Proceedings of the Jangjeon Mathematical Society
26 (2023), No. 3, pp. 341 – 344
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2023.26.3.341

COMMUTATIVITY OF THE DISCONTINUITY GROUP FOR A LOCALLY BOUNDED HOMOMORPHISM OF A CONNECTED LIE GROUP INTO A LINEAR CONNECTED LIE GROUP

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Abstract.

We prove that the discontinuity group of a locally bounded homomorphism from a connected Lie group into a linear connected Lie group is Abelian.

§ 1. Introduction

Let us recall some information needed below.

A (not necessarily continuous) homomorphism π of a topological group G into a topological group H is said to be *relatively compact* if there is a neighborhood $U = U_{e_G}$ of the identity element e_G in G whose image $\pi(U)$ has compact closure in H.

Obviously, a homomorphism into a locally compact group is relatively compact if and only if it is *locally bounded*, i.e., there is a neighborhood U_e whose image is contained in some element of the filter $\mathfrak{V}$ of neighborhoods of e_V having compact closure.

Let us also recall the notion of discontinuity group of a homomorphism π of a topological group G into a topological group H, see [1] and [2].

2020 *Mathematics Subject Classification.* Primary 22E15.

Submitted May 19, 2023.

Key words and phrases. Locally bounded homomorphism, continuity, discontinuity group of a group homomorphism.

Typeset by $\mathcal{AMS}$-TEX

Let $\mathfrak{U} = \mathfrak{U}_G$ be the filter of neighborhoods of e_G in G. For every (not necessarily continuous) locally relatively compact homomorphism π of G into H, the set

$$\mathrm{DG}(\pi) = \bigcap_{U \in \mathfrak{U}} \overline{\pi(U)}$$

is called the discontinuity group of π. Here and further, the bar stands for the closure in the corresponding topology (here the closure is taken in the topology of H). (See Definition 1.1.1 of [1].)

The discontinuity group of a homomorphism has some important properties. Under the above conditions, the set $\mathrm{DG}(\pi)$ is a compact subgroup of the topological group H and a compact normal subgroup of the closed subgroup $\overline{\pi(G)}$ of H. Moreover, the filter basis $\{\overline{\pi(U)} \mid U \in \mathfrak{U}\}$ converges to $\mathrm{DG}(\pi)$, and the homomorphism π is continuous if and only if

$$\mathrm{DG}(\pi) = \{e_H\}.$$

(See Theorem 1.1.2 of [1].)

If G is a connected Lie group, then $\mathrm{DG}(\pi)$ is a compact connected subgroup of H. (See Lemma 1.1.6 of [1].)

Let G be a connected Lie group, let N be a closed normal subgroup of G, and let π be a locally bounded homomorphism of G into a locally compact group H. Let M be the discontinuity group of the restriction $\mathrm{DG}(\pi|_N)$. Then M is a closed normal subgroup of the compact discontinuity group $\mathrm{DG}(\pi)$, and the corresponding quotient group $\mathrm{DG}(\pi)/M$ is isomorphic to the discontinuity group $\mathrm{DG}(\psi)$ of the homomorphism ψ of G obtained as the composition of the homomorphism π and the canonical homomorphism

$$\overline{\pi(G)} \to \overline{\pi(G)}/M.$$

(See Lemma 1.1.7 of [1].)

§ 2. Main result

Theorem 1. *The discontinuity group of a locally bounded homomorphism from a connected Lie group into a connected linear Lie group is Abelian..*

Proof. Let π be a locally bounded homomorphism of a connected Lie group G into a linear connected Lie group H.

Since H is linear, it follows that we may consider a faithful linear finite-dimensional continuous representation ρ of H and topologically identify H

with its image $\rho(H)$ (see [3]). The composition of these mappings can be regarded as a locally bounded finite-dimensional linear representation of G.

To make the presentation more visual, we can assume that the group G is simply connected (combining the canonical homomorphism of the universal covering group of the given group to the given group and the renewed representation π).

As is known (see Lemma 1.1.6 of [1]), for a locally bounded finite-dimensional linear representation, the corresponding group $DG(\pi)$ is a compact connected linear group.

Therefore, it remains to prove that the group $DG(\pi)$ is Abelian, or to prove the (equivalent) assertion claiming that the commutator subgroup of the group $DG(\pi)$ is trivial. However,

$$\{e_H\} \subset DG(\pi)' = DG(\pi|_{G'}),$$

for the discontinuity groups of π and of the restriction of π to the commutator subgroup G' of G; here the last equality follows from Lemma 1.1.9 of [1].

At the same time, the restriction $\pi|_{G'}$ of π to the commutator subgroup G' is continuous by Theorem 1.3.2 of [1], and thus

$$DG(\pi|_{G'}) = \{e_H\}$$

by Theorem 1.1.2 of [1]. This implies that

$$DG(\pi)' = \{e_H\},$$

which proves that the group $DG(\pi)$ is Abelian.

This completes the proof of the theorem.

The general properties of discontinuity groups of connected Lie groups imply the following assertion.

Theorem 2. *Every locally bounded homomorphism from a connected Lie group into a connected linear Lie group without nontrivial connected compact subgroups is continuous.*

Proof. The discontinuity group of such a homomorphism is compact and connected, hence trivial by the condition of the theorem and, therefore, the homomorphism is continuous.

§ 4. Comments

Note that, by Harish-Chandra's theorem [4], any semidirect product of a linear connected perfect Lie group and a vector group is linear. For one of the lists of the linearity conditions for connected Lie groups, see [5].

Acknowledgments

I thank Professor Taekyun Kim for the invitation to publish this paper in the Proceedings of the Jangjeon Mathematical Society.

Funding

The research was partially supported by the Moscow Center for Fundamental and Applied Mathematics.

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