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# NOVEL RESULTS FOR GENERALIZED APOSTOL TYPE POLYNOMIALS ASSOCIATED WITH HERMITE POLYNOMIALS

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Abstract. In this paper, the authors introduce a new class of Hermite-based generalized Apostol-Bernoulli, Apostol-Euler, and Apostol-Genocchi polynomials. The authors then derive some basic properties and several implicit summation formulae by utilizing the series manipulation methods. The authors also investigate several symmetric identities, which are extensions of many earlier well-known results. Moreover, the authors consider a novel class of Hermite-based generalized Apostol-Bernoulli, Apostol-Euler, and Apostol-Genocchi polynomials including geometric and Bell polynomials, and give some basic properties.

## 1. Introduction

Apostol [1] considered a class of the usual Bernoulli numbers and polynomials (called Apostol-Bernoulli numbers and Apostol-Bernoulli polynomials) when he worked the Lipschitz-Lerch Zeta functions and derived several basic properties of these numbers and polynomials. From Apostol's time to now, Apostol type polynomials with many extensions have been studied and investigated by many mathematicians, *cf.* [1,2,6,10,13-17,19-21,23,33] and see also the references cited therein. For example, Luo et al. [19] introduced Apostol-Bernoulli polynomials and numbers of higher order and also, investigated some of their elementary properties and various explicit identities for them including the Gaussian hypergeometric function and the Hurwitz-Zeta function. Luo [16] considered Apostol-Euler polynomials and numbers of higher order and also, derived some of their basic properties and diverse explicit formulas involving the Stirling numbers of the second kind and the Gaussian hypergeometric function. Luo et al. [20] defined Apostol-Genochhi polynomials and numbers of higher order and also, acquired some of their elementary properties and explicit relationships with the Apostol-Bernoulli polynomials and the Apostol-Euler polynomials, and derived various explicit series representations in terms of the Gaussian hypergeometric function and the Hurwitz (or generalized) zeta function. Boyadzhiev [5] investigated some new connections for Apostol-Bernoulli polynomials related to the derivative polynomials and Eulerian polynomials. Khan [10] considered generalized Apostol type Hermite-based polynomials and investigated some properties and relations of them. Kurt [13] presented a new unification of Apostol type polynomials and numbers, and discovered some symmetry identities, explicit relationships and recurrence relation for these unified polynomials. Llanos et al. [14] defined modified generalized Apostol-type polynomials and attained some algebraic and differential properties associated with the Stirling numbers of the second kind, the Jacobi polynomials, the generalized Bernoulli polynomials, the Genocchi polynomials and the Apostol-Euler polynomials. Lu et al. [15] studied some properties of the generalized Apostol-type polynomials given in [20] and provided several recurrence relations, differential equations and some other connected problems. Luo [17] introduced Apostol-Genocchi polynomials and $q$-Apostol-Genocchi polynomials and then, gave some basic properties and several interesting relationships for $q$-Apostol-Genocchi polynomials.

The Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda)$, the Apostol-Euler polynomials $E_n^{(\alpha)}(x;\lambda)$ and the Apostol-Genocchi polynomials $G_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$, are defined by means of the following generating functions

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(see [1,2,6,10,13-17,19-21,23,33]):

$$\left(\frac{t}{\lambda e^t-1}\right)^{\alpha} e^{xt}=\sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!},\ (|t+\ln\lambda|<2\pi, 1^{\alpha}=1), \tag{1.1}$$

$$(|t|<2\pi \ \ when\ \lambda=1;\ \ |t|<|\log\lambda|\ \ when\ \lambda\neq 1)$$

$$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha} e^{xt}=\sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!},\ (|t+\ln\lambda|<\pi, 1^{\alpha}=1), \tag{1.2}$$

$$(|t|<\pi \ \ when\ \lambda=1;\ \ |t|<|\log(-\lambda)|\ \ when\ \lambda\neq 1)$$

and

$$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha} e^{xt}=\sum_{n=0}^{\infty} G_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!},\ (|t+\ln\lambda|<\pi, 1^{\alpha}=1). \tag{1.3}$$

$$(|t|<\pi \ \ when\ \lambda=1;\ \ |t|<|\log(-\lambda)|\ \ when\ \lambda\neq 1).$$

Upon setting $\lambda=1$, the polynomials in (1.1) to (1.3) reduce to the classical Bernoulli, Euler, and Genocchi polynomials of order $\alpha\in\mathbb{C}$, respectively (*cf.* [2,6,9,13,19,20,33]):

$$B_n^{(\alpha)}(x;1):=B_n^{(\alpha)}(x),\ E_n^{(\alpha)}(x;1):=E_n^{(\alpha)}(x) \text{ and } G_n^{(\alpha)}(x;1):=G_n^{(\alpha)}(x).$$

Also note that

$$B_n^{(1)}(x;\lambda):=B_n(x;\lambda),\ E_n^{(1)}(x;\lambda):=E_n(x;\lambda) \text{ and } G_n^{(1)}(x;\lambda):=G_n(x;\lambda),$$

which are familiar Apostol-Bernoulli polynomials, Apostol-Euler polynomials and the Apostol-Genocchi polynomials, respectively.

Moreover, in the following special cases,

$$B_n^{(1)}(x;1):=B_n(x),\ E_n^{(1)}(x;1):=E_n(x) \text{ and } G_n^{(1)}(x;1):=G_n(x)$$

are called, respectively, usual Bernoulli, Euler and Genocchi polynomials (see [1-34]).

A unification of the above polynomials in (1.1) to (1.3) are introduced as follows (see [15]):

$$\left(\frac{2^{\mu}t^{\nu}}{\lambda e^t+1}\right)^{\alpha} e^{xt}=\sum_{n=0}^{\infty} F_n^{(\alpha)}(x;\lambda;\mu,\nu)\frac{t^n}{n!},\quad (|t|<|\log(-\lambda)|), \tag{1.4}$$

where $\alpha\in\mathbb{N}_0, \mu,\nu\in\mathbb{C}$. Upon setting $x=0$,

$$F_n^{(\alpha)}(0;\lambda;\mu,\nu):=F_n^{(\alpha)}(\lambda;\mu,\nu), \tag{1.5}$$

denotes the generalized Apostol type numbers of order $\alpha$.

Notice that

$$(-1)^{\alpha}F_n^{(\alpha)}(x;-\lambda;0,1)=B_n^{(\alpha)}(x;\lambda), \tag{1.6}$$

$$F_n^{(\alpha)}(x;\lambda;1,0)=E_n^{(\alpha)}(x;\lambda) \tag{1.7}$$

and

$$F_n^{(\alpha)}(x;\lambda;1,1)=G_n^{(\alpha)}(x;\lambda). \tag{1.8}$$

Let $c>0$. The generalized 2-variable 1-parameter Hermite Kampé de Fériet polynomials $H_n(x,y,c)$ are defined by (*cf.* [2,3])

$$c^{xt+yt^2}=\sum_{n=0}^{\infty} H_n(x,y,c)\frac{t^n}{n!}. \tag{1.9}$$

Letting $c=e$, it yields 2-variable Hermite Kampé de Fériet polynomials $H_n(x,y)$ (see [2,3,10-12,24-27,33]) given by

$$e^{xt+yt^2}=\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}. \tag{1.10}$$

By (1.9), it is readily derived that

$$H_n(x,y,c)=\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{j}(\ln c)^{n-j}x^{n-2j}y^j. \tag{1.11}$$

The exponential generating function for the geometric polynomials (also known as Fubini polynomials) $F_n(x)$ is given by [5,6,32]:

$$\frac{1}{1-x(e^t-1)}=\sum_{n=0}^{\infty}F_n(x)\frac{t^n}{n!}. \tag{1.12}$$

The geometric polynomials have also close relationships with Apostol-Bernoulli numbers $\beta_n(\lambda)$ and Apostol-Euler numbers $E_n(\lambda)$ as follows (see [6]):

$$\beta_n(\lambda)=\frac{n}{\lambda-1}F_n(\frac{\lambda}{1-\lambda}),\ \ (\lambda\neq 1)\ \text{ and } E_n=F_n(\frac{-1}{2}),$$

where

$$\frac{t}{\lambda e^t-1}=\sum_{n=0}^{\infty}\beta_n(\lambda)\frac{t^n}{n!}\text{ and }\frac{2}{\lambda e^t+1}=\sum_{n=0}^{\infty}E_n(\lambda)\frac{t^n}{n!}. \tag{1.13}$$

The exponential generating function of the Bell (also called exponential) polynomials $\phi_n(x)$ is given by (see [3,4,29-31]):

$$e^{x(e^t-1)}=\sum_{n=0}^{\infty}\phi_n(x)\frac{t^n}{n!} \tag{1.14}$$

and gives the following recurrence relation

$$\phi_{n+1}(x)=x(\phi_n(x)+\frac{d}{dx}\phi_n(x).$$

The geometric and exponential polynomials hold the the following relationship (*cf.* [3,4,29-32])

$$F_n(x)=\int_0^{\infty}\phi_n(x)e^{-\lambda}d\lambda \tag{1.15}$$

Recently, for $\lambda,\mu,\nu\in\mathbb{C}$, Khan [10] introduced the generalized Apostol type Hermite-based polynomials ${}_HF_n^{(\alpha)}(x,y;\lambda,\mu,\nu;c)$ of order $\alpha\in\mathbb{N}_0$ in a suitable neighborhood $t=0$ as follows:

$$\left(\frac{2^{\mu}t^{\nu}}{\lambda c^t+1}\right)^{\alpha}c^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HF_n^{(\alpha)}(x,y;\lambda;\mu,\nu;c)\frac{t^n}{n!}. \tag{1.16}$$

Then,

$${}_HF_n^{(\alpha)}(x,y;\lambda;\mu,\nu;c)=\sum_{m=0}^{n}\binom{n}{m}F_{n-m}^{(\alpha)}(\lambda;\mu,\nu)H_m(x,y;c).$$

In this study, we consider a new class of Hermite-based generalized Apostol-Bernoulli, Apostol-Euler, and Apostol-Genocchi polynomials. We then derive some basic properties and several implicit summation formulae by utilizing the series manipulation methods. We also investigate several symmetric identities, which are extensions of many earlier well-known results. Moreover, we consider a novel class of Hermite-based generalized Apostol-Bernoulli, Apostol-Euler, and Apostol-Genocchi polynomials including geometric and Bell polynomials, and give some basic properties.

## 2. A New Class of Generalized Hermite-Based Apostol Type Polynomials

Recently, for $m \in \mathbb{N}$, $\alpha, \lambda, \mu, \nu \in \mathbb{C}$ and $a, c \in \mathbb{R}^+$, Llanos et al. [14] introduced a new unified presentation of the generalized Apostol type polynomials $Q_n^{[m-1,\alpha]}(x; c, a; \lambda; \mu; \nu)$ of order $\alpha$ by means of the following generating function:

$$\left( \frac{(2^\mu t^\nu)^m}{\lambda c^t + \sum\limits_{h=0}^{m-1} \frac{(t \ln a)^h}{h!}} \right)^\alpha c^{xt} = \sum_{n=0}^{\infty} Q_n^{[m-1,\alpha]}(x; c, a; \lambda; \mu; \nu) \frac{t^n}{n!}, \tag{2.1}$$

where $|t| < 2\pi$ when $\lambda = 1$, $|t| < \pi$ when $\lambda = -1$, $\left|t \ln(\frac{c}{a})\right| < |\log(-\lambda)|$ when $\lambda \in \mathbb{C} \setminus \{-1, 1\}$ and $1^\alpha := 1$. It is noted that setting $x = 0$ the polynomials given in (2.1) reduce to the corresponding numbers:

$$Q_n^{[m-1,\alpha]}(0; c, a; \lambda; \mu; \nu) \frac{t^n}{n!} := Q_n^{[m-1,\alpha]}(c, a; \lambda; \mu; \nu) \frac{t^n}{n!}.$$

Motivated and inspired by above, now, we introduce a unification of generalized Apostol type Hermite-Bernoulli, Apostol type Hermite-Euler and Apostol type Hermite-Genocchi polynomials ${}_HQ_n^{[m-1,\alpha]}(x, y; c, a; \lambda; \mu; \nu)$, $m \geq 1$ (2VHBATP) for a real or complex parameter $\alpha$ in a suitable neighborhood of $t = 0$ by means of the following generating function:

$$\left( \frac{(2^\mu t^\nu)^m}{\lambda c^t + \sum\limits_{h=0}^{m-1} \frac{(t \ln a)^h}{h!}} \right)^\alpha c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HQ_n^{[m-1,\alpha]}(x, y; c, a; \lambda; \mu; \nu) \frac{t^n}{n!}, \tag{2.2}$$

which contains not only generalized Apostol type polynomials $Q_n^{[m-1,\alpha]}(x; c, a; \lambda; \mu; \nu)$, but also Kampé de Fériet generalization of the Hermite polynomials $H_n(x, y)$ in Eq.(1.9).
By substituting $x = y = 0$ in (2.2), we obtain the numbers $Q_n^{[m-1,\alpha]}(c, a; \lambda; \mu; \nu)$, $m \geq 1$ given by

$$\left( \frac{2^\mu t^\nu}{\lambda c^t + \sum\limits_{h=0}^{m-1} \frac{(t \ln a)^h}{h!}} \right)^\alpha = \sum_{n=0}^{\infty} Q_n^{[m-1,\alpha]}(c, a; \lambda; \mu; \nu) \frac{t^n}{n!}. \tag{2.3}$$

**Remark 1.** *When $\lambda \to -\lambda, \mu = 0, \nu = 1$ and $c = e$, the polynomials ${}_HQ_n^{[m-1,\alpha]}(x, y; c, a; \lambda; \mu; \nu)$ reduces to Hermite-based Apostol Bernoulli polynomials. When $\mu = 1, \nu = 0$ and $c = e$, the polynomials ${}_HQ_n^{[m-1,\alpha]}(x, y; c, a; \lambda; \mu; \nu)$ reduces to Hermite-based Apostol Euler polynomials. When $\mu = \nu = 1$ and $c = e$, the polynomials ${}_HQ_n^{[m-1,\alpha]}(x, y; c, a; \lambda; \mu; \nu)$ reduces to Hermite-based Apostol Genocchi polynomials.*

By (1.9) and (2.2), we have the following relationship

$${}_HQ_n^{[m-1,\alpha]}(x, y; c, a; \lambda; \mu; \nu) = \sum_{r=0}^{n} \binom{n}{r} Q_n^{[m-1,\alpha]}(c, a; \lambda; \mu; \nu) H_r(x, y; c), \tag{2.4}$$

which is a generalization of the following familiar results (see Kurt [12]):

$$B_n(x, y) = \sum_{r=0}^{n} \binom{n}{r} B_{n-r} H_r(x, y) \tag{2.5}$$

and

$$H_n(x, y) = \sum_{r=0}^{n} \binom{n}{r} \frac{1}{(n-r+1)} B_r(x, y). \tag{2.6}$$

A summation representation of $H_n(x, y; c)$ in terms of ${}_HQ_n^{[m-1,\alpha]}(x, y; c, a; \lambda; \mu; \nu)$ for $\alpha = 1$ is given by the following theorem.

**Theorem 1.** *For $n \geq 0$, we have*

$$H_n(x,y;c) = \frac{n!}{(n+vm)!}\sum_{k=0}^{n+vm}\binom{n+vm}{k}\lambda\frac{\left((\ln c)^{n+vm-k}+(\ln a)^{n+vm-k}\right)}{2^{\mu m}}{}_HQ_k^{[m-1,1]}(x,y;c,a;\lambda;\mu;\nu)$$
$$-\frac{n!}{(n+vm)!2^{\mu m}}\sum_{k=0}^{n+vm-m}\binom{n+vm}{k}(\ln a)^{k+m}{}_HQ_{n+mv-m-k}^{[m-1,1]}(x,y;c,a;\lambda;\mu;\nu). \tag{2.7}$$

*Proof.* From (2.2) and we gete

$$\begin{aligned} c^{xt+yt^2} &= \frac{\lambda c^t + \sum\limits_{h=0}^{m-1}\frac{(t\ln a)^h}{h!}}{2^{\mu m}t^{\nu m}}\sum_{n=0}^{\infty}{}_HQ_n^{[m-1,1]}(x,y;c,a;\lambda;\mu;\nu)\frac{t^n}{n!} \\ &= \frac{\lambda}{2^{\mu m}t^{\nu m}}\left(\sum_{n=0}^{\infty}(\ln c)^n\frac{t^n}{n!}\right)\sum_{n=0}^{\infty}{}_HQ_n^{[m-1,1]}(x,y;c,a;\lambda;\mu;\nu)\frac{t^n}{n!} \\ &\quad + \frac{1}{2^{\mu m}t^{\nu m}}\sum_{h=0}^{m-1}\frac{(\ln a)^h}{h!}t^h\sum_{n=0}^{\infty}{}_HQ_n^{[m-1,1]}(x,y;c,a;\lambda;\mu;\nu)\frac{t^n}{n!}, \end{aligned}$$

which yields

$$\begin{aligned} c^{xt+yt^2} &= \frac{\lambda}{2^{\mu m}}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(\ln c)^{n-k}{}_HQ_k^{[m-1,1]}(x,y;c,a;\lambda;\mu;\nu)\frac{t^{n-vm}}{n!} \\ &+ \frac{1}{2^{\mu m}}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(\ln a)^{n-k}{}_HQ_k^{[m-1,1]}(x,y;c,a;\lambda;\mu;\nu)\frac{t^{n-vm}}{n!} \\ &- \frac{1}{2^{\mu m}}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+m}{k}(\ln a)^{k+m}{}_HQ_{n-k}^{[m-1,1]}(x,y;c,a;\lambda;\mu;\nu)\frac{t^{n+m-vm}}{(n+m)!}. \end{aligned} \tag{2.8}$$

Then, utilizing (1.11) and (2.8), we have

$$\begin{aligned} \sum_{n=0}^{\infty}H_n(x,y;c)\frac{t^n}{n!} &= \frac{\lambda}{2^{\mu m}}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\left((\ln c)^{n-k}+(\ln a)^{n-k}\right)_H Q_k^{[m-1,1]}(x,y;c,a;\lambda;\mu;\nu)\frac{t^{n-vm}}{n!} \\ &- \frac{1}{2^{\mu m}}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+m}{k}(\ln a)^{k+m}{}_HQ_{n-k}^{[m-1,1]}(x,y;c,a;\lambda;\mu;\nu)\frac{t^{n+m-vm}}{(n+m)!} \end{aligned} \tag{2.9}$$

which means the asserted result (2.7).

□

We give some special cases of Theorem 1 as follows.

**Remark 2.** *For $m = \nu = 1$ in Theorem 1, we have*

$$H_n(x,y;c) = \frac{1}{n+1}\left[\lambda\Psi_{n+1}(x+1,y)+\Psi_{n+1}(x,y)\right], \tag{2.10}$$

*where $\Psi_n(x,y) := {}_HY_n^{[0,1]}(x,y;c,a;\lambda;0;1)$.*

**Remark 3.** *Setting $m = -\lambda = \nu = 1$, $u = a = 0$ and $c = e$ in Theorem 1 gives the well known result of Kurt [12] for $H_n(x,y)$ as follows*

$$H_n(x,y) = \frac{1}{n+1}\left[{}_HB_{n+1}(x+1,y)+{}_HB_{n+1}(x,y)\right], \tag{2.11}$$

*where ${}_HB_n(x,y)$ is the n-th Hermite-Bernoulli polynomial defined by (see [8]):*

$$\frac{t}{e^t-1}e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HB_n(x,y)\frac{t^n}{n!}.$$

**Remark 4.** *For $m=-\lambda=\nu=1$, $y=u=a=0$ and $c=e$, the result in Theorem 1 is an extension of the known result of Kim et al. [11]:*

$$x^n=\frac{B_{n+1}(x+1)+B_{n+1}(x)}{n+1}.$$

**Theorem 2.** *Let $p,q\in\mathbb{R}$, $\alpha\in\mathbb{C}$ and $m\in\mathbb{N}$. The polynomials ${}_HQ_n^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu)$ satisfy the following relation:*

$$\begin{aligned}{}_HQ_n^{[m-1,\alpha]}(px,qy;c,a;\lambda;\mu;\nu)&=n!\sum_{k=0}^{n}\sum_{j=0}^{[\frac{k}{2}]}{}_HQ_{n-k}^{[m-1,\alpha]}(px,qy;c,a;\lambda;\mu;\nu)\\&\times(\ln c(p-1)x)^k(\ln c(q-1)y)^j\frac{1}{(n-k-2j)!j!}.\end{aligned}\tag{2.12}$$

*Proof.* By (2.2), we derive

$$\begin{aligned}\Theta&=\sum_{n=0}^{\infty}{}_HQ_n^{[m-1,\alpha]}(px,qy;c,a;\lambda;\mu;\nu)\frac{t^n}{n!}=\left(\frac{(2^\mu t^\nu)^m}{\lambda c^t+\sum\limits_{h=0}^{m-1}\frac{(t\ln a)^h}{h!}}\right)^\alpha c^{xt+yt^2}c^{(p-1)xt}c^{(q-1)yt^2}\\&=\left(\sum_{n=0}^{\infty}{}_HQ_n^{[m-1,\alpha]}(px,qy;c,a;\lambda;\mu;\nu)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}(\ln c(p-1)x)^k\frac{t^k}{k!}\right)\left(\sum_{j=0}^{\infty}(\ln c(q-1)y)^j\frac{t^{2j}}{j!}\right)\\&=\left(\sum_{n=0}^{\infty}{}_HQ_n^{[m-1,\alpha]}(px,qy;c,a;\lambda;\mu;\nu)\frac{t^n}{n!}\right)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(\ln c(p-1)x)^k(\ln c(q-1)y)^j\frac{t^{k+2j}}{n!k!j!}.\end{aligned}$$

Replacing $k$ by $k-2j$, we have

$$\begin{aligned}\Theta&=\left(\sum_{n=0}^{\infty}{}_HQ_n^{[m-1,\alpha]}(px,qy;c,a;\lambda;\mu;\nu)\frac{t^n}{n!}\right)\sum_{k=2j}^{\infty}(\ln c(p-1)x)^{k-2j}(\ln c(q-1)y)^j\frac{t^k}{(k-2j)!j!}\\&=\sum_{n=0}^{\infty}\sum_{k=2j}^{\infty}{}_HQ_n^{[m-1,\alpha]}(px,qy;c,a;\lambda;\mu;\nu)(\ln c(p-1)x)^{k-2j}(\ln c(q-1)y)^j\frac{t^{n+k}}{(k-2j)!j!n!}.\end{aligned}$$

Again replacing $n$ by $n-k$ , we have

$$\Theta=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{[\frac{k}{2}]}{}_HQ_{n-k}^{[m-1,\alpha]}(px,qy;c,a;\lambda;\mu;\nu)(\ln c(p-1)x)^{k-2j}(\ln c(q-1)y)^j\frac{t^n}{(n-k-2j)!j!k!}.$$

Finally equating the coefficients of $t^n$ of both sides, we get the asserted result (2.12).

□

The second generalization of the Apostol type polynomials, Bell polynomials and Hermite polynomials which we wish to consider is given by the following generating function:

$$\left(\frac{2^\mu t^\nu}{\lambda c^t+\sum\limits_{h=0}^{m-1}\frac{(t\ln a)^h}{h!}}\right)^\alpha c^{x(t+c^t-1)+yt^2}=\sum_{n=0}^{\infty}{}_\phi Q_n^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu)\frac{t^n}{n!}.\tag{2.13}$$

The clue to seeing a relation between Apostol type polynomials, Bell polynomials and Hermite polynomials such as (1.9), (1.10) and (2.1) and the special cases of (2.2) is given in the following theorem.

**Theorem 3.** *For $n \geq 0$, we have*

$$ {}_{\phi}Q_n^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu) = \sum_{r=0}^{n} \binom{n}{r} \phi_r(x;c)_{\phi}Q_{n-r}^{[m-1;\alpha])}(x,y:c,a;\lambda;\mu;\nu). \tag{2.14} $$

*Proof.* The proof is similar to the proof of Theorem 1. So, we omit it. □

The third generalization of the Apostol type polynomials including Fubini polynomials and Hermite polynomials is provided by the following generating function:

$$ \left( \frac{2^{\mu} t^{\nu}}{\lambda c^t + \sum\limits_{h=0}^{m-1} \frac{(t \ln a)^h}{h!}} \right)^{\alpha} \frac{c^{xt+yt^2}}{1 - z(c^t - 1)} = \sum_{n=0}^{\infty} {}_F Q_n^{[m-1;\alpha]}(x,y;z;c,a;\lambda;\mu;\nu) \frac{t^n}{n!}. \tag{2.15} $$

The following theorems are given without their proofs.

**Theorem 4.** *For $n \geq 0$, we have*

$$ {}_F Q_n^{[m-1,\alpha]}(x,y;z;c,a;\lambda;\mu;\nu) = \sum_{r=0}^{n} \binom{n}{r} F_r(z;c) Q_n^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu), \tag{2.16} $$

*where the extended Fubini polynomials $F_n(z;c)$ extensions of (1.12) are given by (cf. [5,6])*

$$ \frac{1}{1 - z(c^t - 1)} = \sum_{n=0}^{\infty} F_n(z;c) \frac{t^n}{n!}. $$

**Theorem 5.** *For $n \geq 0$, we have*

$$ {}_F Q_n^{[m-1,\alpha]}(x,y;z;c;a;\lambda;\mu;\nu) = \sum_{r=0}^{n} \binom{n}{r} {}_F H_r(x,y;z;c) Q_{n-r}^{[m-1,\alpha]}(c,a;\lambda;\mu;\nu), \tag{2.17} $$

*where the extended Fubini-Hermite polynomials ${}_F H_n(x,y,z;c)$ are given by*

$$ \frac{c^{xt+yt^2}}{1 - z(c^t - 1)} = \sum_{n=0}^{\infty} {}_F H_n(x,y;z;c) \frac{t^n}{n!}. \tag{2.18} $$

The formula (2.17) is an extension of the following known formula:

$$ {}_F H_r(x,y;z;c) = \sum_{r=0}^{n} \binom{n}{r} F_r(z;c) H_{n-r}(x,y;c). \tag{2.19} $$

Now, we give the following theorem.

**Theorem 6.** *For $n \geq 0$, we have*

$$ {}_F Q_n^{[m-1,\alpha]}(x,y;z;a;\lambda;\mu;\nu) = \sum_{l=0}^{n} \sum_{k=0}^{l} \binom{n}{l} {}_H Q_{n-l}^{[m-1;\alpha]}(x,y;a;\lambda;\mu;\nu) k! z^k S_2(l,k). \tag{2.20} $$

*Proof.* By (2.2), we obtain

$$ \sum_{n=0}^{\infty} {}_F Q_n^{[m-1,\alpha]}(x,y;z;a;\lambda;\mu;\nu) \frac{t^n}{n!} = \left( \frac{(2^{\mu} t^{\nu})^m}{\lambda e^t + \sum\limits_{h=0}^{m-1} \frac{(t \ln a)^h}{h!}} \right)^{\alpha} \frac{e^{xt+yt^2}}{1 - z(e^t - 1)} $$

$$= \left( \frac{(2^\mu t^\nu)^m}{\lambda e^t + \sum_{h=0}^{m-1} \frac{(t\ln a)^h}{h!}} \right)^\alpha e^{xt+yt^2} \sum_{k=0}^{\infty} z^k \sum_{l=k}^{\infty} k! S_2(l,k) \frac{t^l}{l!}$$

$$= \left( \sum_{n=0}^{\infty} {}_HQ_n^{[m-1;\alpha]}(x,y;a;\lambda;\mu;\nu) \frac{t^n}{n!} \right) \left( \sum_{l=0}^{\infty} z^k \sum_{k=0}^{l} k! S_2(l,k) \frac{t^l}{l!} \right). \tag{2.21}$$

Appliying Cauchy product in (2.21), we attain

$$\sum_{n=0}^{\infty} {}_FQ_n^{[m-1,\alpha]}(x,y;z;a;\lambda;\mu;\nu) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} {}_HQ_{n-l}^{[m-1;\alpha]}(x,y;a;\lambda;\mu;\nu) \sum_{k=0}^{l} z^k k! S_2(l,k) \right) \frac{t^n}{n!},$$

which means the claimed result (2.20).

□

**Theorem 7.** *For $n \geq 0$, we have*

$${}_FQ_n^{[m-1,\alpha]}(x+r,y;z;a;\lambda;\mu;\nu) = \sum_{l=0}^{n} \sum_{k=0}^{l} \binom{n}{l} {}_HQ_{n-l}^{[m-1;\alpha]}(x,y;a;\lambda;\mu;\nu) z^k k! S_2(l+r,k+r), \tag{2.22}$$

*where $S_2(l+r,k+r)$ denotes the $r$-Striling numbers defined by*

$$\sum_{l=k}^{\infty} S_2(l+r,k+r) \frac{t^l}{l!} = \frac{(e^t-1)^k}{k!} e^{rt}, \text{ cf. [2,26].}$$

*Proof.* Replacing $x$ by $x+r$ in (2.21) and using [7, p. 250, Theorem 16], we have

$$\sum_{n=0}^{\infty} {}_FQ_n^{[m-1,\alpha]}(x+r,y;z;a;\lambda;\mu;\nu) \frac{t^n}{n!} = \left( \frac{(2^\mu t^\nu)^m}{\lambda e^t + \sum_{h=0}^{m-1} \frac{(t\ln a)^h}{h!}} \right)^\alpha \frac{e^{(x+r)t+yt^2}}{1-z(e^t-1)}$$

$$= \left( \frac{(2^\mu t^\nu)^m}{\lambda e^t + \sum_{h=0}^{m-1} \frac{(t\ln a)^h}{h!}} \right)^\alpha e^{xt+yt^2} e^{rt} \sum_{k=0}^{\infty} z^k \sum_{l=k}^{\infty} k! S_2(l,k) \frac{t^l}{l!}$$

$$= \left( \sum_{n=0}^{\infty} {}_HQ_n^{[m-1;\alpha]}(x,y;a;\lambda;\mu;\nu) \frac{t^n}{n!} \right) \left( \sum_{l=0}^{\infty} z^k \sum_{k=0}^{l} k! S_2(l+r,k+r) \frac{t^l}{l!} \right),$$

which implies the asserted result (2.22).

□

## 3. Implicit Summation Formulae Involving Generalized Apostol Type Polynomials

In this section, we investigate some imlicit summation formulae for the generalized Apostol type polynomials ${}_HQ_n^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu)$ as follows.

We first give the following theorem.

**Theorem 8.** *The following implicit summation formula*

$$ {}_HQ_{q+l}^{[m-1,\alpha]}(z,y;c,a;\lambda;\mu;\nu) \;=\; \sum_{n,p=0}^{q,l}\binom{q}{n}\binom{l}{p}(\ln c)^{n+p}(z-x)^{n+p}\,{}_HQ_{q+l-p-n}^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu) \tag{3.1}$$

*holds.*

*Proof.* Replacing $t$ by $t + u$ in (2.2), we see that

$$\left(\frac{(2^{\mu}(t+u)^{\nu})^m}{\lambda c^{t+u}+\sum\limits_{h=0}^{m-1}\frac{((t+u)\ln a)^h}{h!}}\right)^{\alpha} c^{y(t+u)^2} = c^{-x(t+u)}\sum_{q,l=0}^{\infty}{}_HQ_{q+l}^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu)\frac{t^q}{q!}\frac{u^l}{l!}. \tag{3.2}$$

Again replacing $x$ by $z$ in the last equation and equating the resulting equation to (3.2), we get

$$c^{(z-x)(t+u)}\sum_{q,l=0}^{\infty}{}_HQ_{q+l}^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu)\frac{t^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty}{}_HQ_{q+l}^{[m-1,\alpha]}(z,y;c,a;\lambda;\mu;\nu)\frac{t^q}{q!}\frac{u^l}{l!}. \tag{3.3}$$

On expanding exponential function (3.3) gives

$$\sum_{N=0}^{\infty}(\ln c)^N\frac{[(z-x)(t+u)]^N}{N!}\sum_{q,l=0}^{\infty}{}_HQ_{q+l}^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu)\frac{t^q}{q!}\frac{u^l}{l!}$$
$$=\sum_{q,l=0}^{\infty}{}_HQ_{q+l}^{[m-1,\alpha]}(z,y;c,a;\lambda;\mu;\nu)\frac{t^q}{q!}\frac{u^l}{l!}. \tag{3.4}$$

Then, using the following series manipulation formula

$$\sum_{N=0}^{\infty}f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty}f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}, \tag{3.5}$$

The equality (3.4) becomes

$$\sum_{n,p=0}^{\infty}(\ln c)^{n+p}\frac{(z-x)^{n+p}t^n u^p}{n!p!}\sum_{q,l=0}^{\infty}{}_HQ_{q+l}^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu)\frac{t^q}{q!}\frac{u^l}{l!}$$
$$=\sum_{q,l=0}^{\infty}{}_HQ_{q+l}^{[m-1,\alpha]}(z,y;c,a;\lambda;\mu;\nu)\frac{t^q}{q!}\frac{u^l}{l!}. \tag{3.6}$$

Now replacing $q$ by $q-n$ and $l$ by $l-p$, and using the lemma [26, p.100] in the left hand side of (3.6), we get

$$\sum_{q,l=0}^{\infty}\sum_{n,p=0}^{q,l}\frac{(z-x)^{n+p}(\ln c)^{n+p}}{n!p!}{}_HQ_{q+l-n-p}^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu)\frac{t^q}{(q-n)!}\frac{u^l}{(l-p)!}$$
$$=\sum_{q,l=0}^{\infty}{}_HQ_{q+l}^{[m-1,\alpha]}(z,y;c,a;\lambda;\mu;\nu)\frac{t^q}{q!}\frac{u^l}{l!}, \tag{3.7}$$

which means the asserted result (3.1).

□

**Remark 5.** *Choosing $l = 0$ in assertion (3.1), the following implicit summation formulae for Apostol type Hermite polynomials holds:*

$$ {}_HQ_q^{[m-1,\alpha]}(z,y;c,a;\lambda;\mu;\nu) \;=\; \sum_{n=0}^{q}\binom{q}{n}(\ln c)^{n+p}(z-x)^{n+p}Q_{q-n}^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu). \tag{3.8}$$

**Remark 6.** *Choosing $l=0$ and replacing $z$ by $z+x$ in (3.1), we acquire*

$$ {}_HQ_q^{[m-1,\alpha]}(z+x,y;c,a;\lambda;\mu;\nu) \;=\; \sum_{n=0}^{q}\binom{q}{n}(\ln c)^{n+p}z^{n+p}\,{}_HQ_{q-n}^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu). \tag{3.9}$$

We provide the following summation formulae stated in Theorem 9, Theorem 10 and Theorem 11.

**Theorem 9.** *The following implicit summation formula*

$$ {}_HQ_n^{[m-1,\alpha]}(x+z,y+u;c,a;\lambda;\mu;\nu) = \sum_{s=0}^{n}\binom{n}{s}{}_HQ_{n-s}^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu)H_s(z,u;c) \tag{3.10}$$

*holds.*

*Proof.* Replacing $x$ by $x+z$ and $y$ by $y+u$ in (2.2), and in view of (1.9), we acquire

$$\left(\frac{(2^\mu t^\nu)^m}{\lambda c^t+\sum\limits_{h=0}^{m-1}\frac{(t\ln a)^h}{h!}}\right)^{\alpha} c^{(x+z)t+(y+u)t^2} = \sum_{n=0}^{\infty}{}_HQ_n^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu)\frac{t^n}{n!}\sum_{s=0}^{\infty}H_s(z,u;c)\frac{t^s}{s!}$$

$$= \sum_{n=0}^{\infty}{}_HQ_n^{[m-1,\alpha]}(x+z,y+u;c,a;\lambda;\mu;\nu)\frac{t^n}{n!}, \tag{3.11}$$

which implies claimed result (3.10). □

**Theorem 10.** *The following formula*

$$ {}_HQ_n^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu) = \sum_{r=0}^{n}\binom{n}{r}{}_HQ_{n-r}^{[m-1,\alpha]}(x-z;c,a;\lambda;\mu;\nu)H_r(z,y;c) \tag{3.12}$$

*is valid.*

*Proof.* By (1.9) and (2.2), we consider

$$\left(\frac{2^\mu t^\nu}{\lambda c^t+\sum\limits_{h=0}^{m-1}\frac{(t\ln a)^h}{h!}}\right)^{\alpha} c^{(x-z)t}c^{zt+yt^2} = \sum_{n=0}^{\infty}Q_n^{[m-1,\alpha]}(x-z;c,a;\lambda;\mu;\nu)\frac{t^n}{n!}\sum_{r=0}^{\infty}H_r(z,y;c)\frac{t^r}{r!}, \tag{3.13}$$

which gives

$$\sum_{n=0}^{\infty}{}_HQ_n^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{r=0}^{n}Q_{n-r}^{[m-1,\alpha]}(x-z;c,a;\lambda;\mu;\nu)H_r(z,y;c)\frac{t^n}{(n-r)!r!},$$

which means the asserted result (3.12). □

**Theorem 11.** *The following implicit summation formula*

$$ {}_HQ_n^{[m-1,\alpha]}(x+1,y;c,a;\lambda;\mu;\nu) = \sum_{r=0}^{n}\binom{n}{r}(\ln c)^r\,{}_HQ_{n-r}^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu) \tag{3.14}$$

*holds.*

*Proof.* From (2.2), we see that

$$\sum_{n=0}^{\infty}{}_HQ_n^{[m-1,\alpha]}(x+1,y;c,a;\lambda;\mu;\nu)\frac{t^n}{n!} - \sum_{n=0}^{\infty}{}_HQ_n^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu)\frac{t^n}{n!}$$

$$= \left( \frac{2^{\mu} t^{\nu}}{\lambda e^{t} + \sum\limits_{h=0}^{m-1} \frac{(tlna)^{h}}{h!}} \right)^{\alpha} (c^{t}-1) c^{xt+yt^{2}} = \sum_{n=0}^{\infty} {}_{H}Q_{n}^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu) \frac{t^{n}}{n!} \left( \sum_{r=0}^{\infty} (\ln c)^{r} \frac{t^{r}}{r!} - 1 \right)$$

$$= \sum_{n=0}^{\infty} {}_{H}Q_{n}^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu) \frac{t^{n}}{n!} \sum_{r=0}^{\infty} (\ln c)^{r} \frac{t^{r}}{r!} - \sum_{n=0}^{\infty} {}_{H}Q_{n}^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu) \frac{t^{n}}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} (\ln c)^{r} {}_{H}Q_{n-r}^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu) \frac{t^{n}}{n!} - \sum_{n=0}^{\infty} {}_{H}Q_{n}^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu) \frac{t^{n}}{n!},$$

which means the desired result (3.14).

□

## 4. Some Symmetric Identities for ${}_{H}Q_{n}^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu)$

In this section, we give some symmetry identities for the generalized Hermite-based Apostol type polynomials ${}_{H}Q_{n}^{[m-1,\alpha]}(x,y;c,a;\lambda;\mu;\nu)$ by applying the generating function (2.1) and (2.2). The results derived in this section extend some known identities provided by Kim et al. [12], Khan [10], Pathan and Khan [24-26].

Let

$$A(t) = \left( \frac{((pq)^{\nu} 2^{2\mu} t^{2\nu})^{m}}{(\lambda c^{pt} + \sum\limits_{h=0}^{m-1} \frac{(tlna)^{h}}{h!})(\lambda c^{qt} + \sum\limits_{h=0}^{m-1} \frac{(tlna)^{h}}{h!})} \right)^{\alpha}, \tag{4.1}$$

$$B(t) = \left( \frac{(2^{\mu} p^{\nu} t^{\nu})^{m}}{(\lambda c^{pt} + \sum\limits_{h=0}^{m-1} \frac{(tlna)^{h}}{h!}} \right)^{\alpha} \tag{4.2}$$

and

$$C(t) = \left( \frac{(2^{\mu} q^{\nu} t^{\nu})^{m}}{(\lambda c^{qt} + \sum\limits_{h=0}^{m-1} \frac{(tlna)^{h}}{h!}} \right)^{\alpha}. \tag{4.3}$$

By means of (4.1), (4.2) and (4.3), we give the following theorem.

**Theorem 12.** *The following symmetric identity*

$$\sum_{r=0}^{n} \binom{n}{r} q^{r} p^{n-r} {}_{H}Q_{n-r}^{[m-1,\alpha]}(qx,q^{2}y;c,a;\lambda;\mu;\nu) {}_{H}Q_{r}^{[m-1,\alpha]}(px,p^{2}y;\lambda;\mu;\nu)$$

$$= \sum_{r=0}^{n} \binom{n}{r} p^{r} q^{n-r} {}_{H}Q_{n-r}^{[m-1,\alpha]}(px,p^{2}y;c,a;\lambda;\mu;\nu) {}_{H}Q_{r}^{[m-1,\alpha]}(qx,q^{2}y;c,a;\lambda;\mu;\nu) \tag{4.4}$$

*holds for* $\alpha \in \mathbb{N}_0$, $x, y \in \mathbb{R}$ *and* $n \geq 0$.

*Proof.* Let

$$\Upsilon = A(t) c^{pqxt + p^{2}q^{2}yt^{2}}.$$

Then, the expression for $\Upsilon$ is symmetric in $a$ and $b$ and we can expand $\Upsilon$ into series in two ways:

$$\Upsilon = \sum_{n=0}^{\infty} {}_{H}Q_{n}^{[m-1,\alpha]}(qx,q^{2}y;c,a;\lambda;\mu;\nu) \frac{(pt)^{n}}{n!} \sum_{r=0}^{\infty} {}_{H}Q_{r}^{[m-1,\alpha]}(px,p^{2}y;c,a;\lambda;\mu;\nu) \frac{(qt)^{r}}{r!}$$

$$= \sum_{n=0}^{\infty} \left( \sum_{r=0}^{n} \binom{n}{r} q^r p^{n-r} {}_HQ_{n-r}^{[m-1,\alpha]}(qx, q^2y; c, a; \lambda; \mu; \nu) {}_HQ_r^{[m-1,\alpha]}(px, p^2y; c, a; \lambda; \mu; \nu) \right) \frac{t^n}{n!} \tag{4.5}$$

and similarly,

$$\Upsilon = \sum_{n=0}^{\infty} {}_HQ_n^{[m-1,\alpha]}(px, p^2y; c, a; \lambda; \mu; \nu) \frac{(qt)^n}{n!} \sum_{r=0}^{\infty} {}_HQ_r^{[m-1,\alpha]}(qx, q^2y; c, a; \lambda; \mu; \nu) \frac{(pt)^r}{r!}$$

$$= \sum_{n=0}^{\infty} \left( \sum_{r=0}^{n} \binom{n}{r} p^r q^{n-r} {}_HQ_{n-r}^{[m-1,\alpha]}(px, p^2y; c, a; \lambda; \mu; \nu) {}_HQ_r^{[m-1,\alpha]}(qx, q^2y; c, a; \lambda; \mu; \nu) \right) \frac{t^n}{n!}, \tag{4.6}$$

which implies the desired result (4.4).

□

Here is another symmetric identity for ${}_HQ_r^{[m-1,\alpha]}(x, y; c, a; \lambda; \mu; \nu)$ as follows.

**Theorem 13.** *Let $\alpha \in \mathbb{N}_0$, $p, q \in \mathbb{R} \setminus \{0\}$; $\lambda \in \mathbb{C}$ and $x, y \in \mathbb{R}$ and $n \geq 0$, then the following identity holds:*

$$\sum_{r=0}^{n} \binom{n}{r} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} p^{n-r} q^r {}_HQ_{n-r}^{[m-1,\alpha]} \left( qx + \frac{q}{p} i + j, q^2 z; c, a; \lambda; \mu; \nu \right) Q_r^{[m-1,\alpha]}(py; c, a; \lambda; \mu; \nu)$$

$$= \sum_{r=0}^{n} \binom{n}{r} \sum_{j=0}^{q-1} q^{n-r} p^r {}_HQ_{n-r}^{[m-1,\alpha]} \left( px + \frac{p}{q} i + j, p^2 z; c, a; \lambda; \mu; \nu \right) Q_r^{[m-1,\alpha]}(qy; c, a; \lambda; \mu; \nu). \tag{4.7}$$

*Proof.* Let

$$\Psi = A(t) \frac{1 + \lambda(-1)^{p+1} c^{pqt}}{(\lambda c^{pt} + 1)(\lambda c^{qt} + 1)} c^{pq(x+y)t + p^2 q^2 z t^2}$$

$$= B(t) c^{pqxt + p^2 q^2 z t^2} \left( \frac{1 - \lambda (c^{-qt})^p}{\lambda c^{qt} + 1} \right) C(t) c^{pqyt} \left( \frac{1 - \lambda (c^{-pt})^q}{\lambda c^{pt} + 1} \right),$$

which yields

$$\Psi = B(t) c^{pqxt + p^2 q^2 z t^2} \sum_{i=0}^{p-1} (-\lambda)^i c^{qti} C(t) e^{cdyt} \sum_{j=0}^{q-1} (-\lambda)^j c^{ptj} \tag{4.8}$$

$$= B(t) c^{p^2 q^2 z t^2} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} (-\lambda)^{i+j} c^{(qx + \frac{q}{p} i + j)pt} \sum_{r=0}^{\infty} Q_r^{[m-1,\alpha]}(py; c, a; \lambda; \mu; \nu) \frac{(qt)^r}{r!}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} p^{n-r} q^r {}_HQ_{n-r}^{[m-1,\alpha]} \left( qx + \frac{q}{p} i + j, q^2 z; c, a; \lambda; \mu; \nu \right)$$

$$\times Q_r^{[m-1,\alpha]}(py; c, a; \lambda; \mu; \nu) \frac{t^n}{n!}. \tag{4.9}$$

On the other hand, we attain

$$\Psi = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} q^{n-r} p^r {}_HQ_{n-r}^{[m-1,\alpha]} \left( px + \frac{p}{q} i + j, p^2 z; c, a; \lambda; \mu; \nu \right)$$

$$\times Q_r^{[m-1,\alpha]}(qy; c, a; \lambda; \mu; \nu) \frac{t^n}{n!}. \tag{4.10}$$

By means of (4.9) and (4.10), we obtain the claimed result (4.7).

□

**Theorem 14.** *Let a and b be two integers and $n \geq 0$. Then the following identity*

$$\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}p^{n-r}q^{r}\,{}_{H}Q_{n-r}^{[m-1,\alpha]}\left(qx+\frac{q}{p}i,q^{2}z;c,a;\lambda;\mu;\nu\right)Q_{r}^{[m-1,\alpha]}(py+\frac{p}{q}j;c,a;\lambda;\mu;\nu)$$

$$=\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}q^{n-r}p^{r}\,{}_{H}Q_{n-r}^{[m-1,\alpha]}\left(px+\frac{p}{q}i,p^{2}z;c,a;\lambda;\mu;\nu\right)Q_{r}^{[m-1,\alpha]}(qy+\frac{q}{p}j;c,a;\lambda;\mu;\nu) \tag{4.11}$$

*holds.*

*Proof.* We rewrite (4.8) in the following form

$$\Psi=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}p^{n-r}q^{r}\,{}_{H}Q_{n-r}^{[m-1,\alpha]}\left(qx+\frac{q}{p}i,q^{2}z;c,a;\lambda;\mu;\nu\right)$$

$$\times Q_{r}^{[m-1,\alpha]}(py+\frac{p}{q}j;c,a;\lambda;\mu;\nu)\frac{t^{n}}{n!} \tag{4.12}$$

and also

$$\Psi=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}q^{n-r}p^{r}\,{}_{H}Q_{n-r}^{[m-1,\alpha]}\left(px+\frac{p}{q}i,p^{2}z;c,a;\lambda;\mu;\nu\right)$$

$$\times Q_{r}^{[m-1,\alpha]}(qy+\frac{q}{p}j;c,a;\lambda;\mu;\nu)\frac{t^{n}}{n!}. \tag{4.13}$$

By (4.12) and (4.13), we get the asserted result (4.11).

□

## 5. Conclusions

In the presented paper, we have considered a new class of Hermite-based generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. Then, we have investigated some basic properties and several implicit summation formulae by using series manipulation methods. Also, we have derived diverse symmetric identities, which are extensions of many earlier well-known results in [10,12,24-26]. Moreover, we have introduced a novel class of Hermite-based generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials including geometric and Bell polynomials, and have given some basic properties.

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