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EQUI EDGE-INCIDENCE AND MAXIMAL EQUI EDGE-INCIDENCE ENERGY OF A GRAPH

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ABSTRACT. In this paper we introduce equi edge-incidence matrix $EI(G)$ and maximal equi edge-incidence matrix $EI_M(G)$ of a simple graph G and obtain bounds for eigenvalues of $EI(G)$ and $EI_M(G)$. We establish equi edge-incidence energy $EEI(G)$, maximal equi edge-incidence energy $EEI_M(G)$ and its bounds. Equi edge-incidence vertex and edge contraction for a simple graph G is defined and applied on Fibonacci difference graph.

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1. INTRODUCTION

The concept of graph energy was introduced by Ivan Gutman for a simple graph in 1978, as the sum of the absolute values of the eigenvalues of its adjacency matrix. Energy of a graph is a much studied concept in the mathematical history. Recent work on energy of graphs can be found in [4] and [5]. In mathematical chemistry a chemical graph is a labelled graph whose vertices correspond to atoms of the compound and edges corresponds to chemical bonds between them. Graphs corresponding to chemical structures represented through matrices play a vital role in analyzing the compound thoroughly and gives an extension to the study of topological indices which is applicable in drug development. For example, the boiling point of alkane is determined by the geometric structure of the alkane which is obtained through the matrix.

Motivated by this, in Section 2 of this paper we introduce equi edge-incidence matrix $EI(G)$ and maximal equi edge-incidence matrix $EI_M(G)$ and establish bounds for eigenvalues of $EI(G)$ and $EI_M(G)$. We also introduce equi-edge incidence energy $EEI(G)$ and maximal equi edge-incidence energy $EEI_M(G)$ and obtain bounds for $EEI(G)$ and $EEI_M(G)$

Chemical compounds represented through graphs in terms of matrices having vertices with equal edge-incidence in particular, corresponds to elements belonging to the same group in the periodic table which gives an insight to the replacement of that element by another element from the same group as it will satisfy the condition of the bond and hence results into different chemical compounds. This plays a vital role in doping techniques for improving the conductivity of semi conductors.

Let G be a simple graph and its vertex set be $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge set be $E(G)$. In G there can be vertices with equal degree and these vertices are either adjacent or not. Considering this factor we define a new variant of energy called equi edge-incidence energy which is denoted by $EEI(G)$.

Definition 1.1. *Let G be a simple graph with n vertices $v_1, v_2, \ldots, v_n$ and let $d(v_i)$ be the degree of v_i for $i = 1, 2, 3, \ldots, n$. The equi-edge incidence matrix of G is the $n \times n$ matrix $EI(G) = (a_{ij})$ whose entries a_{ij} are given by,*

$$a_{ij} = \begin{cases} d(v_i) & if \quad v_iv_j \in E(G) \quad and \quad d(v_i) = d(v_j), \\ -d(v_i) & if \quad v_iv_j \notin E(G) \quad and \quad d(v_i) = d(v_j), \\ 1 & if \quad v_iv_j \in E(G) \quad and \quad d(v_i) \neq d(v_j), \\ 0 & otherwise. \end{cases}$$

The characteristic polynomial of $EI(G)$ is defined by

$$\phi(G; \lambda) = det(\lambda I - EI(G)).$$

Equi edge-incidence energy of a graph is defined as $EEI(G) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$ are the roots of the characteristic polynomial $\phi(G; \lambda) = 0$. Since $EI(G)$ is a real symmetric matrix with zero trace, these eigen values are real with sum equal to zero. Thus, $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 0$.

Example 1.2.

The equi edge-incidence matrix of the graph G in Fig. 1 is

$$EI(G) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 3 & -3 \\ 0 & 1 & 1 & 3 & 0 & -3 \\ 1 & 1 & 1 & -3 & -3 & 0 \end{pmatrix}.$$

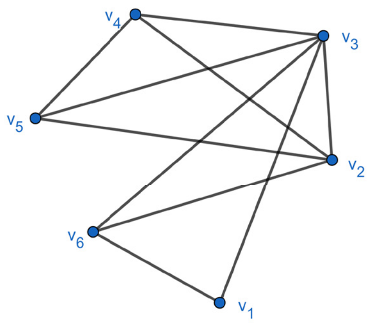

FIGURE 1. G

The characteristic polynomial of equi edge-incidence matrix $EI(G)$ is

$$= \begin{vmatrix} \lambda & 0 & -1 & 0 & 0 & -1 \\ 0 & \lambda & -1 & -1 & -1 & -1 \\ -1 & -1 & \lambda & -1 & -1 & -1 \\ 0 & -1 & -1 & \lambda & -3 & 3 \\ 0 & -1 & -1 & -3 & \lambda & 3 \\ -1 & -1 & -1 & 3 & 3 & \lambda \end{vmatrix}$$

$$= \lambda^6 - 36\lambda^4 - 50\lambda^3 + 183\lambda^2 + 274\lambda + 12$$

and the equi edge-incidence eigenvalues of G are

$$\lambda_1 \approxeq 6.1746, \lambda_2 \approxeq 2.4234, \lambda_3 \approxeq -0.0452,$$

$$\lambda_4 \approxeq -1.4381\lambda_5 \approxeq -3, \lambda_6 \approxeq -4.1148.$$

Equi edge-incidence energy $EEI(G) \approxeq 17.1960$.

Inspired by the minimum covering energy of a graph introduced by C. Adiga et al. [2], in this paper we extend the definition of equi edge-incidence matrix to maximal equi edge-incidence matrix denoted by $EI_M(G)$, by partitioning the vertex set $V(G)$ into sets of equal edge-incidence vertices in G.

If $S_1, S_2, S_3, \ldots, S_k$ are the disjoint subsets of $V(G)$ such that

$$|V(G)| = |S_1| \cup |S_2| \cup |S_3| \cup \ldots \cup |S_k|$$

with each set containing equi edge-incidence vertices then the set with maximum number of vertices with equi edge-incidence or maximum cardinality is denoted by M. The maximal equi edge-incidence matrix of the graph G is then given by $EI_M(G) = (b_{ij})$ where,

$$b_{ij} = \begin{cases} 1 & \text{if} \quad v_iv_j \in E(G) \quad \text{and} \quad i \neq j, \\ 1 & \text{if} \quad v_i \in M \quad \text{and} \quad i = j \\ 0 & otherwise. \end{cases}$$

The characteristic polynomial of $EI_M(G)$ is defined by

$$\psi(G;\mu) = det(\mu I - EI_M(G)).$$

Maximal equi edge-incidence energy of a graph is defined as
$EEI_M(G) = \sum_{i=1}^{n} |\mu_i|$ where $\mu_1 \geq \mu_2 \ldots \geq \mu_n$ are the roots of the characteristic polynomial $\psi(G;\mu) = 0$. Since $EI_M(G)$ is a real symmetric matrix, eigen values are real and also $\mu_1 + \mu_2 + \ldots + \mu_n =| M |$.
The characteristic polynomial of maximal equi edge-incidence matrix of graph G in Fig. 1 is

$$\begin{vmatrix} \mu & 0 & -1 & 0 & 0 & -1 \\ 0 & \mu & -1 & -1 & -1 & -1 \\ -1 & -1 & \mu & -1 & -1 & -1 \\ 0 & -1 & -1 & \mu-1 & -1 & 0 \\ 0 & -1 & -1 & -1 & \mu-1 & 0 \\ -1 & -1 & -1 & 0 & 0 & \mu-1 \end{vmatrix}.$$

$$= \mu^6 - 3\mu^5 - 7\mu^4 + 8\mu^3 + 14\mu^2 + 2\mu,$$

and the maximal equi edge-incidence eigenvalues of G are

$$\mu_1 \approxeq 4.0230, \mu_2 \approxeq 1.7929, \mu_3 \approxeq 0,$$

$$\mu_4 \approxeq -0.1592, \mu_5 \approxeq -1.1766, \mu_6 \approxeq -1.4802.$$

Hence, $EEI_M(G) \approxeq 8.6319$.

We are mainly interested in studying mathematical aspects of maximal equi-edge incidence energy of a graph. It is possible that the maximal equi edge-incidence energy of a graph may have some applications in chemistry as well as in other areas.

2. Bounds for equi edge-incidence matrix and maximal equi edge-incidence matrix

We first compute few coefficients of characteristic polynomial of equi edge-incidence matrix $EI(G)$. Let $\phi(G;\lambda) = det(\lambda I - EI(G)) = c_0\lambda^n + c_1\lambda^{n-1} + \ldots + c_n$ be the characteristic polynomial of G.

Theorem 2.1. *The coefficient c_i of $\lambda^{n-i}, i = 0, 1, 2$ in the characteristic polynomial of the equi edge-incidence matrix $EI(G)$ is,*

(1) $c_0 = 1$.
(2) $c_1 = 0$.
(3) $c_2 = -(\alpha+\beta)$ *where α is the number of pair of adjacent vertices with different edge-incidence and β is the sum of the squares of degree of vertices corresponding to pair of vertices with equal edge-incidence.*

Proof. (1) Directly, from definition of $\phi(G;\lambda) = 0$ it follows that $c_0 = 1$.
(2) Since, the sum of the diagonal elements of $EI(G) = 0$, $c_1 = 0$.
(3) $(-1)^2 c_2$ =sum of the determinants of all 2×2 principal sub matrices of $EI(G)$, that is $\sum_{1\leq i<j\leq n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$

where,

$$a_{ij}a_{ji} = \begin{cases} d^2(v_i) & \text{if} \quad d(v_i) = d(v_j) \\ 1 & \text{if } d(v_i) \neq d(v_j) \quad \text{and} \quad v_iv_j \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$c_2 = -[\alpha + \beta].$$

□

Example 2.2. *For graph G in Figure 1, pair of adjacent vertices with different edge incidence are*

$$v_1v_3, v_1v_6, v_2v_3, v_2v_4, v_2v_5, v_2v_6, v_3v_4, v_3v_5, v_3v_6.$$

Hence $\alpha = 9$. Pair of vertices with equal edge incidence are, v_4v_5, v_4v_6, v_5v_6. Hence $\beta = 3^2 + (-3)^2 + (-3)^2 = 27$. Thus $c_2 = -(9 + 27) = -36$.

Theorem 2.3. *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of $EI(G)$, then $\sum_{i=1}^{n} \lambda_i^2 = 2(\alpha + \beta)$, where α is the number of pair of adjacent vertices with different edge-incidence and β is the sum of the squares of degree of vertices corresponding to pair of vertices with equal edge-incidence.*

Proof.

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}a_{ji} \\ &= 2\sum_{i<j} (a_{ij})^2 + \sum_{i=1}^{n} (a_{ii})^2 \\ &= 2\sum_{i<j} (a_{ij})^2 = 2(\alpha + \beta). \end{aligned}$$

□

Theorem 2.4. *If G is a graph with n vertices, then*

$$EEI(G) \leq \sqrt{2n(\alpha + \beta)}$$

where α is the number of pair of adjacent vertices with different edge-incidence and β is the sum of the squares of degree of vertices corresponding to pair of vertices with equal edge-incidence.

Proof. Let $\lambda_1 \geq \lambda_2 \geq \ldots, \geq \lambda_n$ be the eigen values of $EI(G)$. Now by Cauchy-Schwartz inequality

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \left(\sum_{i=1}^{n} a_i^2 \right) \left(\sum_{i=1}^{n} b_i^2 \right).$$

Putting $a_i = 1$ and $b_i = |\lambda_i|$ in the above inequality we obtain

$$\begin{aligned} EEI^2(G) &= \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \\ &\leq n\left(\sum_{i=1}^{n} |\lambda_i|^2\right) \\ &\leq n\left(\sum_{i=1}^{n} \lambda_i^2\right) \\ &\leq 2n(\alpha+\beta). \end{aligned}$$

□

Theorem 2.5. *If $D = |det(EI(G))|$, then*

$$EEI(G) \geq \sqrt{2(\alpha+\beta) + n(n-1)D^{2/n}}.$$

Proof.

$$\begin{aligned} EEI^2(G) &= \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \\ &= \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i\neq j} |\lambda_i||\lambda_j| \end{aligned}$$

Using Arithmetic and Geometric mean inequality, we have

$$\frac{1}{n(n-1)} \sum_{i\neq j} |\lambda_i||\lambda_j| \geq \left(\prod_{i\neq j} |\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}}$$

$$\begin{aligned} EEI^2(G) &\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i\neq j} |\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}} \\ &\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i=1}^{n} |\lambda_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\ &\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)D^{2/n} \\ &\geq 2(\alpha+\beta) + n(n-1)D^{2/n} \end{aligned}$$

which implies,

$$EEI(G) \geq \sqrt{2(\alpha+\beta) + n(n-1)D^{2/n}}.$$

□

Theorem 2.6. *For the graph G with vertex set $V(G)$ and edge set $E(G)$ and a maximal equi edge-incidence vertex set M with cardinality $|M|$,* *let* $\psi(G,\mu) = c'_0\mu^n + c'_1\mu^{n-1} + \ldots + c'_n$, *be the characteristic polynomial of* $EI_M(G)$ *then,*

(1) $c'_0 = 1$.

(2) $c'_1 = -|M|$.

(3) $c'_2 = \binom{|M|}{2} - |E|$.

(4) $c'_3 = |M||E| - \sum_{v_i\in M}(d(v_i)) - \binom{|M|}{3} - 2\triangle$,

where $\triangle$ *is the number of triangles in the graph* G.

Proof. (1) From the definition it follows that $c_0 = 1$.

(2) Since, sum of diagonal elements of adjacency matrix of $EI_M(G)$ is equal to $|M|$ which is the trace of $EI_M(G)$. Thus, $(-1)^1 c'_1 = |M|$.

(3)

$$\begin{aligned} c'_2 &= \sum_{1\le i<j\le n} \begin{vmatrix} b_{ii} & b_{ij} \\ b_{ji} & b_{jj} \end{vmatrix} \\ &= \sum_{1\le i<j\le n} b_{ii}b_{jj} - \sum_{1\le i<j\le n} (b_{ij}^2) \\ &= \binom{|M|}{2} - |E|. \end{aligned}$$

(4) $c'_3 = (-1)^3 \sum_{1\le i<j<k\le n} \begin{vmatrix} b_{ii} & b_{ij} & b_{ik} \\ b_{ji} & b_{jj} & b_{jk} \\ b_{ki} & b_{kj} & b_{kk} \end{vmatrix}$

$= \binom{|M|}{3} + [\sum_{i=1}^n b_{ii}] \left[\sum_{1\le j<k\le n} b_{jk}\right] - \sum_{i=1}^n b_{ii} \sum_{k=1,k\neq i} b_{ik} - 2\Delta.$

$= |M||E| - \sum_{v_i\in M} d(v_i) - \binom{|M|}{3} - 2\triangle.$

□

Theorem 2.7.

$$EEI_M(G) \le \sqrt{n(2|E| + |M|)}.$$

Proof.

$$
\begin{aligned}
(EEI_M(G))^2 &= \sum_{i=1}^{n} (|\mu_i|)^2 \\
&\leq n(\sum_{i=1}^{n} |\mu_i|^2) \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\
&\leq 2\sum_{i<j}(a_{ij})^2 + \sum_{i=1}^{n}(a_{ii})^2 \\
&\leq n\sum_{i=1}^{n} \mu_i^2 \\
&\leq n(2|E| + |M|).
\end{aligned}
$$

Hence the result. □

Corollary 2.8.

$$EEI_M(G) \geq \sqrt{2|E| + |M| + n(n-1)(D')^{2/n}},$$

where $D' = |det(EI_M(G))|$.

Proof.

$$
\begin{aligned}
(EEI_M(G))^2 &\geq \sum_{i=1}^{n} |\mu_i|^2 + n(n-1)\prod_{i=1}^{n} |\mu_i|^{2/n} \\
&\geq 2|E| + |M| + n(n-1)(D')^{2/n}.
\end{aligned}
$$

□

Theorem 2.9. *Let G be a graph with a maximal equi edge-incidence vertex set M. If $EEI_M(G)$ is a rational number, then*

$$EEI_M(G) \equiv |M|(mod 2).$$

Proof. Let $\mu_1, \mu_2, \ldots, \mu_r$ be the positive eigenvalues and the rest of the eigen values of $EI_M(G)$ be non positive, then

$$
\begin{aligned}
EEI_M(G) &= \sum_{i=1}^{n} |\mu_i| \\
&= (\mu_1 + \mu_2 + \ldots + \mu_r) - (\mu_{r+1} + \mu_{r+2} + \ldots + \mu_n). \\
EEI_M(G) &= 2(\mu_1 + \mu_2 + \ldots + \mu_r) - |M|.
\end{aligned}
$$

Since, $\mu_1, \mu_2, \ldots, \mu_r$ are algebraic integers, so is their sum. Hence the theorem. □

3. Equi edge-incidence energy and maximal equi edge-incidence energy of some standard graphs

Many articles have been published on different types of graph energies. The work of [3], [7] lead to the study of equi edge-incidence energy and maximal equi edge-incidence energy. In this Section we find equi edge-incidence energy $EEI(G)$ and maximal equi edge-incidence energy $EEI_M(G)$ of some well known graphs and their compliments.

Theorem 3.1. *Equi edge-incidence energy of the complete graph K_n is*

$$EEI(K_n) = 2(n-1)^2$$

.

Proof. In K_n , $d(v_1) = d(v_2) = d(v_3) = \ldots = d(v_n) = n-1$.

$$EI(K_n) = \begin{pmatrix} 0 & n-1 & n-1 & \ldots & n-1 & n-1 \\ n-1 & 0 & n-1 & \ldots & n-1 & n-1 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ n-1 & n-1 & n-1 & \ldots & n-1 & 0 \end{pmatrix}$$

$\det[\lambda I - EI(K_n)] = (\lambda + (n-1))^{n-1}(\lambda - (n-1)^2) = 0$.
Hence the result. □

Remark 3.2. *Maximal equi edge-incidence energy of complete graph K_n with respect to the maximal equi edge-incidence vertex set $M = \{v_1, v_2, \ldots, v_n\}$ having $|M| = n$ is $EEI_M(G) = n$.*

Corollary 3.3. *If $\overline{K_n}$ is the compliment of the complete graph, which is a null graph then, $EEI(\overline{K_n}) = 0$*

Remark 3.4. *Maximal equi edge-incidence energy of null graph with respect to the maximal equi-edge incidence vertex set $M = \{v_1, v_2, \ldots, v_n\}$ with $|M| = n$ and $d(v_1) = d(v_2) = d(v_3) = \ldots = d(v_n) = 0$ is*

$$EEI_M(\overline{K_n}) = n.$$

Theorem 3.5. *Equi edge-incidence energy of the cycle graph C_n is*

$$EEI(C_n) = |10 - 2n| + \sum_{m=1}^{n-1} |2 + 8cos(\frac{2\pi m}{n})|.$$

Proof. For C_n, $d(v_1) = d(v_2) = \ldots = d(v_n) = 2$.
Since, $v_iv_j \in C_n$ for $j = 2$ and n and $v_iv_j \notin C_n$ for $j = 3, 4, \ldots, n-1$,

$$EI(C_n) = \begin{pmatrix} 0 & 2 & -2 & \ldots & -2 & 2 \\ 2 & 0 & 2 & \ldots & -2 & -2 \\ -2 & 2 & 0 & \ldots & -2 & -2 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 2 & -2 & -2 & \ldots & 2 & 0 \end{pmatrix}.$$

We can observe that $EI(C_n)$ is a circulant matrix of order $n \times n$. The eigenvalues of such matrices are

$$\lambda_m = \sum_{k=1}^{n} a'_k e^{\frac{2\pi i m(k-1)}{n}}$$

where a'_ks are the entries of the first row and $0 \leq m \leq n-1$. Let $\omega = e^{\frac{2\pi i m}{n}}$. Then,

$$\lambda_m = 2(\omega + \omega^{n-1}) - 2(\omega^2 + \omega^3 + \ldots + \omega^{n-2}) = 2 + 4\omega + 4\omega^{n-1}.$$

Hence,

$$\lambda_m = \begin{cases} 10 - 2n \quad \text{if} \quad m = 0 \\ 2 + 8cos(\frac{2\pi m}{n}) \quad \text{for} \quad 1 \leq m \leq n-1. \end{cases}$$

which completes the proof. □

Remark 3.6. *Maximal equi edge-incidence energy of C_n with respect to $M = \{v_1, v_2, \ldots, v_n\}$ and $|M| = n$ is*

$$EEI_M(C_n) = 3 + \sum_{m=1}^{n-1} |1 + 2cos(\frac{2\pi m}{n})|.$$

Corollary 3.7. *If $\overline{C_n}$ is the compliment of the cycle graph then for $n \geq 4$ it is a regular graph with degree of each vertex being $n-3$, equi edge-incidence energy of $\overline{C_n}$ is*

$$EEI(\overline{C_n}) = |(n-3)(n-5)| + (n-3) \sum_{m=1}^{n-1} |1 + 4cos(\frac{2\pi m}{n})|.$$

Remark 3.8. *Maximal equi edge-incidence energy of $\overline{C_n}$ for $n \geq 4$ with $|M| = n$ is $EEI_M(\bar{C_n}) = |n-2| + \sum_{m=1}^{n-1} |2cos(\frac{2\pi m}{n})|$.*

Theorem 3.9. *Equi edge-incidence energy of a star graph for $n \geq 4$ is*

$$EEI(K_{1,n-1}) = 2(n-1).$$

Proof. The equi edge-incidence adjacency matrix of the star graph

$$EI(K_{1,n-1}) = \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & -1 & \ldots & -1 & -1 \\ 1 & -1 & 0 & \ldots & -1 & -1 \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 1 & -1 & -1 & \ldots & -1 & 0 \end{pmatrix}.$$

The characteristic polynomial of $EI(K_{1,(n-1)})$ is $(\lambda-1)^{n-1}(\lambda+(n-1)) = 0$. Hence the result. □

Remark 3.10. *The maximal equi edge-incidence energy of $K_{1,n-1}$ with maximal equi edge-incidence vertex set $M = \{v_2, v_3, \ldots, v_n\}$ where $|M| = n-1$ is*

$$EEI_M(K_{1,n-1}) = n - 2 + \sqrt{4n-3}.$$

Proof. The maximal equi edge-incidence adjacency matrix of the star graph is given by,

$$EI_M(K_{1,n-1}) = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ 1 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Note:We use the following notations in row and column operations while computing the determinants .

(1) C_i is the i^{th} column, C_i' is the new column obtained after column operation.
(2) Similarly, R_i is the i^{th} row and R_i' is the new row obtained by row operations.

Step 1: Replace R_i with R_i' by applying $R_i - R_2$ for $i = 3, 4, 5, \dots, n$ and taking $(\lambda - 1)^{n-2}$ as a common factor.
Step 2: Replace R_1 by R_1' on applying $R_1 + R_i$ for $i = 3, 4, 5, \dots, n$.
Step 3:On applying C_2+C_i for $i = 3, 4, 5, \dots, n$ results into the characteristic polynomial $(\lambda - 1)^{n-2}(\lambda^2 - \lambda - (n-1))$.
Hence the result. □

Corollary 3.11. *The spectra of Equi edge-incidence matrix of compliment of the star graph $\overline{K_{1,n-1}}$ is $\begin{pmatrix} -2 & 0 & 2n-4 \\ n-2 & 1 & 1 \end{pmatrix}$ and hence its energy is*

$$EEI(\overline{K_{1,n-1}}) = 4(n-2).$$

Remark 3.12. *The Maximal equi edge-incidence energy of complement of $\overline{K_{1,(n-1)}}$ with respect to $M = \{v_2, v_3, \dots, v_n\}$ with $|M| = n-1$ is $EEI_M(\overline{K_{1,n-1}}) = 3 + \sum_{m=1}^{n-1} |1 + 2cos(\frac{2\pi m}{n})|$.*

Theorem 3.13. *Equi edge-incidence energy of complete bipartite graph of order $2n$ is $EEI(K_{n,n}) = 2n(2n-1)$.*

Proof. Let $K_{n,n}$ be the complete bipartite graph with its equi edge-incidence adjacency matrix of order $2n \times 2n$ then,

$$EI(K_{1,n-1}) = \begin{pmatrix} 0 & -n & -n & \dots & -n & n & n & \dots & n \\ -n & 0 & -n & \dots & -n & n & n & \dots & n \\ -n & -n & 0 & \dots & -n & n & n & \dots & n \\ . & . & . & \dots & . & . & . & \dots & . \\ . & . & . & \dots & . & . & . & \dots & . \\ . & . & . & \dots & . & . & . & \dots & . \\ -n & -n & -n & \dots & 0 & n & n & \dots & n \\ n & n & n & \dots & n & 0 & -n & \dots & -n \\ . & . & . & \dots & . & . & . & \dots & . \\ . & . & . & \dots & . & . & . & \dots & . \\ . & . & . & \dots & . & . & . & \dots & . \\ n & n & n & \dots & n & -n & -n & \dots & 0. \end{pmatrix}.$$

Adjacency matrix of the complete bipartite graph is a 2×2 block matrix of the form $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ which is symmetric such that the spectra of the matrix is the union of spectra of the matrices $A+B$ and $A-B$.
The matrix $A+B$ is of the form

$$\begin{pmatrix} n & 0 & \dots & 0 \\ 0 & n & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & n \end{pmatrix},$$

and the matrix $A-B$ is of the form

$$\begin{pmatrix} -n & -2n & \dots & -2n \\ -2n & -n & \dots & -2n \\ \vdots & \vdots & & \vdots \\ -2n & -2n & \dots & -n. \end{pmatrix}.$$

Applying the following row operations on the matrix $A-B$
Step1: Replace R_1 by $R_1' = R_1 + R_2 + \ldots + R_n$ and we get $(\lambda + (2n-1))$ as the common factor.
Step2: Replace C_i by $C_i - C_1$ for $i = 2, 3, \ldots, n$ and replacing R_i by $R_i' = R_i - (2n)R_1'$, for $i = 2, 3, 4, \ldots, n$.
Step3: On taking $(\lambda - n)^{n-1}$ as the common factor from $n-1$ rows yields the following eigenvalues of the matrix $A-B$. $\begin{pmatrix} n & -(2n-1) \\ n-1 & 1 \end{pmatrix}$.
The union of eigen values of $A+B$ and $A-B$ leads to the result. □

Remark 3.14. *The Maximal equi edge-incidence energy of* $K_{n,n} = 2(2n-1)$.

Theorem 3.15. *Equi edge-incidence energy of a ladder rung graph is*

$$EEI(L_{2n}) = 6(n-1).$$

Proof. The equi edge-incidence adjacency matrix of the ladder rung graph with $2n$ vertices is

$$EI(L_{2n}) = \begin{pmatrix} 0 & -1 & -1 & \dots & -1 & 1 & -1 & \dots & -1 & \\ -1 & 0 & -1 & \dots & -1 & -1 & 1 & \dots & -1 & \\ . & . & . & \dots & . & . & . & \dots & . & \\ . & . & . & \dots & . & . & . & \dots & . & \\ . & . & . & \dots & . & . & . & \dots & . & \\ -1 & -1 & -1 & \dots & 0 & -1 & -1 & \dots & 1 & \\ 1 & -1 & -1 & \dots & -1 & 0 & -1 & \dots & -1 & \\ . & . & . & \dots & . & . & . & . & \dots & . \\ . & . & . & \dots & . & . & . & . & \dots & . \\ . & . & . & \dots & . & . & . & . & \dots & . \\ -1 & -1 & -1 & \dots & 1 & -1 & -1 & \dots & 0 & \end{pmatrix}.$$

The equi edge-incidence adjacency matrix of ladder rung graph is a block symmetric matrix of the form discussed earlier. Performing the same row and coloumn operations as in Theorem 3.13 leads to the following spectra

$$\begin{pmatrix} -(2n-3) & 3 & -1 \\ 1 & n-1 & n \end{pmatrix}.$$

Hence the result. □

Remark 3.16. *Maximal equi edge-incidence energy of* $L_{2n} = 2n$.

Corollary 3.17. *If* $\overline{L}_{2n}$ *is the compliment of the ladder rung graph which is a cocktail party graph, then its spectra is*

$$\begin{pmatrix} (2n-1)(4n-3) & -3(2n-1) & 2n-1 \\ 1 & n-1 & n \end{pmatrix}.$$

Hence, equi edge-incidence energy of cocktail party graph is given by

$$EEI(\overline{L}_{2n}) = 2(2n-1)(4n-3).$$

Remark 3.18. *Maximal equi edge-incidence energy of* $\overline{L}_{2n} = 4n - 2$.

Theorem 3.19. *For a* d *regular graph one of the eigenvlaues of equi-edge incidence matrix is* $d(2r - n + 1)$

Proof. If G is a d-regular graph with r adjacent vertices then, there exists $n - r - 1$ non adjacent vertices. Hence the theorem. □

4. Equi edge-incidence graph contraction

In this section we introduce equi edge-incidence vertex contraction and equi edge-incidence edge contraction on a simple connected graph G.

Vertex and edge contraction are fundamental graph operation techniques that are useful in structures where we wish to simplify a graph by identifying vertices or edges that represent essentially equivalent entities. This technique is highly applicable in coalescing performed in graph coloring, register allocation, in creating low polygon models for 3D modelling packages and many other areas. In chemistry during mechanism of Diel's Alder reaction one of the chemically stable products obtained from the substrate follows equi edge-incidence contraction.

Definition 4.1. *Equi edge-incidence vertex contraction is the operation that produces a graph* $G^{\underline{v}}$ *from* G *by identifying vertices* v_i *and* v_j *for* $i \neq j$ *in* G *with* $d(v_i) = d(v_j)$ *which might be adjacent or not, being replaced with a single vertex* w *in* $G^{\underline{v}}$ *such that* w *is adjacent to the union of the vertices to which* v_i *and* v_j *were adjacent in* G. *If* v_i *and* v_j *are connected by an edge that edge is removed in* $G^{\underline{v}}$ *on contraction.*

Definition 4.2. *Equi edge-incidence edge contraction is the special case of equi edge-incidence vertex contraction. The contraction of an edge* v_iv_j *in* $E(G)$ *for* $i \neq j$ *is applied by identifying adjacent vertices* v_i *and* v_j *with* $d(v_i) = d(v_j)$ *and replacing them with a single vertex* w' *in* $G^{\underline{e}}$ *such that any edges that were incident on* v_i *or* v_j *are adjacent to* w'. *The operation is performed in any order on a set of edges of* G *by contracting edges holding equi degree vertices.*

Example 4.3.

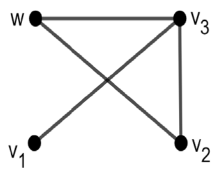

FIGURE 2. $G^{\underline{v}}$

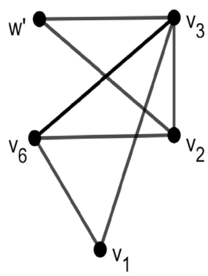

FIGURE 3. $G^{\underline{e}}$

For the graph G in figure 1, equi edge-incidence vertex contraction on G is the graph $G^{\underline{v}}$ (fig. 2), where vertex w in the above graph is obtained on contracting vertices v_4, v_5 and v_6 as $d(v_4) = d(v_5) = d(v_6)$. Similarly, equi edge-incidence edge contraction on G is the graph $G^{\underline{e}}$ (fig.3), where vertex w' in the above graph is obtained on contracting edge v_4v_5 as $d(v_4) = d(v_5)$.

Further, in this section we apply equi edge-incidence vertex contraction corresponding to vertices with equal edge-incidence (equal degree) and equi edge-incidence edge contraction corresponding to adjacent vertices with equal degree on Fibonacci difference graph G_{F_d} introduced by C. Adiga et al. [1].

Theorem 4.4. *For the Fibonacci difference graph G_{F_d}, vertex contraction on equi edge-incidence vertices of $V(G_{F_d})$ results into a graph $G^{\underline{v}}_{F_d} \cong K_{d-2}$.*

Proof. By Theorem 2.15 in [1], the degree sequence of G_{F_d} is monotonically increasing and assume every degree between $\delta = d-1$ and $\Delta = 2d-4$ with G_{F_d} having $\Delta - \delta + 1 = d-2$ different degrees. Now we partition the vertex

set $V(G_{F_d})$ into $(d-2)$ sets $S_1, S_2, \ldots, S_{d-2}$ where,
$S_1 = \{v_{F_1}, v_{F_d}\}$,
$S_{m+1} = \{(v_k | F_m < k \leq F_{m+1}) \cup (v_k | F_d - F_{m+1} + 1 \leq k < F_d - F_m + 1)\}$,
for $1 \leq m \leq (d-4)$
and $S_{d-2} = \{v_k | F_{d-3} + 1 \leq k \leq F_d - F_{d-3}\}$.

Observe that degree of each vertex in S_i for $1 \leq i \leq d-2$ is same and $S_i \bigcap S_j$ is empty for $i \neq j$. Moreover, $\bigcup_{i=1}^{d-2} S_i = V(G_{F_d})$. After applying vertex contraction, we get $G_{F_d}^{\text{V}}$ with $d-2$ vertices $w_1, w_2, \ldots, w_{d-2}$. Here, w_i is obtained by contracting all vertices in S_i.

Now, we shall show that vertices $w_1, w_2, \ldots, w_{d-2}$ in $G_{F_d}^{\text{V}}$ are mutually adjacent and $G_{F_d}^{\text{V}} \cong K_{d-2}$ where K_{d-2} is a complete graph with $d-2$ vertices.

Case 1: Suppose $j = i+1$ and $j < d-2$.
Since $v_{F_i} \in S_i$, $v_{F_j} = v_{F_{i+1}} \in S_j$, as $F_{i+1} - F_i = F_{i-1}$ is a Fibonacci number, there exists an edge between w_i and w_j for $j = i+1$ and $j < d-2$.

Case 2: Suppose $i+1 < j < d-2$.
Since $i+1 < j$, $i \leq j-2$ and hence $F_i \leq F_{j-2}$.
This implies $F_{j-1} + F_i \leq F_{j-1} + F_{j-2} = F_j$.
Hence $v_{F_{j-1}+F_i} \in S_j$, as $F_{j-1} + F_i - F_i = F_{j-1}$ is a Fibonacci number. This guarantees an edge between w_i and w_j for $i+1 < j < d-2$.

Case 3: Suppose $j = d-2$. For $1 \leq i \leq d-3$, we have $v_{F_i} \in S_i$. As $3F_{d-3} \leq F_{d-3} + F_{d-2} + F_{d-2} = F_d$ and $F_{d-3} + F_i \leq F_d - F_{d-3}$.
This implies $v_{F_{d-3}+F_i} \in S_{d-2}$. Moreover $F_{d-3} + F_i - F_i = F_{d-3}$ is a Fibonacci number for $d > 3$, which confirms the existence of an edge between w_i and w_j for $j = d-2$.
Hence the result. □

Remark 4.5. [6] *For a connected graph G we know that $\omega \leq \chi(G) \leq \Delta(G) + 1$ where ω is the clique number and $\Delta(G)$ the maximum degree. For $d \geq 4$ in G_{F_d} graph, $\omega = 4$ and $\Delta(G) = 2d-4$. Hence $\chi(G_{F_d})$ lies between 4 and $2d-3$. On vertex contraction $G_{F_d}^{v} \cong K_{d-2}$ hence $\chi(G_{F_d}^{v}) = d-2$.*

Theorem 4.6. *For the graph G_{F_d} on applying repeated edge contraction on edges holding vertices of equi edge-incidence results into a graph $G_{F_d}^{e}$. For $d \geq 5$ $|V(G_{F_d}^{e})| = 2d-6$.*

Proof. As in Theorem 4.3 we partition the vertex set of $G(F_d)$ into sets S_i such that $\bigcup_{i=1}^{d-2} S_i = V(G_{F_d})$, where degree of each vertex in S_i for $1 \leq i \leq d-2$ is same. Now we apply equi edge-incidence edge contraction on G_{F_d} and obtain the graph $G_{F_d}^{\text{e}}$ with $2d-6$ vertices.
Vertices v_{F_1} and v_{F_d} belonging to the set S_1 represents two distinct vertices in $G_{F_d}^{\text{e}}$ as v_{F_1} and v_{F_d} are non adjacent in G_{F_d}.
Case 1: For $1 \leq m \leq d-5$ the set

$$S_{m+1} = \{(v_k | F_m < k \leq F_{m+1}) \cup (v_k | F_d - F_{m+1} + 1 \leq k < F_d - F_m + 1)\}$$

splits into two distinct sets

$$S'_{m+1} = \{v_k | F_m < k \leq F_{m+1}\}$$

and

$$S''_{m+1} = \{v_{k'} | F_d - F_{m+1} + 1 \leq k' < F_d - F_m + 1\},$$

as $v_k v_{k'} \notin E(G^{\mathrm{e}}_F d)$. It results into $2(d-5)$ vertices.
Case 2: For $m = d-4$, the set S_{m+1} shrinks to a single vertex and

$$v_{F_m+1} \in S'_{m+1}, v_{F_d - F_{m+1}+1} \in S''_{m+1}.$$

Also, $F_d - F_{m+1} + 1 - (F_m + 1) = F_{d-1}$ is a Fibonacci number. Hence, there exists an edge between the vertices of the sets S'_{m+1} and S''_{m+1}.
Case 3: On repeated equi edge-incidence edge contraction, it is obvious that the set S_{m+1} for $m = d-3$ shrinks to a single vertex in $G^{\mathrm{e}}_{F_d}$.
Thus, $|V(G^{\mathrm{e}}_{F_d})| = 2 + 2(d-5) + 1 + 1 = 2d - 6$. □

Corollary 4.7. *$G^{e}_{F_d}$ is a biregular graph with 2 vertices having degree $2d-7$ and the remaining vertices with degree $d-3$ hence $|E(G^{e}_{F_d})| = d^2 - 5(d-1)$.*

Proof. Vertices of the set S_{m+1} for $m = d-4$ are

$$v_{F_{d-4}+1}, v_{F_{d-4}+2}, \ldots, v_{F_{d-3}}, v_{F_d - F_{d-4}}, v_{F_d - F_{d-4} - 1}, \ldots, v_{F_d - F_{d-3}+1}.$$

Vertices of the set S_{m+1} for $m = d-3$ are

$$v_{F_{d-3}+1}, v_{F_{d-3}+2}, \ldots, v_{F_{d-2}}, v_{F_d - F_{d-2}+1} \ldots, v_{F_d - F_{d-3}}.$$

For $1 \leq m \leq d-5$ vertices of the set S'_{m+1} are $v_{F_m+1}, v_{F_m+2} \ldots, v_{F_{m+1}}$ and vertices of the set S''_{m+1} are $v_{F_d - F_{m+1}+1}, v_{F_d - F_{m+1}+2}, \ldots, v_{F_d - F_m}$. It is evident to see that there is an edge between the vertices in $G^{\mathrm{e}}_{F_d}$ obtained on shrinking vertices of the set S_{d-2} in G_{F_d} to vertices of S'_{m+1} and S''_{m+1} for $1 \leq m \leq d-5$ as $F_{d-2} - F_{d-3} = F_{d-4}$ is a Fibonacci number. Also $F_d - F_{d-3} - (F_d - F_{d-4}) = F_{d-2}$ is a Fibonacci number. Thus, the vertex in $G^{\mathrm{e}}_{F_d}$ obtained on shrinking vertices of set S_{d-2} in G_{F_d} is adjacent to all the remaining vertices.
Similarly, the vertex obtained on shrinking the vertices of the set S_{d-3} is of degree $2d-7$, whereas vertices of the set S'_{m+1} and S''_{m+1} for $1 \leq m \leq d-5$ is adjacent to $(d-4) - 1 + 2 = d-3$ vertices in $G^{\mathrm{e}}_{F_d}$. Hence the $2d-8$ vertices in $G^{\mathrm{e}}_{F_d}$ has $d-3$ degree.

$$|E(G^{\mathrm{e}}_{F_d})| = \frac{1}{2}(2(2d-7) + (d-3)(2d-8)) = d^2 - 5(d-1).$$

Hence the result. □

Theorem 4.8. *For $d \geq 8$ energy of $G^{e}_{F_d}$ graph is*

$$3d - 14 + \sqrt{d^2 + 4d - 28}.$$

Proof. For $G^{\mathrm{e}}_{F_d}$ adjacency matrix is

$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 1 & 1 & 0 & \dots & 0 \\ . & . & . & \dots & . & . & . & \dots & . \\ . & . & . & \dots & . & . & . & \dots & . \\ . & . & . & \dots & . & . & . & \dots & . \\ 1 & 1 & 1 & \dots & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 & 1 & \dots & 1 \\ . & . & . & \dots & . & . & . & . & \dots & . \\ . & . & . & \dots & . & . & . & . & \dots & . \\ . & . & . & \dots & . & . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{(2d-6)\times(2d-6)}.$$

The characterisitic polynomial of $G^{\mathrm{e}}_{F_d}$ is given by

$$\begin{vmatrix} \nu & -1 & -1 & \dots & -1 & -1 & 0 & \dots & 0 \\ -1 & \nu & -1 & \dots & -1 & -1 & 0 & \dots & 0 \\ . & . & . & \dots & . & . & . & \dots & . \\ . & . & . & \dots & . & . & . & \dots & . \\ . & . & . & \dots & . & . & . & \dots & . \\ -1 & -1 & -1 & \dots & \nu & -1 & -1 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 & \nu & -1 & \dots & -1 \\ . & . & . & \dots & . & . & . & . & \dots & . \\ . & . & . & \dots & . & . & . & . & \dots & . \\ . & . & . & \dots & . & . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & -1 & -1 & -1 & \dots & \nu \end{vmatrix}_{(2d-6)\times(2d-6)}.$$

We simplify the determinant by applying the following row and column operations.

(1) Replace R_1 by $R'_1 = R_1 + R_2 + \dots + R_{d-4}$ and R_{d-1} by $R'_{d-1} = R_{d-1} + R_d + \dots + R_{2d-6}$.
(2) Replace R'_1 by $R''_1 = R'_1 - R'_{d-1}$, and take $\lambda - (d-5)$ as a common factor from R''_1.
(3) Replace C_i by $C'_i = C_i - C_1$ for $i = 2, 3, \dots, d-3$ and C_i by $C'_i = C_i - C_{d-1}$ for $i = d, d+1, \dots, 2d-6$. This results into $\lambda + 1$ being a common factor from $2d-9$ columns (except from C_1, C_{d-3} and C_{d-1}).
(4) Replace C'_{d-3} by $C''_{d-3} = C'_{d-3} - (C_d + C_{d+1} + \dots C_{2d-6})$. Also replace C'_{d-1} by $C''_{d-1} = C'_{d-1} - (C_d + C_{d+1} + \dots C_{2d-6})$.
(5) Replace R'_{d-3} by $R''_{d-3} = R'_{d-3} + R''_1$, also R'_{d-2} by $R''_{d-2} = R'_{d-2} + R''_1$.
(6) Replace C''_{d-3} by $C'''_{d-3} = C''_{d-3} + C'_1$. Similarly C''_{d-1} by $C'''_{d-1} = C''_{d-1} + C'_1$.

Hence the characteristic polynomial of $G^{\mathrm{e}}_{F_d}$ is

$$(\nu+1)^{2d-9}(\nu - (d-5))(\nu^2 - (d-4)\nu - (3d-11)) = 0.$$

Spectra of $G^{\mathrm{e}}_{F_d}$ is: $\begin{pmatrix} -1 & d-5 & \frac{(d-4)\pm\sqrt{d^2+4d-28}}{2} \\ 2d-9 & 1 & 1 \end{pmatrix}$ and hence energy is given by $3d - 14 + \sqrt{d^2+4d-28}$. □

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