

ENTIRE FUNCTIONS SHARING SOME VALUES WITH THEIR DIFFERENCE OPERATORS AND DERIVATIVES

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Abstract. In this paper we study the uniqueness of entire functions sharing some values with their difference operator and derivatives. The idea of entire and meromorphic functions relies heavily on this direction. Li Ping and Yang improved the result given by Rubel and Yang and proved that if $f(z)$ is a non-constant entire function and a, b are two finite distinct complex values and if $f(z)$ and $f^{(k)}(z)$ share a counting multiplicities and b ignoring multiplicities then $f(z) \equiv f^{(k)}(z)$. By replacing finite distinct complex values by polynomials, we prove the following result: Let $\Delta f(z)$ be transcendental entire functions of finite order, $k \geq 0$, $n \geq 0$ be integers and P_1 and P_2 be two polynomials. If $\Delta f(z)$ and $(f^n)^{(k)}$ share P_1 CM and share P_2 IM, then $\Delta f \equiv (f^n)^{(k)}$.

1. Introduction and Main Results

The reader is presumed to be familiar with the fundamental notations and conclusions of Nevanlinna's value distribution theory of meromorphic functions [1, 2]. $S(r, f)$ means that $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$ outside of a possible exceptional set of finite logarithmic measure, and

$$\lambda(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ N\left(r, \frac{1}{f}\right)}{\log r}, \quad \delta(P_1, f) = \liminf_{r \rightarrow \infty} \frac{m\left(r, \frac{1}{f-a}\right)}{T(r, f)} = 1 - \limsup_{r \rightarrow \infty} \frac{N\left(r, \frac{1}{f}\right)}{T(r, f)}$$

stand for the exponents of convergence of zero sequence of f and the deficiency of f at the point a , respectively. For a nonconstant meromorphic function h , we denote by $T(r, h)$ the Nevanlinna characteristic of h and by $S(r, h)$ any quantity satisfying $S(r, h) = o(T(r, h))$, as r runs to infinity outside of a set $E \subset (0, +\infty)$ of finite linear measure. We say that h is a small function of f , if $T(r, h) = S(r, f)$. In the sequel, we denote by I a set of infinite linear measure not necessarily the same in all its occurrences.

We say that f and g share the value a IM (ignoring multiplicities), if f and g have the same a point. If f and g have the same a point with the same multiplicities, then we say f, g share the value a CM (counting multiplicities).

Definition 1.1. [3] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a; f)$ the set of all a points of $f(z)$ where an a point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, then we say that f, g share the value a with weight k .

We write f and g share (a, k) to mean that f, g share the value a with weight k .

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Rubel and Yang Chung-Chun [4] considered the uniqueness of an entire function and its derivative. They proved the following.

Theorem 1. Let $f(z)$ be a non-constant entire function, let a, b be two finite distinct values. If $f(z)$ and $f'(z)$ share a, bCM , then $f(z) \equiv f'(z)$.

Li Ping and Yang Chung-Chun [5] improved Theorem 1 and proved.

Theorem 2. Let $f(z)$ be a non-constant entire function, and let a, b be two finite distinct complex values. If $f(z)$ and $f^{(k)}(z)$ share aCM , and share b IM. Then $f(z) \equiv f^{(k)}(z)$.

The value distribution of meromorphic functions of finite order with respect to difference analogue has become a subject of some interests, see [6]-[16].

Theorem 3. [17] Suppose that $f(z)$ and $g(z)$ are nonconstant meromorphic functions. If f, g share $0, 1, \infty CM$ and $\bar{N}\left(r, \frac{1}{f}\right) + \bar{N}(r, f) < (d + o(1))T(r, f)$ for $r \in I$ and $r \in \infty$, where d is a positive number satisfying $0 < d < \frac{1}{2}$, which $I \subset (0, +\infty)$ is a subset of infinite linear measure, then $f = g$ or $f \cdot g = 1$.

Theorem 4. [4] Let f be a nonconstant entire function. If f shares two distinct finite values CM with f' , then $f \equiv f'$.

More results on uniqueness of f' with its n -th derivative $f^{(n)}$ were obtained by several authors (see [?]). In view of the progress on the difference analogues of classical Nevanlinna theory of meromorphic functions [21, 22], it is quite natural to investigate the uniqueness problems of meromorphic functions and their difference operators (see [23]-[26]).

Example 1.1. Let $f(z) = e^{Az}$, where $A \neq 0$ is a constant. Then $(f^n)^{(k)} = (A^n)^k e^{Az}$ and $\Delta f = f(z+1) - f(z) = n(e^A - 1)e^{Az}$. Clearly, $\Delta(f)$ and $(f^n)^{(k)}$ share 0 CM and ∞ IM, and that $\mu = 1$. We can choose A such that $e^A - 1 \neq nA^k$, and so $f' \not\equiv \Delta(f)$.

Theorem 5. [27] Let $f(z)$ be a transcendental entire function of finite order, let $\eta \neq 0$ be a finite complex number, $n \geq 1, k \geq 0$ be two integers and let a, b be two distinct finite complex values. If $f(z)$ and $(\Delta_\eta^n(f(z)))^{(k)}$ share aCM and share b IM, then $f(z) \equiv (\Delta_\eta^n(f(z)))^{(k)}$.

Lemma 1.1. [9] Let Δf be a nonconstant meromorphic function of finite order, let $\eta \neq 0$ be a finite complex number. Then

$$m\left(r, \frac{\Delta f(z + \eta)}{\Delta f}\right) = S(r, \Delta f)$$

for all r outside of a possible exceptional set E with finite logarithmic measure.

Lemma 1.2. [28], Lemma 4.3]. Let Δf be a nonconstant meromorphic function. Suppose that the polynomials $P_j, j = 0, 1, \dots, q, q > p$, and let $P(\Delta f) = a_0(\Delta f)^p + a_1(\Delta f)^{p-1} + \dots + a_p$ ($a_0 \neq 0$) is a polynomial of degree p with constant coefficient $a_j, j = 0, 1, \dots, p$. Then

$$m\left(r, \frac{P(\Delta f)(\Delta f)'}{(\Delta f - P_1)(\Delta f - P_2)\dots(\Delta f - P_q)}\right) = S(r, \Delta f).$$

Lemma 1.3. *Let Δf and Δg be two non constant entire functions, and let P_1, P_2 be two polynomials. If*

$$H = \frac{\Delta f'}{(\Delta f - P_1)(\Delta f - P_2)} - \frac{\Delta g'}{(\Delta g - P_1)(\Delta g - P_2)} \equiv 0$$

and Δf and Δg share P_1CM , and share P_2IM , then either

$$2T(r, \Delta f) \leq \bar{N}\left(r, \frac{1}{\Delta f - P_1}\right) + \bar{N}\left(r, \frac{1}{\Delta f - P_2}\right) + S(r, \Delta f)$$

or

$$\Delta f \equiv \Delta g$$

◁ Integrating H which leads to

$$\frac{\Delta g - P_2}{\Delta g - P_1} = C \frac{\Delta f - P_2}{\Delta f - P_1}$$

where C is a nonzero constant.

If $C = 1$, then $\Delta f \equiv \Delta g$. If $C \neq 1$, then from above, we have

$$\frac{P_1 - P_2}{\Delta g - P_1} \equiv \frac{(C - 1)\Delta f - CP_2 + P_1}{\Delta f - P_1}$$

and

$$T(r, \Delta f) = T(r, \Delta g) + S(r, \Delta f) + S(r, \Delta g)$$

Obviously, $\frac{CP_1 - P_2}{C - 1} \neq a$ and $\frac{CP_1 - P_2}{C - 1} \neq b$. It follows that $N\left(r, \frac{1}{\Delta f - \frac{CP_1 - P_2}{C - 1}}\right) = 0$. Then by the Second Fundamental Theorem,

$$\begin{aligned} 2T(r, \Delta f) &= \bar{N}(r, \Delta f) + \bar{N}\left(r, \frac{1}{\Delta f - P_1}\right) + \bar{N}\left(r, \frac{1}{\Delta f - P_2}\right) + \bar{N}\left(r, \frac{1}{\Delta f - \frac{CP_1 - P_2}{C - 1}}\right) \\ &+ S(r, \Delta f) \leq \bar{N}\left(r, \frac{1}{\Delta f - P_1}\right) + \bar{N}\left(r, \frac{1}{\Delta f - P_2}\right) + S(r, \Delta f) \end{aligned}$$

that is

$$2T(r, \Delta f) \leq \bar{N}\left(r, \frac{1}{\Delta f - P_1}\right) + \bar{N}\left(r, \frac{1}{\Delta f - P_2}\right) + S(r, \Delta f)$$

Lemma 1.4. Let Δf be a transcendental entire function of finite order, n, k be positive integer, let P_1 be a nonzero complex value or constant. If Δf and $(f^n)^{(k)}$ share P_1 CM, and $N\left(r, \frac{1}{(f^n)^{(k)}}\right) = S(r, \Delta f)$, then one of the following cases must occur:

- $(f^n)^{(k)} = He^p$, where p is a polynomial, and $H \neq 0$ is a small function of e^p .
- $T(r, e^p) = S(r, \Delta f)$

◁ Since Δf is a transcendental entire function of finite order, Δf and $(f^n)^{(k)}$ share P_1 CM, then there is a polynomial p such that

$$\Delta f - P_1 = e^p \left((f^n)^{(k)} \right) - P_1 e^p. \quad (1.1)$$

Set $g = (f^n)^{(k)}$. It follows by (1.1) that

$$g = ((ge^p)^n)^{(k)} - ((P_1 e^p)^n)^{(k)}. \quad (1.2)$$

Then we rewrite (1.2) as

$$1 + \frac{((P_1 e^p)^n)^{(k)}}{g} = De^p, \quad (1.3)$$

where

$$D = \frac{((ge^p)^n)^{(k)}}{ge^p}. \quad (1.4)$$

Note that $N\left(r, \frac{1}{(f^n)^{(k)}}\right) = N\left(r, \frac{1}{g}\right) = S(r, f)$, then by Lemma 1.1 we have

$$T(r, D) = T\left(r, \frac{((ge^p)^n)^{(k)}}{\Delta ge^p}\right).$$

Next we discuss two cases.

Case 1: $e^{-p} - D \neq 0$. Rewrite (1.3) as

$$ge^p (e^{-p} - D) = ((P_1 e^p)^n)^{(k)}. \quad (1.5)$$

When $D \equiv 0$, (1.5) implies

$$g = nHe^p. \quad (1.6)$$

Here $H \neq 0$ is a small function of e^p .

When $D \neq 0$, it follows from (1.5) that $N\left(r, \frac{1}{e^{-p}-D}\right) = S(r, f)$. Then use the Second Fundamental Theorem to e^p we can obtain

$$T(r, e^p) = T(r, e^{-p}) + O(1) \leq \bar{N}(r, e^p) + \bar{N}\left(r, \frac{1}{e^{-p}}\right) + \bar{N}\left(r, \frac{1}{e^{-p}-D}\right) + O(1) = S(r, \Delta f).$$

Case 2: $e^{-p} - D \equiv 0$. It implies that $T(r, e^p) = T(r, e^{-p}) + O(1) = S(r, \Delta f)$, a contradiction. From above discussions, we get $T(r, e^p) = S(r, \Delta f)$.

Theorem 6. Let $\Delta f(z)$ be a transcendental entire functions of finite order, n, k be integers such that $k \geq 0, n \geq 0$ and let P_1 and P_2 be two polynomials. If $\Delta f(z)$ and $(f^n)^{(k)}$ share P_1CM and share P_2IM , then $\Delta f \equiv (f^n)^{(k)}$.

\triangleleft If $\Delta f \equiv (f^n)^{(k)}$, there is nothing to prove. Solve $\Delta f \not\equiv (f^n)^{(k)}$. Since Δf is a transcendental entire function of finite order, Δf and $(f^n)^{(k)}$ share P_1CM , then we get

$$\frac{(f^n)^{(k)} - P_1}{\Delta f - P_1} = e^Q, \tag{1.7}$$

where Q is a polynomial.

Since Δf and $(f^n)^{(k)}$ share P_1CM and share P_2IM , then by second fundamental theorem and Lemma 1.1 we have

$$\begin{aligned} T(r, \Delta f) &\leq \bar{N}\left(r, \frac{1}{\Delta f - P_1}\right) + \bar{N}\left(r, \frac{1}{\Delta f - P_2}\right) + S(r, \Delta f) \\ &\leq \bar{N}\left(r, \frac{1}{\Delta f - P_2}\right) + \bar{N}\left(r, \frac{1}{(f^n)^{(k)} - P_2}\right) \leq N\left(r, \frac{1}{\Delta f - (f^n)^{(k)}}\right) + S(r, \Delta f) \\ &\leq T\left(r, \Delta f - (f^n)^{(k)}\right) + S(r, \Delta f) \leq m\left(r, \Delta f - (f^n)^{(k)}\right) + S(r, \Delta f) \\ &\leq m(r, \Delta f) + m\left(r, 1 - \frac{(f^n)^{(k)}}{\Delta f}\right) + S(r, \Delta f) \leq nT(r, f) + S(r, f), \\ T(r, \Delta f) &= \bar{N}\left(r, \frac{1}{\Delta f - P_1}\right) + \bar{N}(\Delta f - P_2) + S(r, \Delta f). \end{aligned} \tag{1.8}$$

According to (1.7) and (1.8) we have

$$T(r, \Delta f) = T\left(r, \Delta f - (f^n)^{(k)}\right) + S(r, \Delta f) \leq N\left(r, \frac{1}{\Delta f - (f^n)^{(k)}}\right) + S(r, \Delta f) \tag{1.9}$$

and

$$T(r, e^Q) = m(r, e^Q) = m\left(r, \frac{(f^n)^{(k)} - P_1}{\Delta f - P_1}\right) \leq m\left(r, \frac{1}{\Delta f - P_1}\right) + S(r, \Delta f). \tag{1.10}$$

Then it follows from (1.7) and (1.9) that

$$\begin{aligned} m\left(r, \frac{1}{\Delta f - P_1}\right) &= m\left(r, \frac{e^Q - 1}{\Delta f - (f^n)^{(k)}}\right) \leq m\left(r, \frac{1}{\Delta f - (f^n)^{(k)}}\right) + m(r, e^Q - 1) \\ &\leq T(r, e^Q) + S(r, \Delta f). \end{aligned} \tag{1.11}$$

Then by (1.10) and (1.11)

$$T(r, e^Q) = m\left(r, \frac{1}{\Delta f - P_1}\right) + S(r, \Delta f). \tag{1.12}$$

On the other hand, (1.1) can be rewritten as

$$\frac{(f^n)^{(k)} - \Delta f}{\Delta f - P_1} = e^Q - 1, \tag{1.13}$$

which implies

$$\bar{N}\left(r, \frac{1}{\Delta f - P_2}\right) \leq \bar{N}\left(r, \frac{1}{e^Q - 1}\right) = T(r, e^Q) + S(r, \Delta f). \quad (1.14)$$

Then by (1.8), (1.12) and (1.14)

$$\begin{aligned} m\left(r, \frac{1}{\Delta f - P_1}\right) + N\left(r, \frac{1}{\Delta f - P_1}\right) &= \bar{N}\left(r, \frac{1}{\Delta f - P_1}\right) + \bar{N}\left(r, \frac{1}{\Delta f - P_2}\right) + S(r, \Delta f) \\ &\leq \bar{N}\left(r, \frac{1}{\Delta f - P_1}\right) + \bar{N}\left(r, \frac{1}{e^Q - 1}\right) + S(r, \Delta f) \\ &\leq \bar{N}\left(r, \frac{1}{\Delta f - P_1}\right) + m\left(r, \frac{1}{\Delta f - P_1}\right) + S(r, \Delta f) \\ N\left(r, \frac{1}{\Delta f - P_1}\right) &= \bar{N}\left(r, \frac{1}{\Delta f - P_1}\right) + S(r, \Delta f), \end{aligned} \quad (1.15)$$

and then

$$\bar{N}\left(r, \frac{1}{\Delta f - P_2}\right) = T(r, e^Q) + S(r, \Delta f). \quad (1.16)$$

Set

$$\Phi = \frac{(\Delta f)' (\Delta f - (f^n)^{(k)})}{(\Delta f - P_1) (\Delta f - P_2)}, \quad (1.17)$$

and

$$\Psi = \frac{(f^n)^{(k+1)} (\Delta f - (f^n)^{(k)})}{((f^n)^{(k)} - P_1) ((f^n)^{(k)} - P_2)}. \quad (1.18)$$

Easy to know that Φ is an entire function by Lemma 1.1 and Lemma 1.2 we have

$$\begin{aligned} T(r, \Phi) = m(r, \Phi) &= m\left(r, \frac{(\Delta f)' (\Delta f - (f^n)^{(k)})}{(\Delta f - P_1) (\Delta f - P_2)}\right) + S(r, \Delta f) \\ &\leq m\left(r, \frac{(\Delta f)' (\Delta f)}{(\Delta f - P_1) (\Delta f - P_2)}\right) m\left(r, 1 - \frac{(f^n)^{(k)}}{\Delta f}\right) + S(r, \Delta f) = S(r, \Delta f), \end{aligned}$$

that is

$$T(r, \Phi) = S(r, \Delta f). \quad (1.19)$$

Obviously let $d = P_1 - nk(P_1 - P_2)$, $k \neq 0, n \neq 0$, by Lemmas 1.1 and 1.2, we obtain

$$\begin{aligned} m\left(r, \frac{1}{\Delta f}\right) + m\left(r, \frac{1}{(P_2 - P_1)\Phi}\right) &\left(\frac{(\Delta f)'}{\Delta f - P_1} - \frac{(\Delta f)'}{\Delta f - P_2}\right) \left(\frac{1 - (\Delta f)'}{\Delta f}\right) \\ &\leq m\left(r, \frac{1}{\Phi}\right) + m\left(r, \left(\frac{(\Delta f)'}{\Delta f - P_1} - \frac{(\Delta f)'}{\Delta f - P_2}\right)\right) + m\left(r, \frac{(\Delta f)'}{\Delta f}\right) + S(r, \Delta f) = S(r, \Delta), \end{aligned} \quad (1.20)$$

and

$$\begin{aligned}
 m\left(r, \frac{1}{\Delta f - d}\right) &= m\left(r, \frac{(\Delta f)' (\Delta f - (f^n)^{(k)})}{(\Delta f - P_1)(\Delta f - P_2)(\Delta f - d)}\right) \\
 &\leq m\left(r, \frac{1 - (f^n)^{(k)}}{\Delta f}\right) + m\left(r, \frac{(\Delta f)' (\Delta f - (f^n)^{(k)})}{(\Delta f - P_1)(\Delta f - P_2)(\Delta f - d)}\right) + S(r, \Delta f) = S(r, \Delta f).
 \end{aligned} \tag{1.21}$$

Set

$$\Phi = \frac{(f^n)^{(k+1)}}{((f^n)^{(k)} - P_1)((f^n)^{(k)} - P_2)} - \frac{(\Delta f)'}{(\Delta f - P_1)(\Delta f - P_2)}. \tag{1.22}$$

We discuss two cases

Case 1: $\Phi \equiv 0$. Integrating both side of (1.22) which leads to

$$\frac{\Delta f - P_2}{\Delta f - P_1} = c \frac{(f^n)^{(k)} - P_2}{(f^n)^{(k)} - P_1} \tag{1.23}$$

where c is a non zero constant. Then by Lemma 1.3 we see that

$$2T(r, \Delta f) \leq \bar{N}\left(r, \frac{1}{\Delta f - P_1}\right) + \bar{N}\left(r, \frac{1}{\Delta f - P_2}\right) + S(r, \Delta f), \tag{1.24}$$

which contradicts with (1.8).

Case 2 : $\Phi \not\equiv 0$. By (1.8), (1.19) and (1.22) we can obtain

$$\begin{aligned}
 m(r, \Delta f) &= m\left(r, \Delta f - (f^n)^{(k)}\right) + S(r, \Delta f) = m\left(r, \frac{\Phi (\Delta f - (f^n)^{(k)})}{\Phi}\right) + S(r, \Delta f) \\
 &= m\left(r, \frac{\Psi - \Phi}{\Phi}\right) + S(r, \Delta f) \leq T\left(r, \frac{\Phi}{\Psi - \Phi}\right) + S(r, \Delta f) \\
 &\leq T(r, \Psi - \Phi) + T(r, \Phi) + S(r, \Delta f) \leq T(r, \Psi) + \bar{N}\left(r, \frac{1}{\Delta f - P_2}\right) + S(r, \Delta f).
 \end{aligned} \tag{1.25}$$

On the other hand

$$\begin{aligned}
 T(r, \Psi) &= T\left(r, \frac{(f^n)^{(k+1)} (\Delta f - (f^n)^{(k)})}{((f^n)^{(k)} - P_1)((f^n)^{(k)} - P_2)}\right) \\
 &= m\left(r, \frac{(f^n)^{(k+1)} (\Delta f - (f^n)^{(k)})}{((f^n)^{(k)} - P_1)((f^n)^{(k)} - P_2)}\right) + S(r, \Delta f) \\
 &\leq m\left(r, \frac{(f^n)^{(k+1)}}{(f^n)^{(k)} - P_2}\right) + m\left(r, \frac{\Delta f - (f^n)^{(k)}}{(f^n)^{(k)} - P_1}\right) \\
 &\leq m\left(r, \frac{1}{\Delta f - P_1}\right) + S(r, \Delta f) = \bar{N}\left(r, \frac{1}{\Delta f - P_2}\right) + S(r, \Delta f).
 \end{aligned} \tag{1.26}$$

Hence combining (1.25) and (1.26) we obtain

$$T(r, \Delta f) \leq 2\bar{N}\left(r, \frac{1}{\Delta f - P_2}\right) + S(r, \Delta f). \quad (1.27)$$

Next, case 2 is divided into two subcases.

Subcase 2.1. $P_1 = 0$. Then by (1.7) and Lemma 1.1 we get

$$m(r, e^Q) = m\left(r, \frac{(f^n)^{(k)}}{\Delta f}\right) = S(r, \Delta f). \quad (1.28)$$

Then by (1.16), (1.27) and (1.28) we can have $T(r, \Delta f) = S(r, \Delta f)$ a contradiction.

Subcase 2.2. $P_2 = 0$. Then by (1.16), (1.27) and (1.28) and Lemma 1.1 we get

$$\begin{aligned} T(r, \Delta f) &\leq m\left(r, \frac{1}{\Delta f - P_1}\right) + \bar{N}\left(r, \frac{1}{(f^n)^{(k)}}\right) + S(r, \Delta f) \\ &\leq m\left(r, \frac{1}{(f^n)^{(k)}}\right) + \bar{N}\left(r, \frac{1}{(f^n)^{(k)}}\right) + S(r, \Delta f) \leq \left(T\left(r, (f^n)^{(k)}\right)\right) + S(r, \Delta f). \end{aligned} \quad (1.29)$$

From the fact that

$$T\left(r, (f^n)^{(k)}\right) \leq T(r, \Delta f) + S(r, \Delta f), \quad (1.30)$$

which follows from (1.29) that

$$T(r, \Delta f) = T\left(r, (f^n)^{(k)}\right) + S(r, \Delta f), \quad (1.31)$$

By second Nevanlinna Fundamental theorem, Lemma 1.1, (1.8) and (1.31) we have

$$\begin{aligned} 2T(r, \Delta f) &\leq 2T\left(r, (f^n)^{(k)}\right) + S(r, \Delta f) \\ &\leq \bar{N}\left(r, \frac{1}{(f^n)^{(k)} - P_1}\right) + \bar{N}\left(r, \frac{1}{(f^n)^{(k)}}\right) + \bar{N}\left(r, \frac{1}{(f^n)^{(k)} - d}\right) + S(r, \Delta f) \\ &\leq \bar{N}\left(r, \frac{1}{\Delta f - P_1}\right) + \bar{N}\left(r, \frac{1}{\Delta f}\right) + T\left(r, \frac{1}{(f^n)^{(k)} - d}\right) - m\left(r, \frac{1}{(f^n)^{(k)} - d}\right) + S(r, \Delta f) \\ &\leq 2T(r, \Delta f) - m\left(r, \frac{1}{(f^n)^{(k)} - d}\right) + S(r, \Delta f). \end{aligned}$$

Thus

$$m\left(r, \frac{1}{(f^n)^{(k)} - d}\right) = S(r, \Delta f). \quad (1.32)$$

From the First Fundamental Theorem, Lemma 1.1, (1.20) to (1.21), (1.31), (1.32) and Δf is a transcendental entire function of finite order, we obtain

$$\begin{aligned}
 m\left(r, \frac{\Delta f - d}{(f^n)^{(k)} - d}\right) &\leq m\left(r, \frac{\Delta f}{(f^n)^{(k)} - d}\right) + m\left(r, \frac{d}{(f^n)^{(k)} - d}\right) + S(r, \Delta f) \\
 &\leq T\left(r, \frac{\Delta f - d}{(f^n)^{(k)} - d}\right) - N\left(r, \frac{\Delta f - d}{(f^n)^{(k)} - d}\right) + S(r, \Delta f) \\
 &= m\left(r, \frac{(f^n)^{(k)} - d}{f}\right) + N\left(r, \frac{(f^n)^{(k)} - d}{f}\right) - N\left(r, \frac{\Delta f}{(f^n)^{(k)} - d}\right) + S(r, \Delta f) \\
 &\leq N\left(r, \frac{1}{\Delta f}\right) - N\left(r, \frac{1}{(f^n)^{(k)} - 1}\right) + S(r, \Delta f) = T\left(r, \frac{1}{\Delta f}\right) - T\left(r, \frac{1}{(f^n)^{(k)} - d}\right) + S(r, \Delta f) \\
 &= T(r, \Delta) - T\left(r, (f^n)^{(k)}\right) + S(r, \Delta f) = S(r, \Delta f).
 \end{aligned}$$

Thus, we get

$$m\left(r, \frac{\Delta f - d}{(f^n)^{(k)} - d}\right) = S(r, \Delta f). \tag{1.33}$$

It's easy to see that $N(r, \Psi) = S(r, \Delta f)$ and (1.12) can be rewritten as

$$\Psi = \left[\frac{P_1 - d}{P_1} \frac{(f^n)^{(k+1)}}{(f^n)^{(k)} - P_1} + \frac{d(f^n)^{(k+1)}}{(f^n)^{(k)}} \right] \left[\frac{f - d}{(f^n)^{(k)} - d} - 1 \right]. \tag{1.34}$$

Then by (1.33) and (1.34) we can get

$$T(r, \Psi) = m(r, \Psi) + N(r, \Psi) = S(r, \Delta f). \tag{1.35}$$

By (1.7), (1.25) and (1.35) we get

$$\bar{N}\left(r, \frac{1}{\Delta f - P_1}\right) = S(r, \Delta f) \tag{1.36}$$

Moreover, by (1.7), (1.31) and (1.36), we have

$$m\left(r, \frac{1}{(f^n)^{(k)}}\right) = S(r, \Delta f), \tag{1.37}$$

which implies

$$\bar{N}\left(r, \frac{1}{\Delta f}\right) = m\left(r, \frac{1}{\Delta f - P_1}\right) \leq m\left(r, \frac{1}{(f^n)^{(k)}}\right) = S(r, \Delta f). \tag{1.38}$$

Then by (1.7) we obtain $T(r, \Delta f) = S(r, \Delta f)$, a contradiction. So, by (1.12), (1.16), (1.27) and the Second Fundamental Theorem of Nevanlinna, we can get

$$\begin{aligned}
 T(r, \Delta f) &\leq 2m\left(r, \frac{1}{\Delta f - P_1}\right) + S(r, \Delta f) \leq 2m\left(r, \frac{1}{(f^n)^{(k)}}\right) + S(r, \Delta f) \\
 &\leq 2T\left(r, (f^n)^{(k)}\right) - 2N\left(r, \frac{1}{(f^n)^{(k)}}\right) + S(r, \Delta f) \leq \bar{N}\left(r, \frac{1}{\Delta f - P_1}\right) + \bar{N}\left(r, \frac{1}{\Delta f - P_1}\right) \\
 &\quad + \bar{N}\left(r, \frac{1}{(f^n)^{(k)}}\right) - 2N\left(r, \frac{1}{(f^n)^{(k)}}\right) + S(r, \Delta f)T(r, \Delta f) - N\left(r, \frac{1}{(f^n)^{(k)}}\right) + S(r, f),
 \end{aligned}$$

which deduces that

$$N\left(r, \frac{1}{(f^n)^{(k)}}\right) = S(r, \Delta f). \quad (1.39)$$

It follows from the second theorem of Nevanlinna that

$$\begin{aligned} T\left(r, (f^n)^{(k)}\right) &\leq \bar{N}\left(r, \frac{1}{(f^n)^{(k)}}\right) + \bar{N}\left(r, \frac{1}{(f^n)^{(k)} - P_1}\right) + S(r, \Delta f) \\ &\leq \bar{N}\left(r, \frac{1}{(f^n)^{(k)} - P_1}\right) + S(r, \Delta f) \leq T\left(r, (f^n)^{(k)}\right) + S(r, \Delta f), \end{aligned}$$

which implies that

$$T\left(r, (f^n)^{(k)}\right) = \bar{N}\left(r, \frac{1}{(f^n)^{(k)} - P_1}\right) + S(r, \Delta f). \quad (1.40)$$

Similarly

$$T\left(r, (f^n)^{(k)}\right) = \bar{N}\left(r, \frac{1}{(f^n)^{(k)} - P_2}\right) + S(r, \Delta f). \quad (1.41)$$

Then, by (1.27), we get

$$T(r, \Delta f) = 2T\left(r, (f^n)^{(k)}\right) + S(r, \Delta f). \quad (1.42)$$

By (1.25) and (1.26) we have

$$T(r, \Phi) = T\left(r, (f^n)^{(k)}\right) + S(r, \Delta f). \quad (1.43)$$

When case 1 occurs, we apply Lemma 1.4 and obtain

$$(f^n)^{(k)} = He^t. \quad (1.44)$$

Here $H \neq 0$ is a small function of e^t . Rewrite (1.16) as

$$\Phi = \frac{(f^n)^{(k+1)} (\Delta f - P_1) (\Delta f - P_2) - \Delta f ((f^n)^{(k)} - P_1) ((f^n)^{(k)} - P_2)}{(\Delta f - P_1) (\Delta f - P_2) ((f^n)^{(k)} - P_1) ((f^n)^{(k)} - P_2)}. \quad (1.45)$$

Combining (1.27) with (1.44) we get

$$X = (f^n)^{(k+1)} (\Delta f - P_1) (\Delta f - P_2) - \Delta f' ((f^n)^{(k)} - P_1) ((f^n)^{(k)} - P_2) = \sum_{i=0}^5 \delta_i e^{it}, \quad (1.46)$$

and

$$Y = (\Delta f - P_1) (\Delta f - P_2) - \Delta f' ((f^n)^{(k)} - P_1) ((f^n)^{(k)} - P_2) = \sum_{j=0}^5 \gamma_j e^{jt} \quad (1.47)$$

where δ_i and γ_j are small functions of e^t , $\delta_5 \neq 0$ and $\gamma_6 \neq 0$.

If X and Y are two mutually prime polynomials in e^t , then we can get $T(r, \phi) = 6T(r, e^t) + S(r, \Delta f)$. It follows from (1.16), (1.41)-(1.43) that $T(r, \Delta f)$, a contradiction.

If X and Y are not two mutually prime polynomials in e^t , it's easy to see that the degree of Y is large than X . Then submitting (1.38) into (1.12) implies

$$H = P_2 \tag{1.48}$$

and

$$t = P_1 z + P_2 \tag{1.49}$$

where $P_1 \neq 0$ and P_2 are polynomials.

According to (1.45), (1.48), (1.49) and by simple calculation, we must have

$$\Phi = \frac{C}{(f^n)^{(k)} - P_2}, \tag{1.50}$$

where C is a non-zero constant. Put (1.44) into (1.16) we have

$$\frac{c((f^n)^{(k)} - (f^n)^{(k+1)} - P_1)}{((f^n)^{(k)} - P_1)((f^n)^{(k)} - P_2)} = \frac{-\Delta f'}{(\Delta f - P_1)(\Delta f - P_2)}. \tag{1.51}$$

We claim that $(f^n)^{(k)} \equiv (f^n)^{(k+1)}$.

Otherwise, combining (1.22), (1.44) and (1.51) we can get $T(r, e^t) = S(r, \Delta f)$. It follows from (1.16) and (1.27) that $T(r, \Delta f) = S(r, \Delta f)$, a contradiction. Hence, it is a easy work to verify that

$$P_1 = 1 \tag{1.52}$$

and

$$(f^n)^{(k)} = P_2 e^{z-P_2} = A e^z, \tag{1.53}$$

where A is a nonzero constant and furthermore

$$\Delta f = A e^{2z} - P_1 A e^z + P_1. \tag{1.54}$$

Then rewrite (1.27) as

$$\frac{\Delta f - (f^n)^{(k)}}{(f^n)^{(k)} - P_1} = e^t - 1. \tag{1.55}$$

Put (1.49), (1.52)-(1.54) into (1.55) and a direct calculation deduces

$$A = P_2 = e^{P_1} = 1. \tag{1.56}$$

It follows from (1.1), (1.28), (1.52) and (1.56) that

$$H = -P_1 (e^{\sigma} - 1)^n = 1. \tag{1.57}$$

Since Δf and $(f^n)^{(k)}$ share P_2 IM and (1.41), (1.42) and (1.56) we get

$$e^{2z} - P_1 e^z + (P_1 - 1) = (e^z - 1)^2, \tag{1.58}$$

i. e.,

$$P_1 = 2. \tag{1.59}$$

It follows from (1.57) that

$$e^{\varpi} = (-2)^{-1/n} + 1. \tag{1.60}$$

But we cannot get (1.2) from (1.60), a contradiction. When case 2 occurs we know that $m(r, e^t) = m(r, e^Q) + O(1) = S(r, \Delta f)$. Then by (1.16) and (1.27) we deduce $T(r, \Delta f) = S(r, \Delta f)$ a contradiction.

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