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ON A NEW CLASS OF SUMMATION FORMULAS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS

DONGKYU LIM, VIDHA KULKARNI, YASHOVERDHAN VYAS AND ARJUN K. RATHIE

Abstract. In the theory of generalized hypergeometric series, classical summation theorems such as those of Gauss, Gauss's second, Kummer and Bailey for the series ${}_2F_1$; Watson, Dixon, and Whipple for the series ${}_3F_2$ and others play an important role. In 2015, Eslahchi and Masjed-Jamei applied the above-mentioned classical summation theorems in a very general hypergeometric identity available in the literature and obtained several interesting summation formulas involving generalized hypergeometric functions. In 2010, Kim et al. established the extensions of the above-mentioned classical summation theorems together with a few more extended summation theorems. This paper aims to establish several new and interesting summation formulas involving generalized hypergeometric functions. This is achieved by applying the above-mentioned extended summation theorems in a very general hypergeometric identity available in the literature. The result obtained earlier by Eslahchi and Masjed-Jamei follows special cases of our main findings. The results established in the paper are simple, interesting, easily established, and may be potentially useful.

1. Introduction and Preliminaries

The well-known generalized hypergeometric function with r numerator and s denominator parameters is defined [4, 17, 21] by:

$$ {}_rF_s\left[\begin{matrix} u_1, \ldots, u_r \\ v_1, \ldots, v_s \end{matrix}; w\right] = \sum_{n=0}^{\infty} \frac{(u_1)_n \cdots (u_r)_n}{(v_1)_n \cdots (v_s)_n} \frac{w^n}{n!} \tag{1.1} $$

Also, no denominator parameter $v_j\,(j = 1, ..., s)$ is supposed to be zero or a negative integer . If any parameter $u_i\,(i = 1, ..., r)$ is zero or a negative integer, the series terminates. The power series (1.1) could be examined using the elementary ratio test, which confirms that:

(i) If $r \leq s$, the series is convergent for all finite w.

(ii) If $r = s + 1$, the series is convergent for $|w| < 1$ and diverges for $|w| > 1$.

(iii) If $r > s + 1$, the series diverges for $w \neq 0$.

Key words and phrases. Generalized hypergeometric function, Classical summation theorems, Extensions, Hypergeometric Identities.

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(iv) If $r = s+1$, the series is absolutely convergent on the circle $|w| = 1$ if

$$Re\left(\sum_{j=1}^{s} v_j - \sum_{i=1}^{r} u_i\right) > 0$$

Moreover $(u)_n$ in series (1.1) is widely known as the shifted factorial for any complex number and is defined by:

$$(u)_n = \frac{\Gamma(u+n)}{\Gamma(u)} = \begin{cases} 1 & ;(n=0, u \,\epsilon\, \mathbb{C} \setminus \{0\}) \\ u(u+1)...(u+n-1) & ;(n \,\epsilon\, \mathbb{N}, u \,\epsilon\, \mathbb{C}) \end{cases} \tag{1.2}$$

Also, when a generalized hypergeometric function ${}_{q+1}F_q$ is transformed into a gamma function, the results are highly relevant and beneficial from applications perspective. We will mention here the following classical summation theorems, so the work should stand on its own [2,4,21]:

- Gauss's summation theorem :

$$ {}_2F_1\left[\begin{matrix} a,\, b \\ c \end{matrix} ;1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \tag{1.3}$$

  provided $Re(c-a-b) > 0$.

- Kummer's summation theorem:

$$ {}_2F_1\left[\begin{matrix} a,\, b \\ 1+a-b \end{matrix} ;-1\right] = \frac{\Gamma(1+a-b)\,\Gamma(1+\frac{1}{2}a)}{\Gamma(1-b+\frac{1}{2}a)\,\Gamma(1+a)} \tag{1.4}$$

  provided $Re(b) < 1$.

- Gauss's second summation theorem:

$$ {}_2F_1\left[\begin{matrix} a,\, b \\ \frac{1}{2}(a+b+1) \end{matrix} ;\frac{1}{2}\right] = \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}(a+b+1))}{\Gamma(\frac{1}{2}(a+1))\,\Gamma(\frac{1}{2}(b+1))} \tag{1.5}$$

- Bailey's summation theorem:

$$ {}_2F_1\left[\begin{matrix} a,\, 1-a \\ b \end{matrix} ;\frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}(b+1))}{\Gamma(\frac{1}{2}(a+b))\,\Gamma(\frac{1}{2}(b-a+1))} \tag{1.6}$$

- Watson's summation theorem:

$$ {}_3F_2\left[\begin{matrix} a,\, b,\, c \\ \frac{1}{2}(a+b+1),\, 2c \end{matrix} ;1\right] = \frac{\sqrt{\pi}\,\Gamma(c+\frac{1}{2})\,\Gamma(\frac{1}{2}(a+b+1))\,\Gamma(c-\frac{1}{2}(a+b-1))}{\Gamma(\frac{1}{2}(a+1))\,\Gamma(\frac{1}{2}(b+1))\,\Gamma(c-\frac{1}{2}(a-1))\,\Gamma(c-\frac{1}{2}(b-1))} \tag{1.7}$$

  provided $Re(2c-a-b) > -1$.

- Dixon's summation theorem:

$$
{}_3F_2\left[\begin{matrix} a,\, b,\, c \\ 1+a-b,\, 1+a-c \end{matrix}; 1\right] = \frac{\Gamma(1+\frac{1}{2}a)\,\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1-b-c+\frac{1}{2}a)}{\Gamma(1+a)\,\Gamma(1-b+\frac{1}{2}a)\,\Gamma(1-c+\frac{1}{2}a)\,\Gamma(1+a-b-c)} \tag{1.8}
$$

  provided $Re(a-2b-2c) > -2$.

- Whipple's summation theorem:

$$
{}_3F_2\left[\begin{matrix} a,\, b,\, c \\ e,\, f \end{matrix}; 1\right] = \frac{\pi\,\Gamma(e)\,\Gamma(f)}{2^{2c-1}\,\Gamma(\frac{1}{2}a+\frac{1}{2}e)\,\Gamma(\frac{1}{2}a+\frac{1}{2}f)\,\Gamma(\frac{1}{2}b+\frac{1}{2}e)\,\Gamma(\frac{1}{2}b+\frac{1}{2}f)} \tag{1.9}
$$

  provided $Re(c) > 0$ and $Re(e+f-a-b-c) > 0$ with $a+b=1$ and $e+f=2c+1$.

- The other two hypergeometric identities which we shall consider in this paper are:

$$
{}_3F_2\left[\begin{matrix} a,\, 1+\frac{1}{2}a,\, b \\ \frac{1}{2}a,\, 1+a-b \end{matrix}; -1\right] = \frac{\Gamma(1+a-b)\,\Gamma(\frac{1}{2}a+\frac{1}{2})}{\Gamma(1+a)\,\Gamma(\frac{1}{2}a-b+\frac{1}{2})} \tag{1.10}
$$

$$
{}_4F_3\left[\begin{matrix} a,\, 1+\frac{1}{2}a,\, b,\, c \\ \frac{1}{2}a,\, 1+a-b,\, 1+a-c \end{matrix}; 1\right] = \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(\frac{1}{2}a+\frac{1}{2})\,\Gamma(\frac{1}{2}a-b-c+\frac{1}{2})}{\Gamma(1+a)\,\Gamma(1+a-b-c)\,\Gamma(\frac{1}{2}a-b+\frac{1}{2})\,\Gamma(\frac{1}{2}a-c+\frac{1}{2})} \tag{1.11}
$$

  provided $Re(a-2b-2c) > -1$.

Remark: For other intersting papers in this direction , we refer research papers [3, 5, 6, 8–16, 18–20]

2. Summation formulas involving generalized hypergeometric functions established by Eslahchi and Masjed-Jamei

By employing classical summation theorems (1.3) to (1.11) given in section1 in following very general hypergeometric identity viz.

$$
{}_pF_q\left[\begin{matrix} a_1,\, \ldots,\, a_p \\ b_1,\, \ldots,\, b_q \end{matrix}; z\right] = \sum_{k=0}^{m-1} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!} \times {}_{mp+1}F_{mq+1}\left[\begin{matrix} \overrightarrow{A}_{1,k},\, \ldots,\, \overrightarrow{A}_{p,k}, 1 \\ \overrightarrow{B}_{1,k},\, \ldots,\, \overrightarrow{B}_{q,k},\, \overrightarrow{I}_{1,k} \end{matrix}; m^{(p-q-1)m} z^m\right] \tag{2.1}
$$

provied that the conditions given in (1.1) are satisfied. Also here

$\overrightarrow{A}_{j,k} = \left(\frac{a_j+k}{m}, \frac{a_j+1+k}{m}, \ldots, \frac{a_j+m-1+k}{m}\right)$, For $j = 1, 2, \ldots, p$

$\overrightarrow{B}_{j,k} = \left(\frac{b_j+k}{m}, \frac{b_j+1+k}{m}, \ldots, \frac{b_j+m-1+k}{m}\right)$, For $j = 1, 2, \ldots, q$

and $\overrightarrow{I}_{1,k} = \left(\frac{1+k}{m}, \frac{2+k}{m}, \ldots, \frac{m+k}{m}\right)$

In 2015 Eslahchi and Masjed-Jamei [7] established following summation formula involving generalized hypergeometric functions:

(a).The general case of Gauss's summation theorem (1.3) for any natural number m:

(2.2)

$$\sum_{k=0}^{m-1} \frac{(a)_k(b)_k}{(c)_k k!} \, {}_{2m+1}F_{2m}\left[\begin{matrix} \frac{a+k}{m}, \ldots, \frac{a+m-1+k}{m}, \frac{b+k}{m}, \ldots, \frac{b+m-1+k}{m}, 1 \\ \frac{c+k}{m}, \ldots, \frac{c+m-1+k}{m}, \frac{1+k}{m}, \ldots, \frac{m+k}{m} \end{matrix}; 1\right]$$

$$= \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}$$

provided $Re(c-a-b) > 0$.

(b).The general case of Kummer's summation theorem (1.4) for any natural number m:

(2.3)

$$\sum_{k=0}^{m-1} \frac{(a)_k(b)_k(-1)^k}{(1+a-b)_k k!} \, {}_{2m+1}F_{2m}\left[\begin{matrix} \frac{a+k}{m}, \ldots, \frac{a+m-1+k}{m}, \frac{b+k}{m}, \ldots, \frac{b+m-1+k}{m}, 1 \\ \frac{1+a-b+k}{m}, \ldots, \frac{a-b+m+k}{m}, \frac{1+k}{m}, \ldots, \frac{m+k}{m} \end{matrix}; (-1)^m\right]$$

$$= \frac{\Gamma(1+a-b)\,\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)\,\Gamma(1+\frac{1}{2}a-b)}$$

provided $Re(b) < 1$.

(c).The general case of Gauss's second summation theorem (1.5) for any natural number m:

(2.4)

$$\sum_{k=0}^{m-1} \frac{(a)_k(b)_k(2)^{-k}}{(\frac{1}{2}(1+a+b))_k k!}$$

$$\times \, {}_{2m+1}F_{2m}\left[\begin{matrix} \frac{a+k}{m}, \ldots, \frac{a+m-1+k}{m}, \frac{b+k}{m}, \ldots, \frac{b+m-1+k}{m}, 1 \\ \frac{k+(a+b+1)/2}{m}, \ldots, \frac{m+k+(a+b-1)/2}{m}, \frac{1+k}{m}, \ldots, \frac{m+k}{m} \end{matrix}; 2^{-m}\right]$$

$$= \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}(1+a+b))}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})}$$

(d).The general case of Bailey's summation theorem (1.6) for any natural number m:

$$\sum_{k=0}^{m-1} \frac{(a)_k(1-a)_k(2)^{-k}}{(c)_k k!} \times {}_{2m+1}F_{2m}\left[\begin{matrix} \frac{a+k}{m},\ldots,\frac{a+m-1+k}{m},\frac{1-a+k}{m},\ldots,\frac{-a+m+k}{m},1 \\ \frac{c+k}{m},\ldots,\frac{c+m-1+k}{m},\frac{1+k}{m},\ldots,\frac{m+k}{m} \end{matrix};2^{-m}\right] = \frac{\Gamma(\frac{c}{2})\,\Gamma(\frac{c+1}{2})}{\Gamma(\frac{c+a}{2})\,\Gamma(\frac{c-a+1}{2})} \tag{2.5}$$

(e). The general case of Watson's summation theorem (1.7) for any natural number m:

(2.6)

$$\sum_{k=0}^{m-1} \frac{(a)_k(b)_k(c)_k}{(\frac{1}{2}(a+b+1))(2c)_k k!} \times {}_{3m+1}F_{3m}\left[\begin{matrix} \frac{a+k}{m},\ldots,\frac{a+m-1+k}{m},\frac{b+k}{m},\ldots,\frac{b+m-1+k}{m},\frac{c+k}{m},\ldots,\frac{c+m-1+k}{m},1 \\ \frac{k+(a+b+1)/2}{m},\ldots,\frac{m-1+k+(a+b+1)/2}{m},\frac{2c+k}{m},\ldots,\frac{2c+m-1+k}{m},\frac{1+k}{m},\ldots,\frac{m+k}{m} \end{matrix};1\right]$$
$$= \frac{\sqrt{\pi}\,\Gamma(c+\frac{1}{2})\,\Gamma(\frac{1}{2}(a+b+1))\Gamma(c-\frac{a+b-1}{2})}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})\Gamma(c-\frac{a-1}{2})\Gamma(c-\frac{b-1}{2})}$$

provided $Re(2c-a-b) > -1$.

(f). The general case of Dixon's summation theorem (1.8) for any natural number m:

(2.7)

$$\sum_{k=0}^{m-1} \frac{(a)_k(b)_k(c)_k}{(1+a-b)_k(1+a-c)_k k!} \times {}_{3m+1}F_{3m}\left[\begin{matrix} \frac{a+k}{m},\ldots,\frac{a+m-1+k}{m},\frac{b+k}{m},\ldots,\frac{b+m-1+k}{m},\frac{c+k}{m},\ldots,\frac{c+m-1+k}{m},1 \\ \frac{1+a-b+k}{m},\ldots,\frac{a-b+m+k}{m},\frac{1+a-c+k}{m},\ldots,\frac{a-c+m+k}{m},\frac{1+k}{m},\ldots,\frac{m+k}{m} \end{matrix};1\right]$$
$$= \frac{\Gamma(1+\frac{a}{2})\,\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+\frac{a}{2}-b-c)}{\Gamma(1+a)\,\Gamma(1+\frac{a}{2}-b)\,\Gamma(1+\frac{a}{2}-c)\,\Gamma(1+a-b-c)}$$

provided $Re(a-2b-2c) > -2$.

(g). The general case of Whipple's summation theorem (1.9) for any natural number m:

(2.8)

$$\sum_{k=0}^{m-1} \frac{(a)_k(1-a)_k(b)_k}{(c)_k(2b-c+1)_k k!}$$
$$\times\, {}_{3m+1}F_{3m}\left[\begin{matrix}\frac{a+k}{m},\ldots,\frac{a+m-1+k}{m},\frac{1-a+k}{m},\ldots,\frac{m-a+k}{m},\frac{b+k}{m},\ldots,\frac{b+m-1+k}{m},1\\ \frac{c+k}{m},\ldots,\frac{c+m-1+k}{m},\frac{2b-c+1+k}{m},\ldots,\frac{2b-c+m+k}{m},\frac{1+k}{m},\ldots,\frac{m+k}{m}\end{matrix};1\right]$$
$$=\frac{\pi\, 2^{1-2b}\,\Gamma(c)\,\Gamma(2b-c-1)}{\Gamma(\frac{1}{2}(a+c))\,\Gamma(b+\frac{1}{2}(a-c+1))\,\Gamma(\frac{1}{2}(1+a-c))\,\Gamma(b+1-\frac{1}{2}(a+c))}$$

provided $Re(b) > 0$.

(h). The general case of hypergeometric identity (1.10) for any natural number m:

(2.9)

$$\sum_{k=0}^{m-1} \frac{(a)_k(1+\frac{a}{2})_k(b)_k(-1)^k}{(\frac{a}{2})_k(1+a-b)_k k!}$$
$$\times {}_{3m+1}F_{3m}\left[\begin{matrix}\frac{a+k}{m},\ldots,\frac{a+m-1+k}{m},\frac{1+\frac{a}{2}+k}{m},\ldots,\frac{\frac{a}{2}+m+k}{m},\frac{b+k}{m},\ldots,\frac{b+m-1+k}{m},1\\ \frac{\frac{a}{2}+k}{m},\ldots,\frac{\frac{a}{2}+m-1+k}{m},\frac{1+a-b+k}{m},\ldots,\frac{a-b+m+k}{m},\frac{1+k}{m},\ldots,\frac{m+k}{m}\end{matrix};(-1)^m\right]$$
$$=\frac{\Gamma(1+a-b)\,\Gamma(\frac{1}{2}(a+1))}{\Gamma(1+a)\,\Gamma(\frac{1}{2}(a+1)-b)}$$

provided $Re(b) > 0$.

(i). The general case of hypergeometric identity (1.11) for any natural number m:

(2.10)

$$\sum_{k=0}^{m-1} \frac{(a)_k(1+\frac{a}{2})_k(b)_k(c)_k}{(\frac{a}{2})_k(1+a-b)_k(1+a-c)_k k!}$$
$$\times {}_{4m+1}F_{4m}\left[\begin{matrix}\frac{a+k}{m},\ldots,\frac{a+m-1+k}{m},\frac{1+\frac{a}{2}+k}{m},\ldots,\frac{\frac{a}{2}+m+k}{m},\frac{b+k}{m},\ldots,\frac{b+m-1+k}{m},\frac{c+k}{m},\ldots,\frac{c+m-1+k}{m},1\\ \frac{\frac{a}{2}+k}{m},\ldots,\frac{\frac{a}{2}+m-1+k}{m},\frac{1+a-b+k}{m},\ldots,\frac{a-b+m+k}{m},\frac{1+a-c+k+m}{m},\ldots,\frac{a-c+m+k}{m},\frac{1+k}{m},\ldots,\frac{m+k}{m}\end{matrix};1\right]$$
$$=\frac{\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(\frac{a+1}{2})\,\Gamma(\frac{a+1}{2}-b-c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)\,\Gamma(\frac{a+1}{2}-b)\,\Gamma(\frac{a+1}{2}-c)}$$

provided $Re(a-2b-2c) > -1$.

It is worth noting that for $m = 1$ equations (2.2) to (2.10), reduce to classical summation theorems (1.3) to (1.11) respectively.

Remark: The results (2.9) to (2.10) are obtained by us by the same technique given by Eslahchi and Masjed-Jamei [7].

3. Results Required

In this section, we shall mention the extension of the classical summation theorems due to Kim et al. [11] that will be required in our present investigations. These are valid for $d \notin \mathbb{Z}_0^-$.

(a). Extension of Gauss's summation theorem:

$$ {}_3F_2\left[\begin{matrix} a,\, b,\, d+1 \\ c+1,\, d \end{matrix};1\right] = \frac{\Gamma(c+1)\,\Gamma(c-a-b)}{\Gamma(c-a+1)\,\Gamma(c-b+1)}\left[(c-a-b)+\frac{ab}{d}\right] \tag{3.1}$$

provided $Re(c-a-b)>0$.
For $d=c$ (3.1) reduces to Gauss's summation theorem (1.3).

(b). Extension of Kummer's summation theorem:

$$ {}_3F_2\left[\begin{matrix} a,\, b,\, d+1 \\ 2+a-b,\, d \end{matrix};-1\right] = \frac{\Gamma(1/2)\,\Gamma(2+a-b)}{2^a(1-b)}\left[\frac{((1+a-b)/d)-1}{\Gamma(\frac{a}{2})\Gamma(\frac{1}{2}(a+3)-b)}+\frac{1-(a/d)}{\Gamma(\frac{1}{2}(a+1))\,\Gamma(\frac{a}{2}-b+1)}\right] \tag{3.2}$$

For $d=1+a-b$, we get Kummer's summation theorem (1.4).

(c). Extension of Gauss's second summation theorem:

$$\begin{aligned} {}_3F_2\left[\begin{matrix} a,\, b,\, d+1 \\ \frac{1}{2}(a+b+3),\, d \end{matrix};\frac{1}{2}\right] &= \frac{\sqrt{\pi}\Gamma(\frac{1}{2}(a+b+3)\Gamma(\frac{1}{2}(a-b-1)}{\Gamma(\frac{1}{2}(a-b+3))} \\ &\times\left[\frac{(1+a+b)/2-(ab/d)}{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(b+1)}+\frac{((a+b+1)/d)-2}{\Gamma(\frac{a}{2})\Gamma(\frac{b}{2})}\right] \end{aligned} \tag{3.3}$$

For $d=\frac{1}{2}(a+b+1)$, we get Gauss's second summation theorem (1.5).

(d). Extension of Bailey's summation theorem:

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} a,\, 1-a,\, d+1 \\ c+1,\, d \end{matrix};\frac{1}{2}\right] \\ &= \frac{\Gamma(1/2)\Gamma(c+1)}{2^c}\left[\frac{(2/d)}{\Gamma(\frac{1}{2}(c+a))\Gamma(\frac{1}{2}(c-a+1))}+\frac{1-(c/d)}{\Gamma(\frac{1}{2}(c-a)+1)\Gamma(\frac{1}{2}(c+a+1))}\right] \end{aligned} \tag{3.4}$$

For $d=c$, we get Bailey's summation theorem (1.6).

(e). Extension of Watson's summation theorem:

- First extension:

$$
{}_4F_3\left[\begin{matrix} a,\, b,\, c,\, d+1 \\ \frac{1}{2}(a+b+1),\, 2c+1,\, d \end{matrix};1\right]
= \frac{2^{a+b-2}\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}(a+b+1))\Gamma(c-\frac{1}{2}(a+b-1))}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)}
\times\left[\frac{\Gamma(a/2)\Gamma(b/2)}{\Gamma(c+\frac{1}{2}(1-a))\Gamma(c+\frac{1}{2}(1-b))}+\frac{((2c-d)/d)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(b+1))}{\Gamma(1+\frac{1}{2}(c-a))\Gamma(c-\frac{b}{2}+1)}\right] \tag{3.5}
$$

provided $Re(2c-a-b) > -1$.

- Second extension:

$$
{}_4F_3\left[\begin{matrix} a,\, b,\, c,\, d+1 \\ \frac{1}{2}(a+b+3),\, 2c+1,\, d \end{matrix};1\right]
= \frac{2^{a+b-2}\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}(a+b+3))\Gamma(c-\frac{1}{2}(a+b+1))}{(a-b-1)(a-b+1)\Gamma(1/2)\Gamma(a)\Gamma(b)}
\times\left[\alpha\frac{\Gamma(a/2)\Gamma(b/2)}{\Gamma(c+\frac{1}{2}(1-a))\Gamma(c+\frac{1}{2}(1-b))}+\beta\frac{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(b+1))}{\Gamma(c-\frac{a}{2})\Gamma(c-\frac{b}{2})}\right] \tag{3.6}
$$

provided $Re(2c-a-b) > 1$.

Here α and β are given by:
$\alpha = a(2c-a)+b(2c-b)-2c+1-\frac{ab}{d}(4c-a-b-1)$, $\beta = 8\left[\frac{1}{2d}(a+b+1)-1\right]$.

For $d = 2c$ and $d = \frac{1}{2}(a+b+1)$ in (3.5) and (3.6) respectively, we get Watson's summation theorem (1.7).

(f). Extension of Dixon's theorem:

$$
{}_4F_3\left[\begin{matrix} a,\, b,\, c,\, d+1 \\ 2+a-b,\, 1+a-c,\, d \end{matrix};1\right]
= \frac{\alpha}{b-1}\frac{\Gamma(1+a-c)\Gamma(2+a-b)\Gamma(\frac{1}{2}(a+3)-b-c)\Gamma(1/2)}{2^a\Gamma(\frac{a}{2})\Gamma(\frac{1}{2}(a+1)-c))\Gamma(2+a-b-c)\Gamma(\frac{1}{2}(a+3)-b)}
+\frac{\beta}{b-1}\frac{2^{-a-1}\Gamma(1/2)\Gamma(1+a-c)\Gamma(1+a-b)\Gamma(1+\frac{a}{2}-b-c)}{\Gamma(\frac{1}{2}(a+1))\Gamma(1+\frac{a}{2}-b)\Gamma(1+\frac{a}{2}-c)\Gamma(1+a-b-c)} \tag{3.7}
$$

provided $Re(a-2b-2c) > -2$.

Here α and β is given by:
$\alpha = 1-\frac{1}{d}(1+a-b)$, $\beta = \frac{1+a-b}{1+a-b-c}\left[\frac{a}{d}(1+a-b-2c)-2(\frac{1}{2}a-b-c+1)\right]$.

For $d = 1+a-b$ in (3.7), we get Dixon's summation theorem (1.8).

(g). Extensions of Whipple's summation theorem:

(3.8)

$$
{}_4F_3\left[\begin{matrix} a,\, 1-a,\, c,\, d+1 \\ e+1,\, 2c-e+1,\, d \end{matrix};1\right]
$$
$$
= \frac{2^{-2a}\Gamma(e+1)\Gamma(e-c)\Gamma(2c-e+1)}{\Gamma(e-a+1)\Gamma(e-c+1)\Gamma(2c-e-a+1)}
$$
$$
\times\left[\Big(1-\frac{2c-e}{d}\Big)\frac{\Gamma(\frac{1}{2}(e-a)+1)\Gamma(c-\frac{1}{2}(e+a-1))}{\Gamma(\frac{1}{2}(e+a))\Gamma(c+\frac{1}{2}(1+a-e))}+\Big(\frac{e}{d}-1\Big)\frac{\Gamma(\frac{1}{2}(e-a+1))\Gamma(c+1-\frac{1}{2}(e+a))}{\Gamma(\frac{1}{2}(1+a+e))\Gamma(c+\frac{1}{2}(a-e))}\right]
$$

provided $Re(c) > 0$.

For $d = e$ in (3.8), we get Whipple's summation theorem (1.7).

(h). Extension of hypergeometric identity (1.10):

(3.9)

$$
{}_3F_2\left[\begin{matrix} a,\ b,\, 1+d \\ 1+a-b,\, d \end{matrix};-1\right] = \Big(1-\frac{a}{2d}\Big)\frac{\Gamma(1+a-b)\Gamma(1+\frac{a}{2})}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)} + \Big(\frac{a}{2d}\Big)\frac{\Gamma(1+a-b)\Gamma(\frac{1}{2}(a+1))}{\Gamma(1+a)\Gamma(\frac{1}{2}(a+1)-b)}
$$

(i). Extension of hypergeometric identity (1.11)

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\ b,\, c,\, 1+d \\ 1+a-b,\, 1+a-c,\, d \end{matrix};1\right] \\
&= \Big(1-\frac{a}{2d}\Big)\frac{\Gamma(1+a-b)\Gamma(1+\frac{a}{2})\Gamma(1+a-c)\Gamma(1+\frac{a}{2}-b-c)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+\frac{a}{2}-b)\Gamma(1+\frac{a}{2}-c)} \\
&+ \Big(\frac{a}{2d}\Big)\frac{\Gamma(\frac{1}{2}(a+1))\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(\frac{1}{2}(a+1)-b-c)}{\Gamma(1+a)\Gamma(1+a-b-c))\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)}
\end{aligned}
\tag{3.10}
$$

provided $Re(a-2b-2c) > -1$.

For $d = \frac{a}{2}$ in (3.9) and for $d = 1+\frac{1}{2}a$ in (3.10), we get (1.10) and (1.11) respectively.

The main purpose of this paper is to establish new class of several summation formulas by employing the results (3.1) to (3.10) in a very general identity recorded (2.1). Results (2.2) to (2.7) obtained earlier by Eslahchi and Masjed-Jamei [7] follows special cases of our main findings.

4. Generalization of extended summation theorems

In this section, we shall generalize the summation theorems (3.1) to (3.10). These are:

(a). Generalization of extended Gauss's summation theorem:

(4.1)

$$\sum_{k=0}^{m-1} \frac{(a)_k (b)_k (d+1)_k}{(c+1)_k (d)_k k!} \times {}_{3m+1}F_{3m}\left[\begin{matrix} \frac{a+k}{m}, \ldots, \frac{a+m-1+k}{m}, \frac{b+k}{m}, \ldots, \frac{b+m-1+k}{m}, \frac{d+1+k}{m}, \ldots, \frac{d+m+k}{m}, 1 \\ \frac{c+k}{m}, \ldots, \frac{c+m-1+k}{m}, \frac{d+k}{m}, \ldots, \frac{d+m-1+k}{m}, \frac{1+k}{m}, \ldots, \frac{m+k}{m} \end{matrix}; 1\right]$$
$$= \frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a+1)\Gamma(c-b+1)}\left[(c-a-b)+\frac{ab}{d}\right]$$

provided $Re(c-a-b) > 0$.

(b). Generalization of extended Kummer's summation theorem:

(4.2)

$$\sum_{k=0}^{m-1} \frac{(a)_k (b)_k (d+1)_k (-1)^k}{(2+a-b)_k (d)_k k!} \times {}_{3m+1}F_{3m}\left[\begin{matrix} \frac{a+k}{m}, \ldots, \frac{a+m-1+k}{m}, \frac{b+k}{m}, \ldots, \frac{b+m-1+k}{m}, \frac{d+1+k}{m}, \ldots, \frac{d+m+k}{m}, 1 \\ \frac{2+a-b+k}{m}, \ldots, \frac{a-b+1+m+k}{m}, \frac{d+k}{m}, \ldots, \frac{d+m-1+k}{m}, \frac{1+k}{m}, \ldots, \frac{m+k}{m} \end{matrix}; (-1)^m\right]$$
$$= \frac{\Gamma(1/2)\Gamma(2+a-b)}{2^a(1-b)}\left[\frac{((1+a-b)/d)-1}{\Gamma(\frac{a}{2})\Gamma(\frac{1}{2}(a+3)-b)} + \frac{(1-(a/d))}{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{a}{2}-b+1)}\right]$$

provided $Re(b) < 1$.

(c). Generalization of extended Gauss's second summation theorem:

(4.3)

$$\sum_{k=0}^{m-1} \frac{(a)_k (b)_k (d+1)_k (2)^{-k}}{(\frac{1}{2}(a+b+3))_k (d)_k k!} \times {}_{3m+1}F_{3m}\left[\begin{matrix} \frac{a+k}{m}, \ldots, \frac{a+m-1+k}{m}, \frac{b+k}{m}, \ldots, \frac{b+m-1+k}{m}, \frac{d+1+k}{m}, \ldots, \frac{d+m+k}{m}, 1 \\ \frac{k+(a+b+3)/2}{m}, \ldots, \frac{m-1+k+(a+b+3)/2}{m}, \frac{d+k}{m}, \ldots, \frac{d+m-1+k}{m}, \frac{1+k}{m}, \ldots, \frac{m+k}{m} \end{matrix}; 2^{-m}\right]$$
$$= \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}(a+b+3))\Gamma(\frac{1}{2}(a-b-1))}{\Gamma(\frac{1}{2}(a-b+3))}\left[\frac{(1+a+b)/2-(ab/d)}{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(b+1))} + \frac{(a+b+1)/d-2}{\Gamma(\frac{a}{2})\Gamma(\frac{b}{2})}\right]$$

(d). Generaliztion of extended Bailye's summation theorem:

(4.4)

$$\sum_{k=0}^{m-1} \frac{(a)_k(1-a)_k(d+1)_k(2)^{-k}}{(c+1)_k(d)_k k!} \times {}_{3m+1}F_{3m}\left[\begin{matrix}\frac{a+k}{m},\ldots,\frac{a+m-1+k}{m},\frac{1-a+k}{m},\ldots,\frac{-a+m+k}{m},\frac{d+1+k}{m},\ldots,\frac{d+m+k}{m},1\\ \frac{c+1+k}{m},\ldots,\ldots\frac{c+m+k}{m},\frac{d+k}{m},\ldots,\frac{d+m-1+k}{m},\frac{1+k}{m},\ldots,\frac{m+k}{m}\end{matrix};2^{-m}\right]$$

$$=\frac{\Gamma(1/2)\Gamma(c+1)}{2^c}\left[\frac{(2/d)}{\Gamma(\frac{1}{2}(c+a))\Gamma(\frac{1}{2}(c-a+1))}+\frac{1-(c/d)}{\Gamma(\frac{1}{2}(c-a)+1)\Gamma(\frac{1}{2}(c+a+1))}\right]$$

(e). Generalization of extension Watson's summation theorem:

- Generalization of first extension Watson's summation theorem:

$$\sum_{k=0}^{m-1} \frac{(a)_k(b)_k(c)_k(d+1)_k}{(\frac{1}{2}(a+b+1))_k(2c+1)_k(d)_k k!}\,{}_{4m+1}F_{4m}\left[\begin{matrix}A,\,B,\,C,\,D,\,1\\ E,\,F,\,G,\,H\end{matrix};1\right]$$

(4.5) $$=\frac{2^{a+b-2}\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}(a+b+1))\Gamma(c-\frac{1}{2}(a+b-1))}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)} \times\left[\frac{\Gamma(a/2)\Gamma(b/2)}{\Gamma(c+\frac{1}{2}(1-a))\Gamma(c+\frac{1}{2}(1-b))}+\frac{((2c-d)/d)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(b+1))}{\Gamma(1+\frac{1}{2}(c-a))\Gamma(c-\frac{b}{2}+1)}\right]$$

provided $Re(2c-a-b)>-1$.

Also, where $A=\frac{a+k}{m},...,\frac{a+m-1+k}{m}$, $B=\frac{b+k}{m},...,\frac{b+m-1+k}{m}$, $C=\frac{c+k}{m},...,\frac{c+m-1+k}{m}$, $D=\frac{d+1+k}{m},...,\frac{d+m+k}{m}$ and $E=\frac{k+(a+b+1)/2}{m},...,\frac{m+k+(a+b-1)/2}{m}$, $F=\frac{2c+1+k}{m},...,\frac{2c+m+k}{m}$, $G=\frac{d+k}{m},...,\frac{d+m-1+k}{m}$, $H=\frac{1+k}{m},...,\frac{m+k}{m}$.

- Generalization of second extension Watson's summation theorem:

$$\sum_{k=0}^{m-1} \frac{(a)_k(b)_k(c)_k(d+1)_k}{(\frac{1}{2}(a+b+3))_k(2c)_k(d)_k}\,{}_{4m+1}F_{4m}\left[\begin{matrix}A,\,B,\,C,\,D,\,1\\ E,\,F,\,G,\,H\end{matrix};1\right]$$

(4.6) $$=\frac{2^{a+b-2}\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}(a+b+3))\Gamma(c-\frac{1}{2}(a+b+1))}{(a-b-1)(a-b+1)\Gamma(1/2)\Gamma(a)\Gamma(b)} \times\left[\alpha\frac{\Gamma(a/2)\Gamma(b/2)}{\Gamma(c+\frac{1}{2}(1-a))\Gamma(c+\frac{1}{2}(1-b))}+\beta\frac{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(b+1))}{\Gamma(c-\frac{a}{2})\Gamma(c-\frac{b}{2})}\right]$$

provided $Re(2c-a-b)>1$.

Also, where $A=\frac{a+k}{m},\ldots,\frac{a+m-1+k}{m}$, $B=\frac{b+k}{m},\ldots,\frac{b+m-1+k}{m}$, $C=\frac{c+k}{m},\ldots,\frac{c+m-1+k}{m}$, $D=\frac{d+1+k}{m},\ldots,\frac{d+m+k}{m}$ and $E=\frac{k+(a+b+3)/2}{m},\ldots,\frac{m+k+(a+b+1)/2}{m}$, $F=\frac{2c+k}{m},\ldots,\frac{2c+m-1+k}{m}$, $G=\frac{d+k}{m},\ldots,\frac{d+m-1+k}{m}$, $H=\frac{1+k}{m},\ldots,\frac{m+k}{m}$.

Here α and β are given by:
$\alpha = a(2c-a) + b(2c-b) - 2c + 1 - \frac{ab}{d}(4c-a-b-1)$, $\beta = 8\left[\frac{1}{2d}(a+b+1) - 1\right]$.

(f). Generalization of extended Dixon's summation theorem:
(4.7)

$$\sum_{k=0}^{m-1} \frac{(a)_k(b)_k(c)_k(d+1)_k}{(1+a-b)_k(1+a-c)_k(d)_k k!}$$
$$\times {}_{4m+1}F_{4m}\left[\begin{matrix} \frac{a+k}{m}, \ldots, \frac{a+m-1+k}{m}, \frac{b+k}{m}, \ldots, \frac{b+m-1+k}{m}, \frac{c+k}{m}, \ldots, \frac{c+m-1+k}{m}, \frac{d+1+k}{m}, \ldots, \frac{d+m+k}{m}, 1 \\ \frac{1+a-b+k}{m}, \ldots, \frac{a-b+m+k}{m}, \frac{1+a-c+k}{m}, \ldots, \frac{a-c+m+k}{m}, \frac{d+k}{m}, \ldots, \frac{d+m-1+k}{m}, \frac{1+k}{m}, \ldots, \frac{m+k}{m} \end{matrix}; 1\right]$$
$$= \frac{\alpha}{(b-1)} \frac{2^{-a}\Gamma(1+a-c)\Gamma(2+a-b)\Gamma(\frac{1}{2}(a+3)-b-c)\Gamma(1/2)}{\Gamma(\frac{a}{2})\Gamma(\frac{1}{2}(a+1)-c))\Gamma(2+a-b-c)\Gamma(\frac{1}{2}(a+3)-b)}$$
$$+ \frac{\beta}{(b-1)} \frac{2^{-a-1}\Gamma(1/2)\Gamma(1+a-c)\Gamma(1+a-b)\Gamma(1+\frac{a}{2}-b-c)}{\Gamma(\frac{1}{2}(a+1))\Gamma(1+\frac{a}{2}-b)\Gamma(1+\frac{a}{2}-c)\Gamma(1+a-b-c)}$$

provided $Re(a-2b-2c) > -2$.
Here α and β are given by:
$\alpha = 1 - \frac{1}{d}(1+a-b)$, $\beta = \frac{1+a-b}{1+a-b-c}\left[\frac{a}{d}(1+a-b-2c) - 2(\frac{1}{2}a-b-c+1)\right]$.

(g). Generalization of extended Whipple's summation theorem:
(4.8)

$$\sum_{k=0}^{m-1} \frac{(a)_k(1-a)_k(c)_k(d+1)_k}{(e+1)_k(2c-e+1)_k(d)_k k!}$$
$$\times {}_{4m+1}F_{4m}\left[\begin{matrix} \frac{a+k}{m}, \ldots, \frac{a+m-1+k}{m}, \frac{1-a+k}{m}, \ldots, \frac{m-a+k}{m}, \frac{c+k}{m}, \ldots, \frac{c+m-1+k}{m}, \frac{d+1+k}{m}, \ldots, \frac{d+m+k}{m}, 1 \\ \frac{e+1+k}{m}, \ldots, \frac{e+m+k}{m}, \frac{2c-e+1+k}{m}, \ldots, \frac{2c-e+m+k}{m}, \frac{d+k}{m}, \ldots, \frac{d+m-1+k}{m}, \frac{1+k}{m}, \ldots, \frac{m+k}{m} \end{matrix}; 1\right]$$
$$= \frac{2^{-2a}\Gamma(e+1)\Gamma(e-c)\Gamma(2c-e+1)}{\Gamma(e-a+1)\Gamma(e-c+1)\Gamma(2c-e-a+1)}$$
$$\times \left[\left(1 - \frac{2c-e}{d}\right)\frac{\Gamma(\frac{1}{2}(e-a)+1)\Gamma(c-\frac{1}{2}(e+a-1))}{\Gamma(\frac{1}{2}(e+a))\Gamma(c+\frac{1}{2}(1+a-e))} + \left(\frac{e}{d} - 1\right)\frac{\Gamma(\frac{1}{2}(e-a+1))\Gamma(c+1-\frac{1}{2}(e+a))}{\Gamma(\frac{1}{2}(1+a+e))\Gamma(c+\frac{1}{2}(a-e))}\right]$$

provided $Re(c) > 0$.

(h). Generalization of extended hypergeometric identity (3.9):
(4.9)

$$\sum_{k=0}^{m-1} \frac{(a)_k(b)_k(d+1)_k(-1)^k}{(1+a-b)_k(d)_k k!}$$
$$\times {}_{3m+1}F_{3m}\left[\begin{matrix} \frac{a+k}{m}, \ldots, \frac{a+m-1+k}{m}, \frac{b+k}{m}, \ldots, \frac{b+m-1+k}{m}, \frac{d+1+k}{m}, \ldots, \frac{d+m+k}{m}, 1 \\ \frac{1+a-b+k}{m}, \ldots, \frac{a-b+m+k}{m}, \frac{d+k}{m}, \ldots, \frac{d+m-1+k}{m}, \frac{1+k}{m}, \ldots, \frac{m+k}{m} \end{matrix}; (-1)^m\right]$$
$$= \left(1 - \frac{a}{2d}\right)\frac{\Gamma(1+a-b)\Gamma(1+\frac{a}{2})}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)} + \left(\frac{a}{2d}\right)\frac{\Gamma(1+a-b)\Gamma(\frac{1}{2}(a+1))}{\Gamma(1+a)\Gamma(\frac{1}{2}(a+1)-b)}$$

(i). Generalization of extended hypergeometric identity (3.10):
(4.10)

$$\sum_{k=0}^{m-1} \frac{(a)_k(b)_k(c)_k(d+1)_k}{(1+a-b)_k(1+a-c)_k(d)_k k!}$$
$$\times {}_{4m+1}F_{4m}\left[\begin{matrix} \frac{a+k}{m}, \ldots, \frac{a+m-1+k}{m}, \frac{b+k}{m}, \ldots, \frac{b+m-1+k}{m}, \frac{c+k}{m}, \ldots, \frac{c+m-1+k}{m}, \frac{d+1+k}{m}, \ldots, \frac{d+m+k}{m}, 1 \\ \frac{1+a-b+k}{m}, \ldots, \frac{a-b+m+k}{m}, \frac{1+a-c+k}{m}, \ldots, \frac{a-c+m+k}{m}, \frac{d+k}{m}, \ldots, \frac{d+m-1+k}{m}, \frac{1+k}{m}, \ldots, \frac{m+k}{m} \end{matrix}; 1\right]$$
$$= \left(1 - \frac{a}{2d}\right) \frac{\Gamma(1+a-b)\Gamma(1+\frac{a}{2})\Gamma(1+a-c)\Gamma(1+\frac{a}{2}-b-c)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+\frac{a}{2}-b)\Gamma(1+\frac{a}{2}-c)}$$
$$+ \left(\frac{a}{2d}\right) \frac{\Gamma(\frac{1}{2}(a+1))\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(\frac{1}{2}(a+1)-b-c)}{\Gamma(1+a)\Gamma(1+a-b-c))\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)}$$

provided $Re(a-2b-2c) > -1$.

Proof. The derivations of our results (4.1) to (4.10) are quite straight forward therefore, in order to establish result (4.1), we proceed as follows. In the general identity (2.1), if we set $p=3, q=2, a_1=a, a_2=b, a_3=d+1, b_1=c+1, b_2=d$ and $z=1$ for $m=1,2,3$, then we, respectively obtain:

$${}_3F_2\left[\begin{matrix} a, b, d+1 \\ c+1, d \end{matrix}; 1\right] = \frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a+1)\Gamma(c-b+1)}\left[(c-a-b) + \frac{ab}{d}\right],$$

$${}_5F_4\left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2}, \frac{1}{2}d+1 \\ \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}c+1, \frac{1}{2}d, \frac{1}{2} \end{matrix}; 1\right] + \frac{ab(d+1)}{(c+1)d}\, {}_5F_4\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1, \frac{1}{2}b+\frac{1}{2}, \frac{1}{2}b+1, \frac{1}{2}d+\frac{3}{2} \\ \frac{1}{2}c+1, \frac{1}{2}c+\frac{3}{2}, \frac{1}{2}d+\frac{1}{2}, \frac{3}{2} \end{matrix}; 1\right]$$
$$= \frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a+1)\Gamma(c-b+1)}\left[(c-a-b) + \frac{ab}{d}\right],$$

and

$${}_7F_6\left[\begin{matrix} \frac{1}{3}a, \frac{1}{3}a+\frac{1}{3}, \frac{1}{3}a+\frac{2}{3}, \frac{1}{3}b, \frac{1}{3}b+\frac{1}{3}, \frac{1}{3}b+\frac{2}{3}, \frac{1}{3}d+1 \\ \frac{1}{3}c+\frac{1}{3}, \frac{1}{3}c+\frac{2}{3}, \frac{1}{3}c+1, \frac{1}{3}d, \frac{1}{3}, \frac{2}{3} \end{matrix}; 1\right]$$
$$+ \frac{ab(d+1)}{(c+1)d}\, {}_7F_6\left[\begin{matrix} \frac{1}{3}a+\frac{1}{3}, \frac{1}{3}a+\frac{2}{3}, \frac{1}{3}a+1, \frac{1}{3}b+\frac{1}{3}, \frac{1}{3}b+\frac{2}{3}, \frac{1}{3}b+1, \frac{1}{3}d+\frac{4}{3} \\ \frac{1}{3}c+\frac{2}{3}, \frac{1}{3}c+1, \frac{1}{3}c+\frac{4}{3}, \frac{1}{3}d+\frac{1}{3}, \frac{2}{3}, \frac{4}{3} \end{matrix}; 1\right]$$
$$+ \frac{ab(a+1)(b+1)(d+2)}{2d(c+1)(c+2)}\, {}_7F_6\left[\begin{matrix} \frac{1}{3}a+\frac{2}{3}, \frac{1}{3}a+1, \frac{1}{3}a+\frac{4}{3}, \frac{1}{3}b+\frac{2}{3}, \frac{1}{3}b+1, \frac{1}{3}b+\frac{4}{3}, \frac{1}{3}d+\frac{5}{3} \\ \frac{1}{3}c+1, \frac{1}{3}c+\frac{4}{3}, \frac{1}{3}c+\frac{5}{3}, \frac{1}{3}d+\frac{2}{3}, \frac{4}{3}, \frac{5}{3} \end{matrix}; 1\right]$$
$$= \frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a+1)\Gamma(c-b+1)}\left[(c-a-b) + \frac{ab}{d}\right]$$

Therefore, proceeding like this, we can represent the general case for any natural number m is as follows:

$$\sum_{k=0}^{m-1} \frac{(a)_k (b)_k (d+1)_k}{(c+1)_k (d)_k k!}$$

$$\times\, {}_{3m+1}F_{3m}\left[\begin{matrix} \frac{a+k}{m}, \ldots, \frac{a+m-1+k}{m}, \frac{b+k}{m}, \ldots, \frac{b+m-1+k}{m}, \frac{d+1+k}{m}, \ldots, \frac{d+m+k}{m}, 1 \\ \frac{c+k}{m}, \ldots, \frac{c+m-1+k}{m}, \frac{d+k}{m}, \ldots, \frac{d+m-1+k}{m}, \frac{1+k}{m}, \ldots, \frac{m+k}{m} \end{matrix}; 1\right]$$

$$= \frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a+1)\Gamma(c-b+1)} \left[(c-a-b) + \frac{ab}{d}\right]$$

provided $Re(c-a-b) > 0$.

This completes the proof of our first result (4.1). In exactly same the manner, other results can be proven similarly. We however, prefer to omit the details. □

5. Corollaries

In this section, we shall recover several results (3.1) to (3.10) due to Eslahchi and Masjed-Jamei [7] as follows:

(1) In (4.1), if we take $d = c$, we recover the result (3.1).
(2) In (4.2), if we take $d = 1 + a - b$, we recover the result (3.2).
(3) In (4.3), if we take $d = \frac{1}{2}(a + b + 1)$, we recover the result (3.3).
(4) In (4.4), if we take $d = c$, we recover the result (3.4).
(5) In (4.5), if we take $d = 2c$, we recover the result (3.5).
(6) In (4.6), if we take $d = \frac{1}{2}(a + b + 1)$, we recover the result (3.6).
(7) In (4.7), if we take $d = 1 + a - b$, we recover the result (3.7).
(8) In (4.8), if we take $d = e$, we recover the result (3.8).
(9) In (4.9), if we take $d = \frac{1}{2}a$, we recover the result (3.9).
(10) In (4.10), if we take $d = 1 + \frac{1}{2}a$, we recover the result (3.10).

6. Conclusions

In this paper, we have established several new and interesting summation formulas involving generalized hypergeometric functions. These are achieved by employing the extensions of the classical summation theorems due to Kim et al. [11].

On account of the general nature of the results (because of the presence of the factor $d \notin \mathbb{Z}_0^-$), our results would serve as master formulas from which a large number of new and interesting results as special cases can be obtained. Therefore our results may be potentially useful and it is believed that our results would be a definite contribution in theory of generalized hypergeometric function. Also, intersting applications of the results obtained in this paper are under investigations and will form a part of the subsequent paper in this direction.

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Dongkyu Lim (Corresponding author): Department of Mathematics Education, Andong National University, Andong 36729, Republic of Korea
Email address: dklim@anu.ac.kr

Vidha Kulkarni: Department of Mathematics, School of Engineering, Sir Padampat Singhania University, Bhatewar, Udaipur-313601, Rajasthan State, India
Email address: kvidha@gmail.com

Yashoverdhan Vyas: Department of Mathematics, School of Engineering, Sir Padampat Singhania University, Bhatewar, Udaipur-313601, Rajasthan State, India
Email address: yashoverdhan.vyas@spsu.ac.in

Arjun K. Rathie :Department of Mathematics, Vedant College of Engineering & Technology, (Rajasthan Technical University), Village: TULSI, Post: Jakhmund, District: BUNDI-323021, Rajasthan State, India
Email address: arjunkumarrathie@gmail.com