Advanced Studies in Contemporary Mathematics
33 (2023), No. 4, pp. 283 - 290
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2023.33.4.283

A NOTE ON GENERALIZED BERNOULLI POLYNOMIALS OF THE SECOND KIND

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ABSTRACT. Recently, generalized Bernoulli polynomials and Bernoulli polynomials of the second kind are introduced by Kim-Kim. In this paper, we consider generalized Bernoulli polynomials of the second kind and investigate some identities and properties of those polynomials. Moreover, we represent the generalized Bernoulli numbers in terms of the Stirling number of the first kind, the Stirling number of the second kind, the Daehee numbers and the Harmonic numbers.

1. INTRODUCTION

As is well known, the Bernoulli polynomials are defined by the generating function to be

(1) $$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad (\text{see } [1,3]).$$

When $x=0, B_n=B_n(0)$ are called the Bernoulli number.

For $r\in\mathbb{N}$, the Bernoulli polynomials of order r are defined by the generating function to be

(2) $$\begin{aligned}\left(\frac{t}{e^t-1}\right)^r e^{xt}&=\underbrace{\left(\frac{t}{e^t-1}\right)\times\cdots\times\left(\frac{t}{e^t-1}\right)}_{r-times}e^{xt}\\&=\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!},\quad (\text{see } [3,6,7]).\end{aligned}$$

In particular, if $r=1$, $B_n(x)=B_n(1)$ are the ordinary Bernoulli polynomials. When $x=0$, $B_n^{(r)}=B_n^{(r)}(0)$ are called the Bernoulli numbers of order r.

As is well known, the Bernoulli polynomials of the second kind (or the Cauchy polynomials) are given by the generating function to be

(3) $$\frac{t}{\log(1+t)}(1+t)^x=\sum_{n=0}^{\infty}b_n(x)\frac{t^n}{n!},\quad (\text{see } [2,3,4,7]).$$

When $x=0$, $b_n=b_n(0)$ are called the Bernoulli numbers of the second kind.

From (2) and (3), we easily see that $b_n=B_n^{(n)}(1)$ $(n\geq 1)$.

Key words and phrases. generalized degenerate type 2 Euler polynomials, the generalized degenerate type 2 Euler-Genocchi polynomials of order α, the degenerate Stirling numbers of the second kind.

For $n \geq 0$, the Stirling number of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \quad (\text{see } [1,3,10,11]). \tag{4}$$

Where $(x)_0 = 1, (x)_n = x(x-1)\dots(x-n+1)$, $(n \geq 1)$. From (4), we derive the following

$$\frac{1}{n!}(\log(1+t))^n = \sum_{k=n}^{\infty} S_1(k,n)\frac{t^k}{k!}, \quad (\text{see } [4,16,17,18]). \tag{5}$$

For a given nonnegative integer n, the stirling number of the second kind are defind by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad (\text{see } [3,10,13]). \tag{6}$$

By (6), we obtain

$$(e^t-1)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}, \quad (\text{see } [12,13]). \tag{7}$$

As is known, the Daehee polynomials are defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, \quad (\text{see } [2,4,8,9,10]). \tag{8}$$

When $x=0$, $D_n = D_n(0)$ are called the Daehee numbers.
It is well known that the harmonic numbers are defined by

$$H_0 = 0, H_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \quad (n \geq 1), \quad (\text{see } [5]). \tag{9}$$

From (9), we can derive the generating function of harmonic numbers given by

$$-\frac{1}{1-t}\log(1-t) = \sum_{n=1}^{\infty} H_n t^n \quad (\text{see } [13,14]). \tag{10}$$

The aim of this paper is to study several relations among those four kind of numbers. We develop methods for generalized Bernoulli polynomials of the second kind and represent the generalized Bernoulli numbers in terms of the Stirling number of the first kind,the Stirling number of the second kind, the Daehee numbers and the Harmonic numbers. We deduce a recurrence relation for generalized Bernoulli polynomials.

2. **The generalized Bernoulli polynomials of the second kind**

For $r \in \mathbb{N}$, the generalized Bernoulli polynomials of the second kind are defined by the generating function to be

$$\frac{t^r}{\log(1+t)}(1+t)^x = \sum_{n=r-1}^{\infty} c_n^{(r)}(x)\frac{t^n}{n!}. \tag{11}$$

When $x=0$, $c_n^{(r)} = c_n^{(r)}(0)$ are called the generalized Bernoulli numbers of the second kind. Note that $c_0^{(r)}(x) = c_1^{(r)}(x) = c_2^{(r)}(x) = \cdots = c_{r-2}^{(r)}(x) = 0$. When $r=1$, we get that $c_n^{(1)}(x) = c_n(x)$ for $n = 0,1,2\dots$

By (11), we observe that

$$\begin{aligned}\sum_{n=r-1}^{\infty} c_n^{(r)}(x)\frac{t^n}{n!} &= \frac{t^r}{\log(1+t)}(1+t)^x = t^{r-1}\times\frac{t}{\log(1+t)}(1+t)^x \\ &= t^{r-1}\sum_{n=0}^{\infty} c_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} c_n(x)\frac{t^{n+r-1}}{n!} \\ &= \sum_{n=r-1}^{\infty} c_{n-r+1}(x)\frac{n!}{(n-r+1)!}\frac{t^n}{n!} = \sum_{n=r-1}^{\infty}(r-1)!\binom{n}{r-1}c_{n-r+1}(x)\frac{t^n}{n!}.\end{aligned} \tag{12}$$

Therefore, by comparing the coefficients on both sides of (12), we obtain the following theorem

Theorem 1. *For $n\in\mathbb{N}$, we have*

$$c_n^{(r)} = \begin{cases} 0, & \text{if } n < r-1 \\ (r-1)!\binom{n}{r-1}c_{n-r+1}(x), & \text{if } n \geq r-1 \end{cases}.$$

From (11) and (12), we observe that

$$\begin{aligned}\sum_{n=r-1}^{\infty} c_n^{(r)}(x)\frac{t^n}{n!} &= \frac{t^r}{\log(1+t)}(1+t)^x \\ &= (r-1)!\frac{(\log(1+t))^{r-1}}{(r-1)!}\left(\frac{t}{\log(1+t)}\right)^r(1+t)^x \\ &= (r-1)!\sum_{m=r-1}^{\infty} S_1(m,r-1)\frac{t^m}{m!}\sum_{l=0}^{\infty} B_n^{(r)}(x)\frac{t^l}{l!} \\ &= (r-1)\sum_{n=r-1}^{\infty}\left(\sum_{l=r-1}^{n}\binom{n}{l}B_l^{(r)}(x)S_1(n-l,r-1)\right)\frac{t^n}{n!}.\end{aligned} \tag{13}$$

Thus, by comparing the coefficients on both sides of (13) we obtain the following theorem.

Theorem 2. *For $n \geq 0$,*

$$c_n^{(r)}(x) = (r-1)!\sum_{l=r-1}^{n}\binom{n}{l}B_l^{(r)}(x)S_1(n-l,r-1).$$

From (11), we observe that

$$\begin{aligned}\sum_{n=r-1} c_n^{(r)}(x)\frac{t^n}{n!} &= \frac{t^r}{\log(1+t)}(1+t)^x \\ &= \left(\sum_{m=r-1}^{\infty} c_m^{(r)}\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\binom{x}{l}t^l\right) \\ &= \sum_{n=r-1}^{\infty}\left(\sum_{l=r-1}^{n}\binom{n}{l}\binom{x}{l}c_{n-l}^{(r)}\right)\frac{t^n}{n!}.\end{aligned} \tag{14}$$

Therefore, by comparing coefficient on both sides of (14), we obtain the following theorem.

Theorem 3. *For $n \geq 0$ and $r\in\mathbb{N}$, we have*

$$c_n^{(r)}(x) = \sum_{l=r-1}^{n}\binom{n}{l}\binom{x}{l}c_{n-l}^{(r)}.$$

For $x, y \in \mathbb{R}$, we observe that

$$\begin{aligned}
\sum_{n=r-1}^{\infty} c_n^{(r)}(x+y)\frac{t^n}{n!} &= \frac{t^r}{\log(1+t)}(1+t)^{x+y} \\
&= \frac{t^r}{\log(1+t)}(1+t)^x(1+t)^y \\
&= \left(\sum_{m=r-1}^{\infty} c_m^{(r)}(x)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\binom{y}{l}t^l\right) \\
&= \sum_{n=r-1}^{\infty}\left(\sum_{l=r-1}^{n}\binom{n}{l}\binom{y}{l}c_{n-l}^{(r)}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{15}$$

Therefore, by comparing coefficient on both sides of (15), we obtain the following theorem.

Theorem 4. *For $x, y \in \mathbb{R}$ and $n \geq 0$ we have*

$$c_n^{(r)}(x+y) = \sum_{l=0}^{n}\binom{n}{l}\binom{y}{l}c_{n-l}^{r}(x).$$

By (5) and (11), we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} c_n^{(r)}(x)\frac{t^n}{n!} &= \frac{t^r}{\log(1+t)}(1+t)^x \\
&= \frac{t^r}{\log(1+t)} \times \frac{t}{\log(1+t)}\frac{\log(1+t)}{t}(1+t)^x \\
&= \left(\sum_{m=r-1}^{\infty} c_m^{(r)}\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty} b_l\frac{t^l}{l}\right)\left(\sum_{i=0}^{\infty} D_i(x)\frac{t^i}{i!}\right) \\
&= \sum_{k=r-1}^{\infty}\left(\sum_{m=r-1}^{k}\binom{k}{m}c_m^{(r)}b_{k-m}\right)\frac{t^k}{k!}\left(\sum_{i=0}^{\infty} D_i(x)\frac{t^i}{i!}\right) \\
&= \sum_{n=r-1}^{\infty}\left(\sum_{k=r-1}^{n}\left(\sum_{m}^{k} c_m^{(r)}b_{k-m}D_{n-k}(x)\right)\right)\frac{t^n}{n!}.
\end{aligned} \tag{16}$$

Therefore, by comparing the coefficients on both sides of(16) we get the following theorem

Theorem 5. *For $n \geq 0$, we have*

$$c_n^{(r)}(x) = \left(\sum_{k=r-1}^{n}\left(\sum_{m}^{k} c_m^{(r)}b_{k-m}D_{n-k}(x)\right)\right).$$

We observe that

$$\begin{aligned}
t^r(1+t)^x &= \sum_{n=r-1}^{\infty} c_n^{(r)}(x)\frac{t^n}{n!} \times \log(1+t) \\
&= \left(\sum_{m=r-1}^{\infty} c_m^{(r)}\frac{t^m}{m!}\right)\left(\sum_{l=1}^{\infty}\frac{(-1)^l}{l}t^l\right) \\
&= \sum_{n=r}^{\infty}\left(\sum_{m=r}^{n}\frac{c_m^{(r)}(x)}{m!}\frac{(-1)^{n-m}}{(n-m)}\right)t^n,
\end{aligned} \tag{17}$$

and

(18) $$(1+t)^x t^r = \sum_{n=0}^{\infty} \binom{x}{n} t^{n+r} = \sum_{n=r}^{\infty} \binom{x}{n-r} t^n.$$

By (17) and (18), we obtain the following theorem

Theorem 6. *For $n \geq r$, we have*

$$\sum_{m=r}^{n} \frac{b_m^{(r)}(x)}{m!} \frac{(-1)^{n-m}}{(n-m)} = \binom{x}{n-r}.$$

Replacing t by $e^t - 1$ in (11), we get

(19) $$\begin{aligned} \frac{(e^t-1)}{t} \times e^{tx} &= \sum_{m=r-1}^{\infty} c_m^{(r)}(x) \frac{(e^t-1)^m}{m!} \\ &= \sum_{m=r-1}^{\infty} c_m^{(r)}(x) \sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!} \\ &= \sum_{n=r-1}^{\infty} \left(\sum_{m-r-1}^{n} \binom{n}{m} c_m^{(r)}(x) S_2(n,m) \right) \frac{t^n}{n!} \end{aligned}$$

On the left hand sides of (19), we have

(20) $$\begin{aligned} \frac{(e^t-1)^r}{t} e^{tx} &= \frac{r!}{t} \frac{(e^t-1)^r}{r!} e^{tx} \\ &= \frac{r!}{t} \left(\sum_{m=r}^{\infty} S_2(m,r) \frac{t^m}{m!} \right) \left(\sum_{l=0}^{\infty} \frac{x^l t^l}{l!} \right) \\ &= \frac{r!}{t} \sum_{n=r}^{\infty} \left(\sum_{m=r}^{n} \binom{n}{m} S_2(m,r) x^{n-m} \right) \frac{t^n}{n!} \\ &= r! \sum_{n=r}^{\infty} \left(\sum_{m=r}^{n} \binom{n}{m} S_2(m,r) x^{n-m} \right) \frac{t^{n-1}}{n!} \\ &= r! \sum_{n=r-1}^{\infty} \left(\sum_{m=r-1}^{n+1} \binom{n+1}{m} S_2(m,r) \frac{x^{n-m+1}}{(n+1)} \right) \frac{t^n}{n!}. \end{aligned}$$

By (19) and (20), we obtain the following theorem.

Theorem 7. *For $n \geq r-1$, we have*

$$\sum_{m-r-1}^{n} \binom{n}{m} c_m^{(r)}(x) S_2(n,m) = \sum_{m=r-1}^{n+1} \binom{n+1}{m} S_2(m,r) \frac{x^{n-m+1}}{(n+1)}.$$

From (10) and (11), we can derive the following equation

$$
\begin{aligned}
\sum_{n=r-1}^{\infty} c_n^{(r)}(x)\frac{t^n}{n!} &= \frac{\log(1+t)}{1+t}\frac{t^r}{(\log(1+t))^2}(1+t)^{x+1} \\
&= \left(\sum_{l=1}^{\infty}(-1)^{l-1}H_l t^l\right)\left(\frac{t}{\log(1+t)}\frac{t^r}{\log(1+t))}(1+t)^{x+1}\right) \\
&= \left(\sum_{l=1}^{\infty}(-1)^{l-1}H_l t^l\right)\left(\sum_{l_1=0}^{\infty} b_{l_1}\frac{t^{l_1}}{l_1!}\right)\left(\sum_{l_2=0}^{\infty} c_{l_2}^{(r-1)}(x+1)\frac{t^{l_2}}{l_2!}\right) \qquad (21) \\
&= \left(\sum_{l=1}^{\infty}(-1)^{l-1}H_l t^l\right)\sum_{k=0}^{\infty}\left(\sum_{l_1=0}^{k} b_{l_1}c_{k-l_1}^{(r-1)}(x+1)\binom{k}{l_1}\right)\frac{t^k}{k!} \\
&= \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}\sum_{l_1=0}^{k}\binom{k}{l_1}b_{l_1}c_{k-l_1}^{(r-1)}(x+1)(-1)^{n-k}n!\frac{H_{n-k}}{k!}\right)\frac{t^n}{n!}.
\end{aligned}
$$

Therefore, we obtain the following theorem.

Theorem 8. *For $n \geq 1$ we have,*

$$
c_n^{(r)}(x) = \sum_{k=0}^{n-1}\sum_{l_1=0}^{k}\binom{k}{l_1}b_{l_1}c_{k-l_1}^{(r-1)}(x+1)(-1)^{n-k}n!\frac{H_{n-k}}{k!}.
$$

Where H_n are the Harmonic numbers.
Now we observe that

$$
\begin{aligned}
\frac{t^r}{\log(1+t)}(1+t)^x &= \frac{\log(1+t)}{1+t}\cdot\frac{t^r}{(\log(1+t))^2}(1+t)^{x+1} \\
&= \frac{\log(1+t)}{1+t}\frac{\log(1+t)}{1+t}\cdot\frac{t^r}{(\log(1+t))^3}(1+t)^{x+2} \\
&= \frac{\log(1+t)}{1+t}\frac{\log(1+t)}{1+t}\frac{\log(1+t)}{1+t}\cdot\frac{t^r}{(\log(1+t))^4}(1+t)^{x+3} \\
&= \cdots
\end{aligned}
$$

continuing this process, we get

$$\underbrace{\frac{\log(1+t)}{1+t}\times\cdots\times\frac{\log(1+t)}{1+t}}_{r-1\ times}\cdot\frac{t^r}{(\log(1+t))^r}(1+t)^{x+r-1}$$

(22)

$$\begin{aligned}
&=\left(\sum_{l_1=1}^{\infty}(-1)^{l_1-1}H_{l_1}t^{l_1}\sum_{l_2=1}^{\infty}(-1)^{l_2-1}H_{l_2}t^{l_2}\times\cdots\times\sum_{l_{r-1}=1}^{\infty}(-1)^{l_{r-1}-1}H_{l_{r-1}}t^{l_{r-1}}\right)\\
&\quad\times\left(\sum_{k=r-1}^{\infty}c_n^{(r)}(x+r-1)\frac{t^n}{n!}\right)\\
&=\sum_{m=r-1}^{\infty}\left(\sum_{l_1+\cdots+l_{r-1}=m}(-1)^{m+r-1}H_{l_1}H_{l_2}\cdots H_{l_{r-1}}\right)t^m\left(\sum_{k=r-1}^{\infty}c_n^{(r)}(x+r-1)\frac{t^n}{n!}\right)\\
&=\sum_{n=r-1}^{\infty}\left(\sum_{m=r-1}^{n}\sum_{l_1+\cdots+l_{r-1}=m}(-1)^{m+r-1}H_{l_1}H_{l_2}\cdots H_{l_{r-1}}\frac{c_{n-m}^{(r)}(x+r-1)n!m!}{m!(n-m!)}\right)\frac{t^n}{n!}\\
&=\sum_{n=r-1}^{\infty}\left(\sum_{m=r-1}^{n}\sum_{l_1+\cdots+l_{r-1}=m}(-1)^{m+r-1}H_{l_1}H_{l_2}\cdots H_{l_{r-1}}\binom{n}{m}c_{n-m}^{(r)}(x+r-1)m!\right)\frac{t^n}{n!}
\end{aligned}$$

Therefore, we obtain the following Theorem.

Theorem 9. *For $n\geq 1$ we have,*

$$c_n^{(r)}(x)=\begin{cases}0, & \text{if } n<r-1,\\ \sum_{m=r-1}^{n}\sum_{l_1+\cdots l_{r-1}=m}(-1)^{m-r+1}H_{l_1}H_{l_2}\cdots H_{l_{r-1}}\binom{n}{m}m!c_{n-m}^{(r)}(x+r-1), & \text{if } n\geq r-1.\end{cases}$$

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