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NOTE ON THE DEGENERATE r-DOWLING POLYNOMIALS AND NUMBERS OF THE FIRST KIND INCLUDING THE GENERALIZED DEGENERATE JINDALRAE-STIRLING NUMBERS AND THE WEAK GANARI POLYNOMIALS

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ABSTRACT. Kim et al. introduced the Jindalrae-Stirling numbers of the first and second kind, the Jindalrae and Gaenari polynomials (Adv. Differ. Eq., 2020:245 (2020)). In this paper, we introduce the generalized degenerate Jindalrae-Stirling numbers of the first and second kind, and the weak Jindalrae and Gaenari polynomials, respectively. We explore some properties for these numbers. and some combinatorial identities for the degenerate r-Dowling polynomials and numbers of the first kind including the generalized degenerate Jindalrae-Stirling numbers and the weak Ganari polynomials.

1. INTRODUCTION

In [16], Kim et al. introduced the degenerate Jindalrae-Stirling numbers of the first and second kind, respectively, give by

$$\frac{1}{k!}(\log_\lambda(\log_\lambda(1+t)+1))^k=\sum_{n=k}^{\infty}S_{J,\lambda}^{(1)}(n,k)\frac{t^n}{n!},\quad (k\ge 0), \tag{1}$$

and

$$\frac{1}{k!}(e_\lambda(e_\lambda(t)-1)-1)^k=\sum_{n=k}^{\infty}S_{J,\lambda}^{(2)}(n,k)\frac{t^n}{n!}\quad (k\ge 0). \tag{2}$$

They also considered the Gaenari and Jindalrae polynomials in connection with the degenerate Jindalrae-Stirling numbers of the first and second kind, respectively, as

$$e_\lambda^x(\log_\lambda(\log_\lambda(1+t)+1))=\sum_{n=0}^{\infty}G_{n,\lambda}(x)\frac{t^n}{n!}, \tag{3}$$

and

$$e_\lambda^x(e_\lambda(e_\lambda(t)-1)-1)=\sum_{n=0}^{\infty}J_{n,\lambda}(x)\frac{t^n}{n!},\quad \text{(see [16])}. \tag{4}$$

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When $x=1, G_{n,\lambda} := G_{n,\lambda}(1)$ and $J_{n,\lambda} := J_{n,\lambda}(1)$ are called the Gaenari and Jindalrae numbers, respectively.

Dowling [8] constructed Dowling lattice $Q_n(G)$ for a finite group of order m using the Möbious function and introduced the Whitney numbers of the first and second kind, $w_m(n,k)$ and $W_m(n,k) \quad (n \geq k \geq 0, m \geq 1)$, respectively, which are independent if the group G itself, but depend only on its order.

In this manuscript, we focus the r-Whitney numbers $w_{m,r}(n,k)$ of first kind.

As a generalization of the Whitney numbers of the first kind, the r-Whitney numbers $w_{m,r}(n,k)$ of the first kind associated with $Q_n(G)$ is given by

$$m^n(x)_n = \sum_{k=0}^{n}(-1)^{n-k}w_{m,r}(n,k)(mx+r)^k, \quad \text{(see [6])}. \tag{5}$$

From (5), we note that the generating function of $w_{m,r}(n,k)$ is

$$\sum_{k=0}^{\infty}(-1)^{n-k}w_{m,r}(n,k)\frac{t^n}{n!} = (1+mx)^{-\frac{r}{m}}\frac{(\log(1+mt))^k}{m^k k!}, \quad \text{(see [6])}.$$

In [9], Gyimesi-Nyule introduced $w_{m,r}(n,k)$ given by the number of coloured permutations in $S_1^{(r)}(n,k)$ which are the product of $k+r$ disjoint cycles such that

- the distinguished elements $1,\cdots,r$ belong to distinct cycles,
- the smallest elements of the cycles are not coloured,
- an element in a cycle containing a distinguished element is not coloured if there are no smaller numbers on the arc from the distinguished element to this element,
- the remaining elements are coloured with m colours.

for $n \geq k \geq 0, r \geq 0, n+r \geq 1$ and $m \geq 1$. Moreover, let $w_{m,0}(0,0) = 1$.

They also showed

$$(x+r|m)^{\bar{n}} = \sum_{k=0}^{n} w_{m,r}(n,k)x^k, \quad \text{(see [9])}. \tag{6}$$

where $(x+r|m)^{\bar{n}} = (x+r)(x+r+m)\cdots(x+r+(n-1)m)$ and $(x+r|m)^{\bar{0}} = 0$.

Further information for the (r-)Whitney numbers can be found in [1, 2, 6-9, 13,14].

In this paper, we first introduce the generalized degenerate Jindalrae-Stirling numbers of the first and second kind, respectively, and investigate some properties for these numbers. Moreover, we consider weak Jindalrae and Gaenari polynomials, respectively. In addition, we explore some combinatorial identities for the degenerate r-Dowling polynomials and numbers of the first kind including the generalized degenerate Jindalrae-Stirling numbers and the weak Ganari polynomials.

From (6), by replacing x with $-x$, we have

$$m^n\left(\frac{x-r}{m}\right)_n = \sum_{k=0}^{n}(-1)^{n-k}w_{m,r}(n,k)x^k, \quad \text{(see [6])}. \tag{7}$$

From (7), by replacing x with $mx+r$, we have

$$m^n(x)_n=\sum_{k=0}^{n}(-1)^{n-k}w_{m,r}(n,k)(mx+r)^k, \quad \text{(see [6])}.$$

We note that

$$w_{m,1}(n,k)=w_m(n,k), \quad w_{1,r}(n,k)=S_1^{(r)}(n,r),$$
$$w_{m,0}(n,k)=S_1(n,r), \quad w_{m,0}(n,k)=m^{n-k}S_1(n,r)$$

For $n\geq 0$, the Stirling numbers of the first and second kind are defined by respectively

(8) $$(x)_n=\sum_{l=0}^{n}S_1(n,l)x_l, \quad \text{and} \quad \frac{1}{k!}(\log(1+t))^k=\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!}, \quad \text{(see [3, 4, 7, 12])}$$

and

(9) $$x^n=\sum_{l=0}^{n}S_2(n,l)(x)_l, \frac{1}{k!} \quad \text{and} \quad (e^t-1)^k=\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!}, \quad \text{(see [3, 4, 7, 12])},$$

where $(x)_0=1$ and $(x)_n=x(x-1)(x-2)\cdots(x-n+1)$.

It is well known that the ordinary Bell polynomials and the generating function of them are given by

(10) $$bel_n(x)=\sum_{k=0}^{n}S_2(n,k)x^k, \quad \text{and} \quad e^{x(e^t-1)}=\sum_{n=0}^{\infty}bel_n(x)\frac{t^n}{n!}, \quad \text{(see [4, 7, 10])},$$

respectively.

For any $\lambda\in\mathbb{R}$,

(11) $$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \quad \text{(see [10-17])},$$

where $(x)_{0,\lambda}=1$ and $(x)_{n,\lambda}=x(x-\lambda)\cdots\big(x-(n-1)\lambda\big)$, $(n\geq 1)$.

In particular, for $m\in\mathbb{N}$, $e_m^l(t)=(1+mt)^{\frac{l}{m}}=\sum_{n=0}^{\infty}(l)_{n,m}\frac{t^n}{n!}$,
where $(l)_{n,m}=l(l-m)(l-2m)\cdots(l-(n-1)m)$.

The degenerate logarithm function $\log_\lambda(1+t)$ is given by

(12) $$\begin{aligned}\log_\lambda(1+t)&=\sum_{n=1}^{\infty}\lambda^{n-1}(1)_{n,1/\lambda}\frac{t^n}{n!}\\&=\frac{1}{\lambda}\sum_{n=1}^{\infty}(\lambda)_n\frac{t^n}{n!}=\frac{1}{\lambda}\big((1+t)^\lambda-1\big), \quad \text{(see [10, 11])}.\end{aligned}$$

Here, $\log_\lambda(t)=\frac{1}{\lambda}(t^\lambda-1)$ is the compositional inverse of $e_\lambda(t)$ satisfying $\log_\lambda(e_\lambda(t)=e_\lambda(\log_\lambda(t))=t$.
When $\lambda=1$, $(x)_0=1$ and $(x)_n=x(x-1)\cdots\big(x-(n-1))\big)$, $(n\geq 1)$.

The degenerate Stirling numbers of the first and second kind are given by as follows:

(13)
$$(x)_{n,\lambda}=\sum_{l=0}^{n}S_{2,\lambda}(n,l)(x)_l \quad \text{and} \quad (x)_n=\sum_{l=0}^{n}S_{1,\lambda}(n,l)(x)_{l,\lambda} \quad (n\geq 0), \quad \text{(see [12, 17])}.$$

From (13), it is well known that

$$\frac{1}{k!}\big(e_\lambda(t)-1\big)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!}\quad (k\geq 0),\qquad \text{(see [12, 17])}, \tag{14}$$

and

$$\frac{1}{k!}\big(\log_\lambda(1+t)\big)^k=\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!}\quad (k\geq 0),\qquad \text{(see [12, 17])}. \tag{15}$$

The Bell polynomials associated with degenerate Stirling numbers of the second kind are given by

$$bel_{n,\lambda}^{(2)}(x)=\sum_{j=1}^{n}S_{2,\lambda}(n,j)x^j,\quad (n\geq 0),\qquad \text{(see [13, 15])}. \tag{16}$$

When $x=1$, we denote $bel_{n,\lambda}^{(2)}(1)$ simply by $bel_{n,\lambda}^{(2)}$.

From (16), the generating function of $bel_{n,\lambda}^{(2)}(x)$ is

$$e^{x(e_\lambda(t)-1)}=\sum_{n=0}^{\infty}bel_{n,\lambda}^{(2)}(x)\frac{t^n}{n!},\qquad \text{(see [13, 15])}. \tag{17}$$

In view of (10), we consider the Bell polynomials associated with degenerate Stirling numbers of the first kind given by

$$bel_{n,\lambda}^{(1)}(x)=\sum_{j=1}^{n}S_{1,\lambda}(n,j)x^j,\quad (n\geq 0). \tag{18}$$

When $x=1$, we denote $bel_{n,\lambda}^{(1)}(1)$ simply by $bel_{n,\lambda}^{(1)}$.

From (18), the generating function of $bel_{n,\lambda}^{(1)}(x)$ is

$$e^{x(\log_\lambda(t+1))}=\sum_{n=0}^{\infty}bel_{n,\lambda}^{(1)}(x)\frac{t^n}{n!}. \tag{19}$$

In addition, we note that

$$(1-t)^{-m}=\sum_{l=0}^{\infty}\binom{-m}{l}(-1)^l t^l=\sum_{l=0}^{\infty}<m>_l\frac{t^l}{l!},\qquad \text{(see [7])}. \tag{20}$$

where $<x>_0=1$, $<x>_n=x(x+1)(x+2)\cdots(x+n-1)$, $(n\geq 1)$.

2. Generalized degenerate Jindalrae-Stirling numbers of the first and second kind

In this section, we introduce the generalized degenerate Jindalrae-Stirling numbers of the first and second kind respectively and investigate some properties for these numbers.

Now, we introduce the generalized degenerate Jindalrae-Stirling numbers of the first and second kind respectively, as follows :

$$\frac{1}{k!}(\log_{\lambda_1}(\log_{\lambda_2}(1+t)+1)^k=\sum_{n=k}^{\infty}S_{J,\lambda_1,\lambda_2}^{(1)}(n,k)\frac{t^n}{n!},\quad (\lambda_1,\lambda_2\in\mathbb{R},\ k\geq 0) \tag{21}$$

and

$$\frac{1}{k!}(e_{\lambda_1}(e_{\lambda_2}(t)-1)-1)^k=\sum_{n=k}^{\infty}S^{(2)}_{J,\lambda_1,\lambda_2}(n,k)\frac{t^n}{n!},\quad (\lambda_1,\lambda_2\in\mathbb{R},\ k\geq 0). \tag{22}$$

When $\lambda_1=\lambda_2=\lambda$, $S^{(1)}_{J,\lambda}:=S^{(1)}_{J,\lambda,\lambda}$ and $S^{(2)}_{J,\lambda}:=S^{(2)}_{J,\lambda,\lambda}$ are the degenerate Jindalrae-Stirling numbers of the first and second kind, respectively.

Theorem 1. *For $n\geq k\geq 0$ with $n\geq k$, we have*

$$S^{(1)}_{J,\lambda_1,\lambda_2}(n,k)=\sum_{l=k}^{n}S_{1,\lambda}(l,k)S_{1,\lambda_2}(n,l) \tag{23}$$

and

$$S^{(2)}_{J,\lambda_1,\lambda_2}(n,k)=\sum_{l=k}^{n}S_{2,\lambda_1}(n,l)S_{2,\lambda_2}(l,k). \tag{24}$$

Proof. From (15) and (21), we observe that

$$\begin{aligned}\frac{1}{k!}(\log_{\lambda_1}(\log_{\lambda_2}(1+t)+1))^k&=\sum_{l=k}^{\infty}S_{1,\lambda_1}(l,k)\frac{1}{l!}(\log_{\lambda_2}(1+t))^l\\&=\sum_{l=k}^{\infty}S_{1,\lambda_1}(l,k)\sum_{n=l}^{\infty}S_{1,\lambda_2}(n,l)\frac{t^n}{n!}\\&=\sum_{n=k}^{\infty}\left(\sum_{l=k}^{n}S_{1,\lambda_1}(l,k)S_{1,\lambda_2}(n,l)\right)\frac{t^n}{n!}.\end{aligned} \tag{25}$$

By (21) and (25), we have the first identity (23).
In the same way, from (14) and (22), we easily get the second identity (24).

□

Remark. When $k=1$, we note that

$$S^{(1)}_{J,\lambda_1,\lambda_2}(n,l)=\sum_{k=1}^{n}S_{1,\lambda_1}(l,1)S_{1,\lambda_2}(n,l)=\sum_{k=1}^{n}(l-1)!\binom{\lambda_1-1}{l-1}S_{1,\lambda_2}(n,l).$$

In view of (10), we naturally consider the Bell polynomials of degenerate Jindalrae-Stirling numbers of the first and second kind, respectively given by

$$\tilde{g}_{n,\lambda_1,\lambda_2}(x)=\sum_{k=0}^{n}S^{(1)}_{J,\lambda_1,\lambda_2}(n,k)x^k \tag{26}$$

and

$$\tilde{J}_{n,\lambda_1,\lambda_2}(x)=\sum_{k=0}^{n}S^{(2)}_{J,\lambda_1,\lambda_2}(n,k)x^k. \tag{27}$$

When $x=1$, $\tilde{g}_{n,\lambda_1,\lambda_2}:=\tilde{g}_{n,\lambda_1,\lambda_2}(1)$ and $\tilde{J}_{n,\lambda_1,\lambda_2}:=\tilde{J}_{n,\lambda_1,\lambda_2}(1)$ are called the Bell numbers of the degenerate Jindalrae-Stirling numbers of the first and second kind, respectively.
Throughout this paper, $\exp(t)=e^t$.

From (21) and (26), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\tilde{g}_{n,\lambda_1,\lambda_2}(x)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}S^{(1)}_{J,\lambda_1,\lambda_2}(n,k)x^k\right)\frac{t^n}{n!}\\&=\sum_{k=0}^{\infty}x^k\frac{1}{k!}(\log_{\lambda_1}(\log_{\lambda_2}(1+t)+1))^k\\&=\exp(x(\log_{\lambda_1}(\log_{\lambda_2}(1+t)+1))).\end{aligned}\tag{28}$$

In the same way, from (22) and (27), we easily get

$$\sum_{n=0}^{\infty}\tilde{J}_{n,\lambda_1,\lambda_2}(x)\frac{t^n}{n!}=\exp(x(e_{\lambda_1}(e_{\lambda_2}(t)-1)-1)).\tag{29}$$

From (28) and (29), we have the generating function of $\tilde{g}_{n,\lambda_1,\lambda_2}(x)$ as

$$\exp(x(\log_{\lambda_1}(\log_{\lambda_2}(1+t)+1))=\sum_{n=0}^{\infty}\tilde{g}_{n,\lambda_1,\lambda_2}(x)\frac{t^n}{n!},\tag{30}$$

and the generating function of $\tilde{J}_{n,\lambda_1,\lambda_2}(x)$ as

$$\exp(xe_{\lambda_1}(e_{\lambda_2}(t)-1))=\sum_{n=0}^{\infty}\tilde{J}_{n,\lambda_1,\lambda_2}(x)\frac{t^n}{n!},\tag{31}$$

respectively.

From now on, we call the Bell polynomials of degenerate Stirling numbers of the first and second kind simply as the weak Gaenari polynomials and the weak Jindalrae polynomials respectively.

Theorem 2. *For $n\geq 0$, we have*

$$bel^{(2)}_{n,\lambda_1}(x)=\sum_{l=0}^{n}S_{1,\lambda_2}(n,l)\tilde{J}_{l,\lambda_1,\lambda_2}(x),\tag{32}$$

and the inverse formula of (32) *as*

$$\tilde{J}_{l,\lambda_1,\lambda_2}(x)=\sum_{l=0}^{n}S_{2,\lambda_2}(n,l)bel^{(2)}_{l,\lambda_1}(x).\tag{33}$$

Proof. By replacing t by $\log_{\lambda_2}(1+t)$ in (31), we have

$$\begin{aligned}\exp(x(e_{\lambda_1}(t)-1))&=\sum_{l=0}^{\infty}\tilde{J}_{l,\lambda_1,\lambda_2}(x)\frac{1}{l!}(\log_{\lambda_2}(1+t))^l\\&=\sum_{l=0}^{\infty}\tilde{J}_{l,\lambda_1,\lambda_2}(x)\sum_{n=l}^{\infty}S_{1,\lambda_2}(n,l)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\tilde{J}_{l,\lambda_1,\lambda_2}(x)S_{1,\lambda_2}(n,l))\frac{t^n}{n!}\right).\end{aligned}\tag{34}$$

From (17) and (34), we have

$$bel_{n,\lambda_1}^{(2)}(x) = \sum_{l=0}^{n} \tilde{J}_{l,\lambda_1,\lambda_2}(x) S_{1,\lambda_2}(n,l).$$

From (14) and (17), we observe that

$$\begin{aligned} \exp(x(e_{\lambda_1}(e_{\lambda_2}(t)-1)) &= \sum_{l=0}^{\infty} bel_{l,\lambda_1}^{(2)}(x) \frac{1}{l!}(e_{\lambda_2}(t)-1)^l \\ &= \sum_{l=0}^{\infty} bel_{l,\lambda_1}^{(2)}(x) \sum_{n=l}^{\infty} S_{2,\lambda_2}(n,l) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} bel_{l,\lambda_1}^{(2)}(x) S_{2,\lambda_2}(n,l) \right) \frac{t^n}{n!}. \end{aligned} \tag{35}$$

By (31) and (35), we have the inverse formula (33) of (32). □

Remark. In Theorem 2, when $k=1$, we have

$$bel_{n,\lambda_1}^{(2)} = \sum_{l=0}^{n} S_{1,\lambda_2}(n,l) \tilde{J}_{l,\lambda_1,\lambda_2} \quad \text{and} \quad \tilde{J}_{l,\lambda_1,\lambda_2} = \sum_{l=0}^{n} S_{2,\lambda_2}(n,l) bel_{l,\lambda_1}^{(2)}.$$

Theorem 3. *For $n \geq 0$, we have*

$$bel_{n,\lambda_1}^{(1)}(x) = \sum_{l=0}^{n} \tilde{g}_{l,\lambda_1,\lambda_2}(x) S_{2,\lambda_2}(n,l), \tag{36}$$

and the inverse formula of (36) *as*

$$\tilde{g}_{n,\lambda_1,\lambda_2}(x) = \sum_{l=0}^{n} S_{1,\lambda_2}(n,l) bel_{l,\lambda_1}^{(1)}(x). \tag{37}$$

Proof. By replacing t by $e_{\lambda_2}(t)-1$ in (30), from (14), we get

$$\begin{aligned} \exp(x \log_{\lambda_1}(1+t)) &= \sum_{l=0}^{\infty} \tilde{g}_{l,\lambda_1,\lambda_2}(x) \frac{1}{l!}(e_{\lambda_2}(t)-1)^l \\ &= \sum_{l=0}^{\infty} \tilde{g}_{l,\lambda_1,\lambda_2}(x) \sum_{n=l}^{\infty} S_{2,\lambda_2}(n,l) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \tilde{g}_{l,\lambda_1,\lambda_2}(x) S_{2,\lambda_2}(n,l) \right) \frac{t^n}{n!}. \end{aligned} \tag{38}$$

On the other hand, from (15) and (19), we get

$$\begin{aligned} \exp(x\log_{\lambda_1}(1+t)) &= \sum_{l=0}^{\infty} x^l \frac{1}{l!}(\log_{\lambda_1}(1+t))^n \\ &= \sum_{l=0}^{\infty} x^l \sum_{n=l}^{\infty} S_{1,\lambda_1}(n,l)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} S_{1,\lambda_1}(n,l)x^l\right)\frac{t^n}{n!} = \sum_{n=0}^{\infty} bel_{n,\lambda_1}^{(1)}(x)\frac{t^n}{n!}. \end{aligned} \tag{39}$$

By (38) and (39), we have the first identity (36).

From (15) and(19), we observe that

$$\begin{aligned} \exp(x(\log_{\lambda_1}(\log_{\lambda_2}(1+t)+1)) &= \sum_{l=0}^{\infty} bel_{l,\lambda_1}^{(1)}(x)\frac{1}{l!}(\log_{\lambda_2}(1+t))^l \\ &= \sum_{l=0}^{\infty} bel_{l,\lambda_1}^{(1)}(x)\sum_{n=l}^{\infty} S_{1,\lambda_2}(n,l)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} S_{1,\lambda_2}(n,l)bel_{l,\lambda_1}^{(1)}(x)\right)\frac{t^n}{n!}. \end{aligned} \tag{40}$$

By (30) and (40), we have the second identity (37).

□

Remark. In Theorem 3, when $x = 1$, we have

$$bel_{n,\lambda_1}^{(1)} = \sum_{l=0}^{n} S_{2,\lambda_2}(n,l)\tilde{g}_{l,\lambda_1,\lambda_2} \quad \text{and} \quad \tilde{g}_{n,\lambda_1,\lambda_2} = \sum_{l=0}^{n} S_{1,\lambda_2}(n,l)bel_{l,\lambda_1}^{(1)}.$$

Theorem 4. *For $n \geq 0$, we have*

$$x^n = \sum_{l=0}^{n} \tilde{g}_{l,\lambda_1,\lambda_2}(x)S_{J,\lambda_2,\lambda_1}^{(2)}(n,l). \tag{41}$$

In particular,

$$1 = \sum_{l=0}^{n} \tilde{g}_{l,\lambda_1,\lambda_2}S_{J,\lambda_2,\lambda_1}^{(2)}(n,l).$$

Proof. By replacing t by $e_{\lambda_2}(e_{\lambda_1}(t)-1)-1$ in (30), we observe that

$$\begin{aligned} e^{xt} &= \sum_{l=0}^{\infty} \tilde{g}_{l,\lambda_1,\lambda_2}(x)\frac{1}{l!}(e_{\lambda_2}(e_{\lambda_1}(t)-1)-1)^l \\ &= \sum_{l=0}^{\infty} \tilde{g}_{l,\lambda_1,\lambda_2}(x)\sum_{n=l}^{\infty} S_{J,\lambda_2,\lambda_1}^{(2)}(n,l)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} \tilde{g}_{l,\lambda_1,\lambda_2}(x)S_{J,\lambda_2,\lambda_1}^{(2)}(n,l)\right) \end{aligned} \tag{42}$$

From (32), we get the desired identity (41).

□

3. DEGENERATE r-DOWLING POLYNOMIALS AND NUMBERS OF THE FIRST KIND

As further generalizations of the r-Whitney numbers of the first kind, Kim-Kim [16] introduced only the definition of the degenerate r-Whitney numbers of the first kind. In this section, we derive some combinatorial identities for the degenerate r-Dowling polynomials and numbers of the first kind containing the generalized Jindalrae-Stirling numbers and the Ganari polynomials.

From (5), we naturally consider the degenerate r-Whitney numbers $w_{m,r,\lambda}(n,k)$ of the first kind by

$$m^n(x)_n=\sum_{k=0}^{n}(-1)^{n-k}w_{m,r,\lambda}(n,k)(mx+r)_{k,\lambda},\qquad \text{(see [6])} \tag{43}$$

We note that $\lim_{\lambda\to 0}w_{m,r,\lambda}(n,k)=w_{m,r}(n,k)$.

Lemma 5. [16] *For $n\geq k\geq 0$ and $m,r\in\mathbb{N}$, we have*

$$\sum_{n=k}^{\infty}(-1)^{n-k}w_{m,r,\lambda}(n,k)\frac{t^n}{n!}=e_m^{-r}(t)\frac{1}{k!}\big(\log_\lambda e_m(t)\big)^k.$$

Proof. By replacing x with $\frac{x-r}{m}$ in (43), we have

$$m^n\left(\frac{x-r}{m}\right)_n=\sum_{k=0}^{n}(-1)^{n-k}w_{m,r,\lambda}(n,k)(x)_{k,\lambda}. \tag{44}$$

From (44), we observe that

$$(1+mt)^{\frac{x-r}{m}}=\sum_{n=0}^{\infty}\left(\frac{x-r}{m}\right)_n m^n\frac{t^n}{n!}=\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}(-1)^{n-k}w_{m,r,\lambda}(n,k)\frac{t^n}{n!}(x)_{k,\lambda}. \tag{45}$$

On the other hand, by (11), we have

$$\begin{aligned}(1+mt)^{\frac{x-r}{m}}&=(1+mt)^{\frac{x}{m}}(1+mt)^{-\frac{r}{m}}=e_\lambda^x\big(\log_\lambda(1+mt)^{\frac{1}{m}}\big)(1+mt)^{-\frac{r}{m}}\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\big(\log_\lambda(1+mt)^{\frac{1}{m}}\big)^k(1+mt)^{-\frac{r}{m}}(x)_{k,\lambda}\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\big(\log_\lambda e_m(t)\big)^k e_m^{-r}(t)(x)_{k,\lambda}.\end{aligned} \tag{46}$$

By comparing with the coefficients of (45) and (46), we obtain the desired identity. □

In view of (10), we consider the degenerate r-Dowling polynomials $\tilde{d}_{m,r,\lambda}(n,x)$ of the first kind given by

$$\tilde{d}_{m,r,\lambda}(n,x)=\sum_{j=0}^{n}(-1)^{n-j}w_{m,r,\lambda}(n,j)x^j,\quad (n\geq 0). \tag{47}$$

When $x=1$, $\tilde{d}_{m,r,\lambda}(n)=\tilde{d}_{m,r,\lambda}(n,1)$ are called the degenerate r-Dowling polynomials of the first kind.

When $r=1$, $\tilde{d}_{m,\lambda}(n,x)=\tilde{d}_{m,1,\lambda}(n,1)$ are called the degenerate Dowling polynomials of the first kind, and $\tilde{d}_{m,\lambda}(n)=\tilde{d}_{m,\lambda}(n,1)$ are called the degenerate Dowling numbers of the first kind.

Theorem 6. *For $m,r\in\mathbb{N}$, the generating function of $\tilde{d}_{m,r,\lambda}(n,x)$ is*

$$\sum_{n=0}^{\infty}\tilde{d}_{m,r,\lambda}(n,x)\frac{t^n}{n!}=e_m^{-r}(t)\exp\big(x\log_\lambda e_m(t)\big).$$

Proof. From Lemma 1 and (47), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\tilde{d}_{m,r,\lambda}(n,x)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}(-1)^{n-j}w_{m,r,\lambda}(n,j)x^j\right)\frac{t^n}{n!}\\&=\sum_{j=0}^{\infty}x^j\sum_{n=j}^{\infty}(-1)^{n-j}w_{m,r,\lambda}(n,j)\frac{t^n}{n!}\\&=e_m^{-r}(t)\sum_{j=0}^{\infty}x^j\frac{1}{j!}\big(\log_\lambda e_m(t)\big)^j=e_m^{-r}(t)\exp\big(x\log_\lambda e_m(t)\big).\end{aligned}\tag{48}$$

By (48), we have the desired the identity. □

Theorem 7. *For $r,m\in\mathbb{N}$, we have*

$$\sum_{l=0}^{n}\binom{n}{l}m^{n-l}\left(\frac{r}{m}\right)_{n-l}\tilde{d}_{m,r,\lambda}(l,x)=\sum_{l=0}^{n}S_{J,\lambda,m}^{(2)}(n,l)\tilde{g}_{l,\lambda,\lambda}(x),$$

where $S_{J,\lambda,m}^{(2)}(n,l)$ and $\tilde{g}_{l,\lambda,\lambda}(x)$ are the generalized Jindalrae-Stirling numbers of the second kind and the weak Gaenari polynomials, respectively.

Proof. Replacing t by $e_\lambda(e_m(t)-1)-1$ in (30), we get

$$\begin{aligned}\exp(x\log_\lambda e_m(t))&=\sum_{l=0}^{\infty}\tilde{g}_{l,\lambda,\lambda}(x)\frac{(e_\lambda(e_m(t)-1)-1)^l}{l!}\\&=\sum_{l=0}^{\infty}\tilde{g}_{l,\lambda,\lambda}(x)\sum_{n=l}^{\infty}S_{J,\lambda,m}^{(2)}(n,l)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}S_{J,\lambda,m}^{(2)}(n,l)\tilde{g}_{l,\lambda,\lambda}(x)\right)\frac{t^n}{n!}.\end{aligned}\tag{49}$$

On the other hand, by Theorem 6, we note that

$$\begin{aligned}\exp(x\log_\lambda e_m(t))&=e_m^{r}(t)\sum_{l=0}^{\infty}\tilde{d}_{m,r,\lambda}(l,x)\frac{t^l}{l!}\\&=\sum_{j=0}^{\infty}(r)_{j,m}\frac{t^j}{j!}\sum_{l=0}^{\infty}\tilde{d}_{m,r,\lambda}(l,x)\frac{t^l}{l!}\\&=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}(r)_{n-l,m}\tilde{d}_{m,r,\lambda}(l,x)\frac{t^n}{n!},\end{aligned}\tag{50}$$

where $(r)_{n-l,m} = r(r-m)\cdots(r-(n-l-1)m)$.
From (49) and (50), we have the desired identity. □

Theorem 8. *For $n \geq k \geq 0$ and $m, r \in \mathbb{N}$, we have*

$$w_{m,r,\lambda}(n,j) = w_{m,r,\lambda}(n-1,j-1) + (m-nm-1)w_{m,r,\lambda}(n-1,j) + j\lambda w_{m,r,\lambda}(n-1,j).$$

Proof. From (43), we observe that

(51)

$$\begin{aligned}
&\sum_{j=0}^{n}(-1)^{n-j}w_{m,r,\lambda}(n,j)(mx+r)_{j,\lambda} = m^n(x)_n = m(x-n+1)m^{n-1}(x)_{n-1}\\
&\quad= \sum_{j=0}^{n-1}(-1)^{n-j}w_{m,r,\lambda}(n-1,j)(mx+r)_{j,\lambda}(mx+r-j\lambda+j\lambda+m-nm-r)\\
&\quad= \sum_{j=0}^{n-1}(-1)^{n-j}w_{m,r,\lambda}(n-1,j)(mx+r)_{j+1,\lambda}\\
&\qquad\qquad+\sum_{j=0}^{n-1}(-1)^{n-j}(j\lambda+m-nm-r)w_{m,r,\lambda}(n-1,j)(mx+r)_{j,\lambda}\\
&\quad= \sum_{j=0}^{n}(-1)^{n-j}\Big(w_{m,r,\lambda}(n-1,j-1)+(j\lambda+m-nm-r)w_{m,r,\lambda}(n-1,j)\Big)(mx+r)_{j,\lambda}.
\end{aligned}$$

By (51), we have what we want. □

In Theorem 8, $w_{m,r,\lambda}(n,n) = w_{m,r,\lambda}(n-1,n-1) = \cdots = w_{m,r,\lambda}(0,0) = 1$.
When $\lambda \to 0$, we have

$$w_{m,r}(n,j) = w_{m,r}(n-1,j-1) + (m-nm-1)w_{m,r}(n-1,j).$$

Theorem 9. *For $m,\ r, \in \mathbb{N}$, we have*

$$(-1)^{j+d}\sum_{d=j}^{n}\sum_{l=j}^{d}\binom{n}{d}m^{n-d}\left\langle\frac{r}{m}\right\rangle_{n-d}S_{1,\lambda}(l,j)S_{2,m}(n,l) = \begin{cases} w_{m,r,\lambda}(n,j) & \text{if} \quad n \geq j \geq 0,\\ 0 & \text{if} \quad \textit{otherwise}, \end{cases}$$

where $\langle x\rangle_n = x(x+1)\cdots(x+(n-1))$ and $\langle x\rangle_0 = 1$.

Proof. From (8), (9), (44) and Lemma 1, we observe that

$$
\begin{aligned}
\sum_{n=j}^{\infty}(-1)^{n-j}w_{m,r,\lambda}(n,j)\frac{t^n}{n!} &= e_m^{-r}(t)\frac{1}{j!}\left(\log_\lambda e_m(t)\right)^j \\
&= e_\lambda^{-r}\sum_{l=j}^{\infty}S_{1,\lambda}(l,j)\frac{1}{l!}\left(e_m(t)-1\right)^l \\
&= e_\lambda^{-r}\sum_{l=j}^{\infty}S_{1,\lambda}(l,j)\sum_{d=l}^{\infty}S_{2,m}(n,l)\frac{t^d}{d!} \qquad (52)\\
&= \sum_{v=0}^{\infty}(-r)_{i,m}\frac{t^v}{v!}\sum_{d=j}^{\infty}\sum_{l=j}^{d}S_{1,\lambda}(l,j)S_{2,m}(n,l)\frac{t^d}{d!} \\
&= \sum_{n=j}^{\infty}\sum_{d=j}^{n}\sum_{l=j}^{d}\binom{n}{d}(-1)^{n-d}m^{n-d}\left\langle\frac{r}{m}\right\rangle_{n-d}S_{1,\lambda}(l,j)S_{2,m}(n,l)\frac{t^n}{n!}.
\end{aligned}
$$

By comparing with the coefficients of both side of (52), we obtain the result. □

Theorem 10. *For $n \geq j \geq 0$ and $m, r \in \mathbb{N}$, we have*

$$\tilde{d}_{m,r,\lambda}(n,x) = \sum_{\alpha=0}^{n}\sum_{j=0}^{\alpha}\sum_{l=j}^{\alpha}\binom{n}{\alpha}(-1)^{n-\alpha}m^{n-\alpha}\left\langle\frac{r}{m}\right\rangle_{n-\alpha}S_{1,\lambda}(l,j)S_{2,m}(\alpha,l)x^j.$$

In addition,

$$
\begin{aligned}
\tilde{d}_{m,r,\lambda}(n,x) = \sum_{\alpha=0}^{n}\sum_{j=0}^{\alpha}\sum_{l=j}^{\alpha}\sum_{d=0}^{j}\binom{n}{\alpha}(-1)^{n-\alpha}m^{n-\alpha}\left\langle\frac{r}{m}\right\rangle_{n-\alpha} \\
\times S_{1,\lambda}(l,j)S_{2,m}(\alpha,l)\tilde{g}_{d,\lambda,\lambda}(x)S^{(2)}_{J,\lambda,\lambda}(j,d),
\end{aligned}
$$

where $\tilde{g}_{l,\lambda,\lambda}(x)$ are the weak Gaenari polynomials and $S^{(2)}_{J,\lambda,\lambda}(j,d)$ are the generalized Jindalrae-Stirling numbers of the second kind.

Proof. From (13) and Theorem 2, we get

(53)
$$\begin{aligned}\sum_{n=0}^{\infty}\tilde{d}_{m,r,\lambda}(n,x)\frac{t^n}{n!} &= e_m^{-r}(t)\exp\left(x\log_\lambda e_m(t)\right)\\ &= e_m^{-r}(t)\sum_{j=0}^{\infty}x^j\frac{1}{j!}(\log_\lambda((e_m(t)-1)+1))^j\\ &= e_m^{-r}(t)\sum_{j=0}^{\infty}x^j\sum_{l=j}^{\infty}S_{1,\lambda}(l,j)\frac{1}{l!}(e_m(t)-1)^l\\ &= e_m^{-r}(t)\sum_{j=0}^{\infty}x^j\sum_{l=j}^{\infty}S_{1,\lambda}(l,j)\sum_{\alpha=l}^{\infty}S_{2,m}(\alpha,l)\frac{t^\alpha}{\alpha!}\\ &= \sum_{i=0}^{\infty}(-r)_{i,m}\frac{t^i}{i!}\sum_{\alpha=0}^{\infty}\sum_{j=0}^{\alpha}\sum_{l=j}^{\alpha}S_{1,\lambda}(l,j)S_{2,m}(\alpha,l)x^j\frac{t^\alpha}{\alpha!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{\alpha=0}^{n}\sum_{j=0}^{\alpha}\sum_{l=j}^{\alpha}\binom{n}{\alpha}(-1)^{n-\alpha}m^{n-\alpha}\left\langle\frac{r}{m}\right\rangle_{n-\alpha}S_{1,\lambda}(l,j)S_{2,m}(\alpha,l)x^j\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing with the coefficients of both side of (53), we have the first identity.
Moreover, by using Theorem 4, we get the second identity. □

Corollary 11. *For $n\geq j\geq 0$, we have*

$$(-1)^{n-j}\sum_{\alpha=j}^{n}\sum_{l=j}^{\alpha}\binom{n}{\alpha}(-r)_{n-\alpha,m}S_{1,\lambda}(l,j)S_{2,m}(\alpha,l)=\begin{cases} w_{m,r,\lambda}(n,j) & \text{if} \quad n\geq j\\ 0 & \text{if} \quad \textit{otherwise}\end{cases}$$

Proof. From (13) and Lemma 1, we note that

(54)
$$\begin{aligned}\sum_{n=j}^{\infty}(-1)^{n-j}w_{m,r,\lambda}(n)\frac{t^n}{n!} &= e_m^{-r}(t)\frac{1}{j!}\left(\log_\lambda(1+(e_m(t)-1))\right)^j\\ &= e_m^{-r}(t)\sum_{l=j}^{\infty}S_{1,\lambda}(l,j)\sum_{\alpha=l}^{\infty}S_{2,m}(\alpha,l)\frac{t^\alpha}{\alpha!}\\ &= \sum_{i=0}^{\infty}(-r)_{i,m}\frac{t^i}{i!}\sum_{\alpha=j}^{\infty}\sum_{l=j}^{\alpha}S_{1,\lambda}(l,j)S_{2,m}(\alpha,l)\frac{t^\alpha}{\alpha!}\\ &= \sum_{n=j}^{\infty}\left(\sum_{\alpha=j}^{n}\sum_{l=j}^{\alpha}\binom{n}{\alpha}(-r)_{n-\alpha,m}S_{1,\lambda}(l,j)S_{2,m}(\alpha,l)\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing with the coefficients of both side of (54), we have the desired result. □

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