any complete graph K_n, $\chi(G) = n$.*

Theorem 1.4. *[26] Let T be a tree with n vertices and l levels. Then $h(T) = n - l$.*

Theorem 1.5. *[26] Let S be a subset of $V(G)$. Then G/S is a complete if and only if S is a hub set of G.*

2 Main results

Proposition 2.1. *The hop hub numbers of some specific classes of graphs are as below:*

1. For any path P_n,

$$h_h(P_n) = \begin{cases} 2, & \text{if } n = 2, 3, \\ n-2, & \text{if } n \geq 4. \end{cases}$$

2. For any complete graph K_n, $h_h(K_n) = n$.

3. For the wheel graph $W_{1,n-1}$,

$$h_h(W_{1,n-1}) = \begin{cases} 4, & \text{if } n = 4, \\ 3, & \text{if } n \geq 5, \end{cases}$$

4. For the complete bipartite graph $K_{n,m}$, $h_h(K_{n,m}) = 2$.

5. For the double star $S_{n,m}$, $h_h(S_{n,m}) = 2$.

6. For any cycle C_n,

$$h_h(C_n) = \begin{cases} 2, & \text{if } n = 4, \\ 3, & \text{if } n = 3, \\ n-3, & \text{if } n \geq 5. \end{cases}$$

Proof. 1. Let $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ be the vertices of a path P_n the following three cases are considered:
Case 1: When n = 2. Suppose that $\{v_1, v_2\}$ be the vertices of a path P_2, see Figure 1, then $S = \{v_1\}$ is not hop hub set because $d(v_1, v_2) = 1$, similarly $S = \{v_2\}$ is not hop hub set, so $S = \{v_1, v_2\}$ is a minimum hop hub set and hence $h_h(P_2) = 2$.

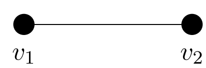

Figure 1

Case 2: When n = 3. Since $|S| \geq 2$ for any hop hub set S of a graph G. Consider $\{v_1, v_2, v_3\}$ be the vertices of a path P_3 see Figure 2, we have $S = \{v_1, v_2\}$ is a hop hub set of P_3, since $v_3 \in V(P_3) - S$ and $v_1 \in S$ such that $d(v_1, v_3) = 2$, similarly for $S = \{v_2, v_3\}$ is hop hub set, since $v_1 \in V(P_3) - S$ and $v_3 \in S$ such that $d(v_1, v_3) = 2$. Then S is a hop hub set of P_3 and it is clear that S is a minimum hop hub set. Therefore $h_h(P_3) = 2$.

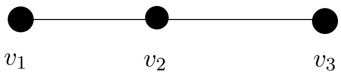

Figure 2

Case 3: When $n \geq 4$. Suppose that $\{v_1, v_2, \cdots, v_n\}$ is a vertex set of P_n and $S = \{v_2, v_3, \cdots, v_{n-1}\}$ is a hop hub set of P_n such that $|S| = n - 2$. We have $V(P_n) - S = \{v_1, v_n\}$ and $d(v_1, v_3) = d(v_n, v_{n-2}) = 2$. To show that S is a minimum hop hub set of P_n, if v_i, $2 \leq i \leq n-1$ is removed form the set S, then there does not exist S-path between v_1 and v_n. Thus S is a minimum hop hub set. and hence $h_h(P_n) = n - 2$, this complete the proof.

2. Since all vertices in K_n are adjacent and the distance between them equal to one, so we must choose all vertices as hop hub set of K_n, and hence $h_h(K_n) = n$.

3. Let $V(W_{1,n-1}) = \{v, v_1, v_2, \cdots, v_{n-1}\}$ be the vertex set of $W_{1,n-1}$, the following two cases are considered:
Case 1: When $n = 4$, then $W_{1,3} \cong K_4$, since $h_h(K_4) = 4$, we get the result.
Case 2: When $n \geq 5$. Let $V(W_{1,n-1}) = \{v, v_1, v_2, \cdots, v_{n-1}\}$ and consider $S = \{v, v_1, v_2\}$ a S-set of $W_{1,n-1}$. We claim that S is a minimum hop hub set of $W_{1,n-1}$, if for any $v_i, v_j \in V - S$, $i \neq j$, there exists S-path between them. Now if v is removed form S, there is no S-path between any two vertices outside the set S. Thus the set S is not hub set, and removal of v_1 or v_2 form S, leads to existence some vertex $v_i \in V - S$ such that $d(v_1, v_i) = 1$ or $d(v_2, v_i) = 1$. Thus S is a minimum hop hub set. Hence, $h_h(W_{1,n-1}) = 3$.

4. Let $V(K_{n,m}) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_m\}$. Consider $S = \{v_1, u_1\}$ is a hop hub set of $K_{n,m}$ such that $|S| = 2$. To show that S is a minimum hop hub set of $K_{n,m}$, if we remove v_1 of S, the set $S = \{u_1\}$ is not hop hub set because there exist $v_i, 2 \leq i \leq n$ such that $d(v_i, u) \neq 2$ and this does not achieve the definition of hop hub set. If we remove u_1 of S, the set $S = \{v_1\}$ is not hop hub set because there exist $u_i, 2 \leq i \leq m$ such that $d(u_i, v) \neq 2$ and this does not achieve the definition of a hop hub set. Therefore, $h_h(K_{n,m}) = 2$.

5. Let $V(S_{n,m}) = \{v, v_1, v_2, \ldots, v_n, u, u_1, u_2, \ldots, u_m\}$. Consider $S = \{v, u\}$ we have S is a minimum hub set of $S_{n,m}$ and hence $h(S_{n,m}) = 2$. We prove that S-set is a hop hub set of $S_{n,m}$, for any $v_i \in V(S_{n,m}) - S$, there

exists $u \in S$ such that $d(v_i, u) = 2$ and for any $u_i \in V(S_{n,m}) - S$, there exists $v \in S$ such that $d(u_i, v) = 2$, then S is a hop hub set of $S_{n,m}$, and $h_h(G) \geq h(G)$, then S-set is a minimum hop hub set of $S_{n,m}$, and hence $h_h(S_{n,m}) = 2$.

6. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ be the vertices of C_n, we have the three following cases:
Case 1: For $n = 3$. Since $C_3 \cong K_3$, we get the result by this Proposition 2.1 part 2, $h_h(C_3) = 3$.
Case 2: When $n = 4$. Consider $S = \{v_1, v_2\}$ is a hop hub set of C_4 as shown in Figure 3. It is clear that S is a minimum hop hub set, so $h_h(C_4) = 2$.

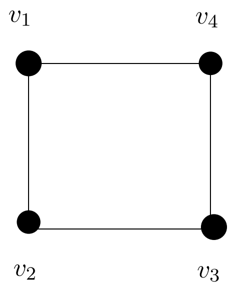

Figure 3

Case 3: When $n \geq 5$. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Consider $S = \{v_4, v_5, \ldots, v_n\}$ is a hop hub set of C_n, such that $|S| = n - 3$. To show that S is a minimum hop hub set of C_n. By Theorem 1.2, $h(C_n) = n - 3$ and since $h_h(C_n) \geq h(C_n)$, also any vertex $v \in V(C_n) - S$, there exist v_i, $4 \leq i \leq n$ such that $d(v_i, v) = 2$, it follows that S is a minimum hop hub set of C_n, and hence $h_h(C_n) = n - 3$.

□

Theorem 2.2. *For any connected graph G, $2 \leq h_h(G) \leq n$.*

Proof. By the definition of a hop hub set S of a graph G we have $|S| \geq 2$ and the upper bound is achieved of $G \cong K_n$. Therefore, $2 \leq h_h(G) \leq n$. □

Theorem 2.3. *Let G be a disconnected graph having $M_1, M_2, ..., M_l$ components. Then $h_h(G) = \min_{1 \leq t \leq l}\{X_t\}$, where $X_t = h_h(M_t) + \sum_{i=1, i \neq t}^{l} |V(M_i)|$.*

Proof. Any hop hub set S of a graph G must contains all the vertices of $t-1$ components and the vertices of hop hub set of the remaining component. To show that S is a minimum. The union of all components except one and taking the hop hub set of the remaining component, we can compute all hop hub sets of G, and more detailed $S = \bigcup_{i=1, i\neq j}^{t} M_i \cup H_h^t$, where H_h^t is a hop hub set of M_t.

Let $X_t = h_h(M_t) + \sum_{i=1, i\neq t}^{l} |V(M_i)|$. Then $\min_{1\leq t\leq l}\{X_t\} = h_h(G)$. □

Remark 2.4. *In general, the inequality $h_h(G') \leq h_h(G)$ is not true for a subgraph G' of G, for the graph G and a subgraph G' shown in Figure 4, we have $h_h(G) = 2$, while $h_h(G') = 4$.*

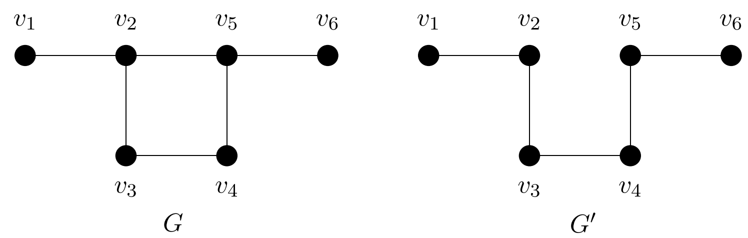

Figure 4

Theorem 2.5. *Let G be a connected graph of order n, $h_h(G) = n$ if and only if $G \cong K_n$.*

Proof. Suppose that $h_h(G) = n$, this means that all vertices of a graph G are adjacent and hence $G \cong K_n$.
Conversely, if $G \cong K_n$, the proof follow form Proposition 2.1. □

Lemma 2.6. *Let S be a subset of $V(G)$. Then G/S is a complete if and only if S is a hop hub set of G.*

Proof. By the definition of a hop hub set and since any hop hub set is a hub set of G. By Theorem 1.5, it follows that G/S is a complete graph. □

Theorem 2.7. *If G is a complete graph, then $\chi(G) = h_h(G)$.*

Proof. Assume that G is a complete graph and by Proposition 2.1, and Theorem 1.3, we get the result. □

Remark 2.8. *The converse of Theorem 2.7 is not true, for example $G \cong K_{1,3}$ as shown in Figure 5.*

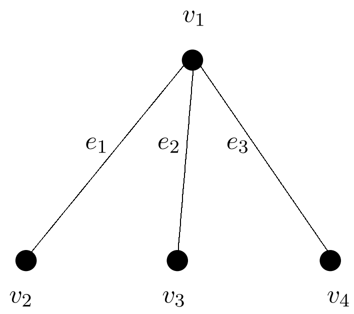

Figure 5

Note that $\chi(G) = h_h(G) = 2$, but G is not complete graph.

Theorem 2.9. *For any connected graph G, if $h_h(G) = 2$ then $diam(G) \leq 3$.*

Proof. Let $h_h(G) = 2$. We prove that $diam(G) \leq 3$, suppose that $diam(G) > 3$, then by the definition of $diam(G)$, there exists a path between at least five vertices and we get $h_h(G) \geq 3$, but this contradiction that $h_h(G) = 2$, then $diam(G) \leq 3$. □

Remark 2.10. *The converse of Theorem 2.9 is not true. For example for $G \cong K_3$ we have $diam(K_3) = 1$ and $h_h(K_3) = 3$.*

Theorem 2.11. *Let T be a tree. Then $h_h(T) = 2$ if and only if $diam(T) \leq 3$.*

Proof. Suppose that $h_h(T) = 2$, then $diam(T) \leq 3$ by Theorem 2.9.
Converse, suppose that $diam(T) \leq 3$ and we prove that $h_h(T) = 2$. Since $diam(T) \leq 3$ and a tree T has not closed path, then the largest distance in T contains four vertices. Let T be P_4 since $h_h(P_4) = 2$, by Proposition 2.1, and without loss of generally, $h_h(T) = 2$. □

Theorem 2.12. *For any connected graph G, $h_h(G) \geq h(G)$ and the inequality is sharp if $G \cong T$, and $h(G) \geq 3$.*

Proof. Form the definition of the hop hub set of G, $h_h(G) \geq h(G)$ and if $h(G) \geq 3$ and $G \cong T$ we have $diam(G) \geq 3$ and for any hop hub set S of T, there exists a vertex $v \in S$ such that $d(u, v) = 2$ for any vertex $u \in V - S$. Therefore, $h_h(G) = h(G)$. □

Lemma 2.13. *Let T be a tree with n vertices, l leaves and k internal vertices, $k \geq 3$., Then $h_h(T) = h(T) = n - l$.*

Proof. Since $k \geq 3$, then $h(T) \geq 3$. By using Theorem 2.12 and by Theorem 1.4, we get $h_h(T) = n - l$. □

3 Hop hub number of line graphs

Definition 3.1. *[7] The line graph $L(G)$ of a graph G has the edges of G as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G.*

Proposition 3.2. *The hop hub number of some specific classes of line graphs are as below*

1. *For any path P_n, $n \geq 5$, $h_h(L(P_n)) = n - 3$.*

2. *For any cycle C_n,*

$$h_h(L(C_n)) = \begin{cases} 2, & \text{if } n = 4, 5, \\ 3, & \text{if } n = 3, \\ n-3, & \text{if } n \geq 6. \end{cases}$$

3. *For the wheel graph $W_{1,n-1}$, $h_h(L(W_{1,n-1})) = \lceil \frac{n-1}{2} \rceil + 1$*

4. *For any double star $S_{n,m}$, $h_h(L(S_{n,m})) = 3$.*

Proof. 1. Since $L(P_n) \cong P_{n-1}$, and by Proposition 2.1, $h_h(L(P_n)) = n - 3$.

2. Since $L(C_n) \cong C_n$, and by Proposition 2.1, $h_h(L(C_n)) = h_h(C_n)$.

3. Let $S = S_1 \cup S_2$ be the set of vertices of $L(W_{1,n-1})$ such that $S_1 = \{e_1, e_2, ..., e_{n-1}\}$ and $S_2 = \{e'_1, e'_2, ..., e'_{n-1}\}$, we have the induced subgraph $\langle S_1 \rangle$ of S is a complete graph and any vertex of S_1 is adjacent to exactly two vertices of S_2, see the Figure 6. So, the set $S' = S'_1 \cup S'_2$ such that $S'_1 = \{e_1, e_2, ..., e_{\lceil \frac{n-1}{2} \rceil}\}$ and $S'_2 = \{e' : e' \in S_2\}$ is a hop hub set of $L(W_{1,n-1})$. To show that S' is a minimum hop hub set of $L(W_{1,n-1})$, if removed one vertex form the set S'_1, then there does not exist S'-path between at least two different vertices e'_i, e'_j, $i \neq j$ of $S_2 - S'_2$ and if removed that vertex form the set S'_2, then $d(e_i, e_j) \neq 2$, $i \neq j$ of $S_1 - S'_1$. Thus S' is a minimum hop hub set and hence $h_h(L(W_{1,n-1})) = \lceil \frac{n-1}{2} \rceil + 1$, this complete the proof.

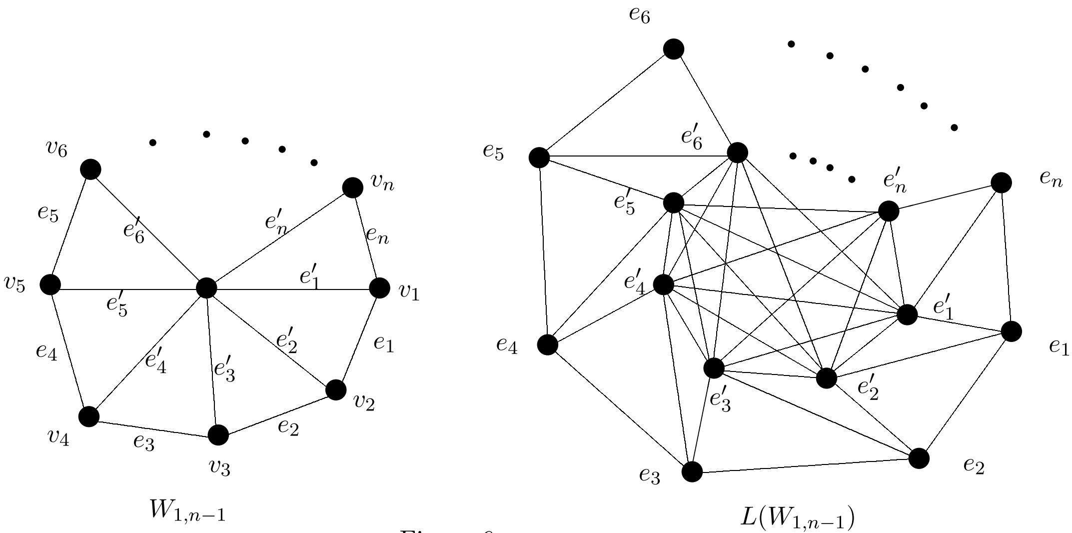

Figure 6

4. The graph $L(S_{n,m})$ consists of two complete graphs of orders n, m respectively, and the vertex e that is adjacent to all vertices in $L(S_{n,m})$. The number of vertices of $L(S_{n,m})$ is $n+m$. The graphs $S_{n,m}$ and $L(S_{n,m})$ are shown in Figure 7, Consider $S = \{e, e_n, e'_m\}$ is a hop hub set of $L(S_{n,m})$. Since e is adjacent to all vertices in $L(S_{n,m})$, then for any vertex $e_i \in L(S_{n,m}), 1 \leq i \leq n-1$, there exists $e'_m \in S$ such that $d(e_i, e'_m) = 2$ and also for any vertex $e_j \in L(S_{n,m}), 1 \leq j \leq m-1$, there exists $e_n \in S$ such that $d(e'_j, e_n) = 2$. If we remove it from the graph $L(S_{n,m})$, there is no S-path between the vertices that are not adjacent. So S is a minimum hop hub set. Therefore $h_h(L(S_{n,m})) = 3$.

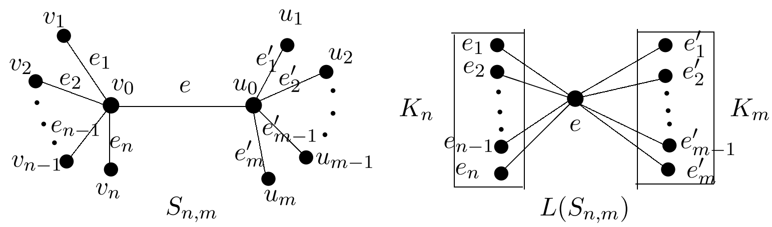

Figure 7

□

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