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# On the solutions of some equations with index matrices

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**Abstract.** Index Matrices (IMs) are extensions of the ordinary matrices. The solutions of some equations with IMs, which elements are real numbers, are described. The conditions for existing of solutions are given.
**Keywords:** Index matrix, Matrix equation.

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## 1 Introduction

The concept of the Index Matrix (IM) is developed sequentially in [3, 4, 5]. It is an object with the form

$$[K, L, \{a_{k_i,l_j}\}] \equiv \begin{array}{c|ccccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & a_{k_1,l_1} & \dots & a_{k_1,l_j} & \dots & a_{k_1,l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & a_{k_i,l_1} & \dots & a_{k_i,l_j} & \dots & a_{k_i,l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & a_{k_m,l_1} & \dots & a_{k_m,l_j} & \dots & a_{k_m,l_n} \end{array},$$

where K $= \{k_1, k_2, ..., k_m\}$, L $= \{l_1, l_2, ..., l_n\}$, and for $1 \leq$ i $\leq$ m, $1 \leq$ j $\leq$ n: $a_{k_i,l_j} \in \mathcal{R}$, where $\mathcal{R}$ is some fixed set.

Let us assume that everywhere the IMs $A = [K, L, \{a_{k_i,l_j}\}]$ and $B = [P, Q, \{b_{p_r,q_s}\}]$ be given.

Over two IMs, in [3, 4, 5] a lot of operations and operators are defined. Here, we will use the following three of them: $\oplus_{(\circ)}$, $\otimes_{(\circ)}$, $\ominus_{(\circ)}$, where $\circ : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is the sub-operation of the basic operation.

**Addition**

$$A \oplus_{(\circ)} B = [K \cup P, L \cup Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \\ a_{k_i,l_j} \circ b_{p_r,q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ \\ e_\circ, & \text{otherwise} \end{cases},$$

where $e_\circ$ is the unit element of $\mathcal{R}$ related to operation $\circ$. For example, if $\mathcal{R}$ is the set of real numbers and $\circ \in \{+, -\}$, then $e_\circ$ is "0"; while if $\circ \in \{\times, :\}$, then $e_\circ$ is "1".

**Termwise multiplication**

$$A \otimes_{(\circ)} B = [K \cap P, L \cap Q, \{c_{t_u,v_w}\}],$$

where for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$

$$c_{t_u,v_w} = a_{k_i,l_j} \circ b_{p_r,q_s},$$

**Structural subtraction**

$$A \ominus B = [K - P, L - Q, \{c_{t_u,v_w}\}],$$

where "–" is the set–theoretic difference operation and

$$c_{t_u,v_w} = a_{k_i,l_j}, \text{ for } t_u = k_i \in K - P \text{ and } v_w = l_j \in L - Q.$$

Let us assume that everywhere the IMs $A$ and $B$ be given. We will determine the condition for solving the equations

$$A \oplus_{(\circ)} X = B, \quad (1)$$

$$A \otimes_{(\circ)} X = B, \quad (2)$$

$$A \ominus X = B, \quad (3)$$

and the form of the IMs $X$, when the elements of all matrices are $\mathcal{R}$.

## 2 The equation with operation $\oplus_{(\circ)}$

We will consider the condition when the elements that constructs the IMs are real numbers and $\circ \in \{+, -\}$ and $\circ \in \{\times, :\}$.

Let us suppose that

$$X = [Y, Z, \{c_{t_u,v_w}\}].$$

When we must solve equation (1), we see that if $K - P \neq \emptyset$, i.e., when there is $k \in K - P$, then $k \in (K \cup Y) - P$ that is impossible. Therefore, one of the conditions for existing of solutions of (1) is $K \subseteq P$. By analogy, a second condition will be: $L \subseteq Q$.

### 2.1 The equation with operation $\oplus_{(+)}$

For the solution of equation (1)

$$X = [Y, Z, \{c_{t_u,v_w}\}]$$

we have

$$X = [Y, Z, \{c_{t_u,v_w}\}] = B \oplus_{(-)} A = [P, Q, \{c_{t_u,v_w}\}]$$

and

$$c_{t_u,v_w} = \begin{cases} b_{p_r,q_s}, & \text{if } t_u = p_r \in P \\ & \text{and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \\ & \text{and } v_w = q_s \in Q \\ b_{p_r,q_s} - a_{k_i,l_j} & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_u = l_j = q_s \in L \cap Q \\ e_\circ, & \text{otherwise} \end{cases},$$

Therefore,

$$A \oplus_{(+)} X = [P, Q, \{b_{p_r,q_s}\}] = B.$$

### 2.2 The equation with operation $\oplus_{(-)}$

The check of the validity of this solution, when the sub-operation is "−", is the following:

$$X = [Y, Z, \{c_{t_u,v_w}\}] = B \oplus_{(+)} A = [P, Q, \{c_{t_u,v_w}\}]$$

and

$$c_{t_u,v_w} = \begin{cases} b_{p_r,q_s}, & \text{if } t_u = p_r \in P \\ & \text{and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \\ & \text{and } v_w = q_s \in Q \\ a_{k_i,l_j} - b_{p_r,q_s} & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_u = l_j = q_s \in L \cap Q \\ e_\circ, & \text{otherwise} \end{cases},$$

Therefore,

$$A \oplus_{(-)} X = [P, Q, \{b_{p_r,q_s}\}] = B.$$

### 2.3 The equation with operation $\oplus_{(\times)}$

The check of the validity of this solution, when the sub-operation is "×", is the following:

$$X = [Y, Z, \{c_{t_u,v_w}\}] = B \oplus_{(:)} A = [P, Q, \{c_{t_u,v_w}\}],$$

where the additional condition for existence of solution is that for every $k_i \in K$ and $l_j \in L$:

$$a_{k_i,l_j} \neq 0$$

and

$$c_{t_u,v_w} = \begin{cases} b_{p_r,q_s}, & \text{if } t_u = p_r \in P \\ & \text{and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \\ & \text{and } v_w = q_s \in Q \\ b_{p_r,q_s} : a_{k_i,l_j} & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_u = l_j = q_s \in L \cap Q \\ e_\circ, & \text{otherwise} \end{cases},$$

Therefore,

$$A \oplus_{(\times)} X = [P, Q, \{b_{p_r,q_s}\}] = B.$$

### 2.4 The equation with operation $\oplus_{(:)}$

Now, we have the solution

$$X = [Y, Z, \{c_{t_u,v_w}\}] = B \oplus_{(\times)} A = [P, Q, \{c_{t_u,v_w}\}],$$

where the additional condition for existence of solution is that for every $p_r \in P$ and $q_s \in Q$:

$$b_{p_r,q_s} \neq 0$$

and

$$c_{t_u,v_w} = \begin{cases} b_{p_r,q_s}, & \text{if } t_u = p_r \in P \\ & \text{and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \\ & \text{and } v_w = q_s \in Q \\ a_{k_i,l_j} : b_{p_r,q_s} & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_u = l_j = q_s \in L \cap Q \\ e_\circ, & \text{otherwise} \end{cases},$$

Therefore,

$$A \oplus_{(:)} X = [P, Q, \{b_{p_r,q_s}\}] = B.$$

## 3 The equation with operation $\otimes_{(\circ)}$

As in Section 2, we see that if $P \cap K = 0$ or $Q \cap L = 0$, or if $K \subset P$, i.e. $K \neq P$ or $L \subset Q$, i.e. $L \neq Q$, then the equation

$$A \otimes_{(\circ)} X = B$$

does not have any solution.

Therefore, the condition for existing of solution of (2) is $P \subseteq K$ and $Q \subseteq L$.

### 3.1 The equation with operation $\otimes_{(+)}$

If the sub-operation is "+", we have

$$X = [Y, Z, \{c_{t_u,v_w}\}] = B \otimes_{(-)} A = [P, Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = b_{p_r,q_s} - a_{k_i,l_j}$$

for $t_u = k_i = p_r \in P$ and $v_w = l_j = q_s \in Q$.

### 3.2 The equation with operation $\otimes_{(-)}$

If the sub-operation is "–", we have

$$X = [Y, Z, \{c_{t_u,v_w}\}] = B \otimes_{(+)} A = [P, Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = a_{k_i,l_j} - b_{p_r,q_s}$$

for $t_u = k_i = p_r \in P$ and $v_w = l_j = q_s \in Q$.

### 3.3 The equation with operation $\otimes_{(\times)}$

If the sub-operation is "×", we have

$$X = [Y, Z, \{c_{t_u,v_w}\}] = B \otimes_{(:)} A = [P \cap K, Q \cap L, \{c_{t_u,v_w}\}],$$

where the additional condition for existence of solution is that for $k_i \in P$ and $l_j \in Q$:

$$a_{k_i,l_j} \neq 0$$

and

$$c_{t_u,v_w} = b_{p_r,q_s} : a_{k_i,l_j}$$

for $t_u = k_i = p_r \in P$ and $v_w = l_j = q_s \in Q$.

### 3.4 The equation with operation $\otimes_{(:)}$

If the sub-operation is ":", we have

$$X = [Y, Z, \{c_{t_u,v_w}\}] = B \otimes_{(\times)} A = [P \cap K, Q \cap L, \{c_{t_u,v_w}\}],$$

where the additional condition for existence of solution is that for $p_r \in P$ and $q_s \in Q$:

$$b_{p_r,q_s} \neq 0$$

and

$$c_{t_u,v_w} = a_{k_i,l_j} : b_{p_r,q_s}$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$.

## 4 The equation with operation $\ominus$

In this case, we see that IMs $A$ and $B$ must satisfy the conditions $P \subseteq K, Q \subseteq L$.

If $K = P$ or $L = Q$, then

$$A \ominus X = B$$

has the solution

$$X = [\emptyset, \emptyset, \{*\}],$$

where symbol "$*$" denotes a lack of elements.

When $P \subset K$, i.e., $P \neq K$ and $Q \subset L$, i.e., $Q \neq L$, then equation (3)

$$X = [K - P, L - Q, \{c_{t_u,v_w}\}],$$

where for every $t_u \in K - P$ and $v_w \in L - Q$, $c_{t_u,v_w}$ are arbitrary real numbers.

## 5 Conclusion

In the present research, we studied the solutions of equations (1) - (3), when operation "$\circ$" is "$+$" or "$-$" or "$*$" or ":". In future, the research can have some directions.

First, it can be checked these equations with different forms of operation "$\circ$", e.g., using the operations from [1, 2].

Second, these equations can use operation $\odot_{(\circ,*)}$.

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