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λ-WHITNEY NUMBERS AND λ-DOWLING POLYNOMIALS

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ABSTRACT. Recently, degenerate Whitney numbers and degenerate Dowling polynomials are introduced by Kim-Kim. In this paper, we consider λ-Whitney numbers and λ-Dowling polynomials whose limits as λ tends 1 are respectively the Whitney numbers and the Dowling polynomials. They are different from the degenerate Whitney numbers and the degenerate Dowling polynomials whose limits as λ tends to 0 are the Whitney numbers and the Dowling polynomials. The aim of this paper is to derive, for these numbers and polynomials, some properties, generating functions, recurrence relations, and explicit expressions in terms of λ-analogues of Stirling numbers.

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1. INTRODUCTION

Dowling constructed a finite geometric lattice for any finite set with n elements and any finite multiplicative group G of order m, called Dowling lattice of rank n over a finite group of order m and denoted by $Q_n(G)$. The Whitney numbers of the first and second kind are defined for any finite geometric lattice. These numbers for Dowling lattice $Q_n(G)$ are the Whitney numbers of the first kind $V_m(n,k)$ and those of the second kind $W_m(n,k)$, which satisfy Stirling number-like relations (see [4]). The degenerate Whitney numbers of the first kind and those of the second kind are introduced by degenerating such relations (see [7]). Moreover, the degenerate r-Whitney numbers of both kinds are considered, as further generalizations of the degenerate Whitney numbers of both kinds (see [7]).

As is well-known, the normal ordering of an integral power of the number operator in terms of boson operators is expressed with the help of the Stirling numbers of the second kind. It is shown in [8] that the normal ordering of a certain quantity involving the number operator is expressed in terms of the degenerate r-Whitney numbers of the second kind. In addition, the degenerate r-Dowling polynomials are considered as a natural extension of the degenerate r-Whitney numbers of the second kind (see [8]). In [10], it is shown that the degenerate Dowling (see [7]) and the degenerate r-Dowling polynomials are connected with Poisson degenerate central moments for a Poisson random variable with a certain parameter and with Charlier polynomials.

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In this paper, we use λ-analogues of Stirling numbers of the first kind $\mathbf{S}_{1,\lambda}(n,k)$ (see [9]) and λ-analogues of Stirling numbers of the second kind $\mathbf{S}_{2,\lambda}(n,k)$ (see [5]), whose limits as λ tends to 1 are respectively the Stirling numbers of the first kind and the Stirling numbers of the second kind. Notice here that they are different from the degenerate Stirling numbers of the first kind $S_{1,\lambda}(n,k)$ and the degenerate Stirling numbers of the second $S_{2,\lambda}(n,k)$, whose limits as λ tends 0 are respectively Stirling numbers of the first kind and Stirling numbers of the second kind.

The aim of this paper is to derive, for the λ-Whitney numbers and the λ-Dowling polynomials, some properties, generating functions, recurrence relations, and explicit expressions in terms of the λ-analogues of Stirling numbers. The novelty of this paper is that several new polynomials and numbers are introduced as analogues of existing polynomials and numbers. They are the λ-Whitney numbers of both kinds, the λ-Dowling polynomials and numbers, the λ-Bell polynomials and numbers, and the λ-Tanny-Dowling polynomials.

In more detail, the outline of this paper is as follows. In Section 1, we recall the Whitney numbers of the first kind, the Whitney numbers of the second kind and the Dowling polynomials. We remind the reader of the degenerate exponentials, the degenerate Whitney numbers of the first kind and the degenerate Whitney numbers of the second. We also recall the λ-analogues of Stirling numbers of the first kind $\mathbf{S}_{1,\lambda}(n,k)$ and the λ-analogues of Stirling numbers of the second kind $\mathbf{S}_{2,\lambda}(n,k)$. Section 2 is the main result of this paper. We define the λ-Whitney numbers of the first kind $\mathbf{V}_{m,\lambda}(n,k)$ and the λ-Whitney numbers of the second kind $\mathbf{W}_{m,\lambda}(n,k)$. In Theorem 1, we show that $\mathbf{W}_{1,\lambda}(n,k) = \mathbf{S}_{2,\lambda}(n+1,k+1)$. We introduce the λ-Dowling polynomials $\mathbf{D}_{m,\lambda}(n,x)$ and derive the generating function of them in Theorem 2. Then we introduce λ-Bell polynomials $\Phi_{n,\lambda}(x)$. In Theorem 3, we show $\mathbf{D}_{1,\lambda}(n,1) = \Phi_{n+1,\lambda}(1)$. We deduce the generating function of $\mathbf{W}_{m,\lambda}(n,k)$ in Theorem 4 and that of the $\mathbf{V}_{m,\lambda}(n,k)$ in Theorem 5. We derive a recurrence relation for $\mathbf{W}_{m,\lambda}(n,k)$ in Theorem 6 and that for $\mathbf{V}_{m,\lambda}(n,k)$ in Theorem 7. $\mathbf{V}_{m,\lambda}(n,k)$ and $\mathbf{W}_{m,\lambda}(n,k)$ are respectively expressed as a finite sum involving $\mathbf{S}_{1,\lambda}(n,k)$ in Theorem 8 and that involving $\mathbf{S}_{2,\lambda}(n,k)$ in Theorem 13. The λ-Tanny-Dowling polynomials are defined and the generating function of them are obtained in Theorem 9. In Theorem 10, $\mathbf{D}_{m,\lambda}(n,x)$ are expressed as an infinite sum in Theorem 10. In Theorem 11, we show $x\mathbf{D}_{1,\lambda}(n,x) = \Phi_{n+1,\lambda}(x)$. We find an expression for $\mathbf{W}_{m,\lambda}(n,k)$ in Theorem 12. We find an expression for $\mathbf{W}_{m,\lambda}(n,k)$ in terms of the difference operator in Theorem 14. In Theorem 15, we derive another recurrence relation for $\mathbf{W}_{m,\lambda}(n,k)$. Finally, we conclude our paper in Section 3. In the rest of this section, we recall the necessary facts that are needed throughout this paper.

For any positive integer m, the Whitney numbers of the first kind $V_m(n,k)$ and the Whitney numbers of the second kind $W_m(n,k)$ are respectively defined by

(1) $$m^n\left(\frac{x-1}{m}\right)_n=\sum_{k=0}^{n}V_m(n,k)x^k,$$

and

(2) $$x^n=\sum_{k=0}^{n}W_m(n,k)m^k\left(\frac{x-1}{m}\right)_k,\quad \text{(see [3, 6-8, 10, 16])}.$$

where $(x)_n$ is given by $(x)_0=1,\ (x)_n=x(x-1)\ \cdots\ (x-n+1)),\quad (n\geq 1)$.

For $n\geq 0$, Dowling polynomials are defined by

(3) $$D_m(n,x)=\sum_{k=0}^{n}W_m(n,k)x^k,\quad \text{(see [3, 6-8, 10, 16])}.$$

When $x=1,\ D_m(n)=D_m(n,1)$ are called the Dowling numbers.

Equivalently, the relations (1) and (2) are given by

(4) $$\begin{aligned} m^n(x)_n=&\sum_{k=0}^{n}V_m(n,k)(mx+1)^k,\ (n\geq 0),\\ &\frac{1}{k!}\Big(\frac{\log(1+mt)}{m}\Big)^k(1+mt)^{-\frac{1}{m}}=\sum_{n=k}^{\infty}V_m(n,k)\frac{t^n}{n!},\ (k\geq 0),\end{aligned}$$

(5) $$\begin{aligned} (mx+1)^n=&\sum_{k=0}^{n}W_m(n,k)m^k(x)_k,\ (n\geq 0),\\ &e^t\frac{1}{k!}\Big(\frac{e^{mt}-1}{m}\Big)^k=\sum_{n=k}^{\infty}W_m(n,k)\frac{t^n}{n!},\ (k\geq 0),\end{aligned}$$

(see [3, 6, 8]).

For any nonzero $\lambda\in\mathbb{R}$, the degenerate exponentials are defined by

(6) $$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\frac{(x)_{n,\lambda}}{n!}t^n,\quad \text{(see [1, 2, 7, 8, 10-12, 17])},$$

where

$$(x)_{0,\lambda}=1,\quad\text{and}\quad (x)_{n,\lambda}=x(x-\lambda)\ \cdots\ (x-(n-1)\lambda),\quad (n\geq 1).$$

For $x=1$, for brevity we write $e_\lambda(t)=e_\lambda^1(t)$.

In [7], the degenerate Whitney numbers of the first kind $V_{m,\lambda}(n,k)$ and the degenerate Whitney numbers of the second kind $W_{m,\lambda}(n,k)$ are respectively given by

(7) $$\begin{aligned} m^n(x)_n=&\sum_{k=0}^{n}V_{m,\lambda}(n,k)(mx+1)_{k,\lambda},\ (n\geq 0),\\ &\frac{1}{k!}\big(\log_\lambda e_m(t)\big)^k e_m^{-1}(t)=\sum_{n=k}^{\infty}V_{m,\lambda}(n,k)\frac{t^n}{n!},\ (k\geq 0),\end{aligned}$$

and

$$(8)\qquad \begin{aligned}(mx+1)_{n,\lambda} &= \sum_{k=0}^{n} W_{m,\lambda}(n,k) m^k (x)_k, \ (n \geq 0),\\ e_\lambda(t)\frac{1}{k!}\bigg(\frac{e_\lambda^m(t)-1}{m}\bigg)^k &= \sum_{n=k}^{\infty} W_{m,\lambda}(n,k)\frac{t^n}{n!}, \ (k \geq 0).\end{aligned}$$

Note that $\lim_{\lambda\to 0} W_{m,\lambda}(n,k) = W_m(n,k)$, $\lim_{\lambda\to 0} V_{m,\lambda}(n,k) = V_m(n,k)$.

The λ-analogues of Stirling numbers of the first kind are defined by

$$\begin{aligned}(x)_{n,\lambda} &= \sum_{k=0}^{n} \mathbf{S}_{1,\lambda}(n,k)x^k, \ (n \geq 0),\\ \frac{1}{k!}\frac{1}{\lambda^k}\big(\log(1+\lambda t)\big)^k &= \sum_{n=k}^{\infty} \mathbf{S}_{1,\lambda}(n,k)\frac{t^n}{n!}, \ (k \geq 0), \quad \text{(see [9])}.\end{aligned}$$

Note that $\lim_{\lambda\to 1} \mathbf{S}_{1,\lambda}(n,k) = S_1(n,k)$, where $S_1(n,k)$ are the Stirling numbers of the first kind defined by

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k, \quad \text{(see [3, 15, 16])}.$$

As the inversion formula of (8), the λ-analogues of Stirling numbers of the second kind are defined by

$$(9)\qquad \begin{aligned}x^n &= \sum_{k=0}^{n} \mathbf{S}_{2,\lambda}(n,k)(x)_{k,\lambda}, \ (n \geq 0),\\ \frac{1}{k!}\frac{1}{\lambda^k}\big(e^{\lambda t}-1\big)^k &= \sum_{n=k}^{\infty} \mathbf{S}_{2,\lambda}(n,k)\frac{t^n}{n!}, \ (k \geq 0), \quad \text{(see [5])}.\end{aligned}$$

Note that $\lim_{\lambda\to 1} \mathbf{S}_{2,\lambda}(n,k) = S_2(n,k)$, where $S_2(n,k)$ are the Stirling numbers of the second kind defined by

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \ (n \geq 0), \quad \text{(see [2,3,13-17])}.$$

2. λ-Whitney numbers and λ-Dowling polynomials

In view of (1) and (2), we consider the λ-Whitney numbers of the first kind $\mathbf{V}_{m,\lambda}(n,k)$ and the λ-Whitney numbers of the second kind $\mathbf{W}_{m,\lambda}(n,k)$ which are different from (7) and (8).

We define the λ-*Whitney numbers of the first kind* as

$$(10)\qquad m^n(x)_{n,\lambda} = \sum_{k=0}^{n} \mathbf{V}_{m,\lambda}(n,k)(mx+\lambda)^k, \quad (n \geq 0).$$

As the inversion formula of (10), we define the λ-*Whitney numbers of the second kind* as

$$(11)\qquad (mx+\lambda)^n = \sum_{k=0}^{n} \mathbf{W}_{m,\lambda}(n,k)m^k(x)_{k,\lambda}, \quad (n \geq 0).$$

From (10) and (11), we will show that

$$\frac{1}{m^k\lambda^k}\frac{1}{k!}\big(\log(1+mt\lambda)\big)^k(1+mt\lambda)^{-\frac{1}{m}}=\sum_{n=k}^{\infty}\mathbf{V}_{m,\lambda}(n,k)\frac{t^n}{n!}, \tag{12}$$

and

$$\frac{1}{\lambda^k k!}\left(\frac{e^{\lambda mt}-1}{m}\right)^k e^{\lambda t}=\sum_{n=k}^{\infty}\mathbf{W}_{m,\lambda}(n,k)\frac{t^n}{n!},\quad (n,k\geq 0). \tag{13}$$

Note that $\lim_{\lambda\to 1}\mathbf{W}_{m,\lambda}(n,k)=W_m(n,k), \lim_{\lambda\to 1}\mathbf{V}_{m,\lambda}(n,k)=V_m(n,k)$.
From (9), we note that

$$\begin{aligned}\sum_{n=k}^{\infty}\mathbf{S}_{2,\lambda}(n+1,k+1)\frac{t^n}{n!}&=\frac{d}{dt}\left(\sum_{n=k}^{\infty}\mathbf{S}_{2,\lambda}(n+1,k+1)\frac{t^{n+1}}{(n+1)!}\right)\\&=\frac{d}{dt}\left(\frac{1}{(k+1)!}\frac{1}{\lambda^{k+1}}(e^{\lambda t}-1)^{k+1}\right)\\&=\frac{1}{k!}\frac{1}{\lambda^k}(e^{\lambda t}-1)^k e^{\lambda t}=\sum_{n=k}^{\infty}\mathbf{W}_{1,\lambda}(n,k)\frac{t^n}{n!}.\end{aligned} \tag{14}$$

Thus, by comparing the coefficients on both sides of (14), we obtain the following theorem.

Theorem 2.1. *For $n,k\geq 0$, with $n\geq k$, we have*

$$\mathbf{W}_{1,\lambda}(n,k)=\mathbf{S}_{2,\lambda}(n+1,k+1).$$

Now, we define the *λ-Dowling polynomials* by

$$\mathbf{D}_{m,\lambda}(n,x)=\sum_{k=0}^{n}\mathbf{W}_{m,\lambda}(n,k)x^k,\ (n\geq 0). \tag{15}$$

For $x=1$, $\mathbf{D}_{m,\lambda}(n)=\mathbf{D}_{m,\lambda}(n,1)$ are called the *λ-Dowling numbers*.
From (15), we note that

(16)
$$\begin{aligned}\sum_{n=0}^{\infty}\mathbf{D}_{m,\lambda}(n,x)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\mathbf{W}_{m,\lambda}(n,k)x^k\right)\frac{t^n}{n!}=\sum_{k=0}^{\infty}x^k\sum_{n=k}^{\infty}\mathbf{W}_{m,\lambda}(n,k)\frac{t^n}{n!}\\&=e^{\lambda t}\sum_{k=0}^{\infty}\frac{x^k}{k!}\left(\frac{e^{\lambda mt}-1}{\lambda m}\right)^k=e^{\lambda t}e^{\frac{x}{\lambda m}(e^{\lambda mt}-1)}.\end{aligned}$$

Theorem 2.2. *For $m\in\mathbb{N}$, we have*

$$e^{\lambda t}e^{\frac{x}{\lambda m}(e^{\lambda mt}-1)}=\sum_{n=0}^{\infty}\mathbf{D}_{m,\lambda}(n,x)\frac{t^n}{n!}.$$

Note that $\lim_{\lambda\to 1}\mathbf{D}_{m,\lambda}(n,x)=D_m(n,x),\quad (n\geq 0)$.
The *λ-Bell polynomials* are defined by

$$\Phi_{n,\lambda}(x)=\sum_{k=0}^{n}\mathbf{S}_{2,\lambda}(n,k)x^k,\ (n\geq 0). \tag{17}$$

Thus, by (17), we get

$$e^{\frac{x}{\lambda}(e^{\lambda t}-1)} = \sum_{n=0}^{\infty} \Phi_{n,\lambda}(x)\frac{t^n}{n!}. \tag{18}$$

Note that $\lim_{\lambda\to 1}\Phi_{n,\lambda}(x) = \phi_n(x), \quad (n \geq 0)$, where $\phi_n(x)$ are the ordinary Bell polynomials defined by

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} \phi_n(x)\frac{t^n}{n!}, \quad \text{(see [3, 5, 11, 15, 16]).}$$

For $x = 1$, $\Phi_{n,\lambda} = \Phi_{n,\lambda}(1)$ are called the λ-*Bell numbers.*
From (17) and (18), we note that

(19)

$$\begin{aligned}\sum_{n=0}^{\infty}\Phi_{n+1,\lambda}(x)\frac{t^n}{n!} &= \frac{d}{dt}e^{\frac{x}{\lambda}(e^{\lambda t}-1)} = \frac{d}{dt}\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}\mathbf{S}_{2,\lambda}(n,k)x^k\Big)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=1}^{n+1}\mathbf{S}_{2,\lambda}(n+1,k)x^k\Big)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\mathbf{S}_{2,\lambda}(n+1,k+1)x^{k+1}\Big)\frac{t^n}{n!}.\end{aligned}$$

Thus, by comparing the coefficients on both sides of (19), we obtain the following equation.

$$\begin{aligned}\Phi_{n+1,\lambda}(x) &= x\sum_{k=0}^{n}\mathbf{S}_{2,\lambda}(n+1,k+1)x^k, \\ \Phi_{n+1,\lambda} &= \sum_{k=0}^{n}\mathbf{S}_{2,\lambda}(n+1,k+1), \quad (n \geq 0).\end{aligned} \tag{20}$$

Here, from (9), we note $\mathbf{S}_{2,\lambda}(n+1,0) = 0$, for $n \geq 0$.
From Theorem 1 and (15), we have

$$\mathbf{D}_{1,\lambda}(n) = \sum_{k=0}^{n}\mathbf{W}_{1,\lambda}(n,k) = \sum_{k=0}^{n}\mathbf{S}_{2,\lambda}(n+1,k+1) = \Phi_{n+1,\lambda}. \tag{21}$$

Therefore, by (21), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$\mathbf{D}_{1,\lambda}(n) = \Phi_{n+1,\lambda}.$$

Now, we show (13). From (11), we see that

$$\begin{aligned}e^{(mx+\lambda)t} &= \sum_{n=0}^{\infty}(mx+\lambda)^n\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\mathbf{W}_{m,\lambda}(n,k)m^k(x)_{k,\lambda}\Big)\frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty}\Big(\sum_{n=k}^{\infty}\mathbf{W}_{m,\lambda}(n,k)\frac{t^n}{n!}\Big)m^k(x)_{k,\lambda}.\end{aligned} \tag{22}$$

On the other hand, we also have

$$
\begin{aligned}
e^{(mx+\lambda)t} &= e^{\lambda t}\big(e^{\lambda mt}-1+1\big)^{\frac{x}{\lambda}} \\
&= e^{\lambda t}\sum_{k=0}^{\infty}\Big(\frac{x}{\lambda}\Big)_k\frac{1}{k!}\big(e^{\lambda mt}-1\big)^k \\
&= e^{\lambda t}\sum_{k=0}^{\infty}\frac{1}{\lambda^k}\frac{1}{k!}\bigg(\frac{e^{\lambda mt}-1}{m}\bigg)^k m^k(x)_{k,\lambda}.
\end{aligned}
\tag{23}
$$

From (22) and (23), we obtain the following theorem.

Theorem 2.4. *For $k \geq 0$, we have*

$$
e^{\lambda t}\frac{1}{\lambda^k}\frac{1}{k!}\bigg(\frac{e^{\lambda mt}-1}{m}\bigg)^k = \sum_{n=k}^{\infty}\mathbf{W}_{m,\lambda}(n,k)\frac{t^n}{n!}.
$$

Next, we show (12). From (10), we see that

$$
\begin{aligned}
(1+m\lambda t)^{\frac{x}{\lambda}} &= \sum_{n=0}^{\infty}m^n(x)_{n,\lambda}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\bigg(\sum_{k=0}^{n}\mathbf{V}_{m,\lambda}(n,k)(mx+\lambda)^k\bigg)\frac{t^n}{n!} \\
&= \sum_{k=0}^{\infty}\bigg(\sum_{n=k}^{\infty}\mathbf{V}_{m,\lambda}(n,k)\frac{t^n}{n!}\bigg)(mx+\lambda)^k.
\end{aligned}
\tag{24}
$$

On the other hand, we also have

$$
\begin{aligned}
(1+m\lambda t)^{\frac{x}{\lambda}} &= (1+m\lambda t)^{\frac{x}{\lambda}+\frac{1}{m}}(1+m\lambda t)^{-\frac{1}{m}} \\
&= (1+m\lambda t)^{\frac{1}{\lambda}(\frac{mx+\lambda}{m})}(1+m\lambda t)^{-\frac{1}{m}} \\
&= e^{\frac{1}{\lambda}(\frac{mx+\lambda}{m})\log(1+m\lambda t)}(1+m\lambda t)^{-\frac{1}{m}} \\
&= \sum_{k=0}^{\infty}\bigg(\frac{1}{\lambda^k m^k}\frac{1}{k!}(\log(1+m\lambda t))^k(1+m\lambda t)^{-\frac{1}{m}}\bigg)(mx+\lambda)^k.
\end{aligned}
\tag{25}
$$

From (24) and (25), we get the next theorem.

Theorem 2.5. *For $k \geq 0$, we have*

$$
\frac{1}{\lambda^k m^k}\frac{1}{k!}\big(\log(1+m\lambda t)\big)^k(1+m\lambda t)^{-\frac{1}{m}} = \sum_{n=k}^{\infty}\mathbf{V}_{m,\lambda}(n,k)\frac{t^n}{n!}.
$$

By (11), we get

(26)

$$\begin{aligned}\sum_{k=0}^{n}\mathbf{W}_{m,\lambda}(n,k)m^k(x)_{k,\lambda} &= (mx+\lambda)^n = (mx+\lambda)^{n-1}(mx+\lambda)\\
&=\sum_{k=0}^{n-1}\mathbf{W}_{m,\lambda}(n-1,k)m^k(x)_{k,\lambda}(mx+\lambda)\\
&=\sum_{k=0}^{n-1}\mathbf{W}_{m,\lambda}(n-1,k)m^{k+1}(x)_{k,\lambda}(x-k\lambda+k\lambda)\\
&\qquad +\lambda\sum_{k=0}^{n-1}\mathbf{W}_{m,\lambda}(n-1,k)m^k(x)_{k,\lambda}\\
&=\sum_{k=0}^{n-1}\mathbf{W}_{m,\lambda}(n-1,k)m^{k+1}(x)_{k+1,\lambda}\\
&\qquad +\sum_{k=0}^{n-1}\mathbf{W}_{m,\lambda}(n-1,k)m^k(x)_{k,\lambda}(mk\lambda+\lambda)\\
&=\sum_{k=1}^{n}\mathbf{W}_{m,\lambda}(n-1,k-1)m^k(x)_{k,\lambda}\\
&\qquad +\sum_{k=0}^{n-1}\mathbf{W}_{m,\lambda}(n-1,k)m^k(x)_{k,\lambda}(mk\lambda+\lambda)\\
&=\sum_{k=0}^{n}\Big(\mathbf{W}_{m,\lambda}(n-1,k-1)+(mk\lambda+\lambda)\mathbf{W}_{m,\lambda}(n-1,k)\Big)m^k(x)_{k,\lambda}.\end{aligned}$$

By comparing the coefficients on both sides of (26), we obtain the following theorem.

Theorem 2.6. *For $n,\ k\geq\ 1$ with $n\geq k$, we have*

$$\mathbf{W}_{m,\lambda}(n,k)=\mathbf{W}_{m,\lambda}(n-1,k-1)+(mk\lambda+\lambda)\mathbf{W}_{m,\lambda}(n-1,k).$$

In Theorem 6, by taking the limit as $\lambda\to 1$, we note that

$$W_m(n,k)=W_m(n-1,k-1)+(mk+1)W_m(n-1,k).$$

From (10), we note that

(27)

$$\begin{aligned}\sum_{k=0}^{n}\mathbf{V}_{m,\lambda}(n,k)(mx+\lambda)^k &= m^n(x)_{n,\lambda}=m(x-(n-1)\lambda)m^{n-1}(x)_{n-1,\lambda}\\
&=\sum_{k=0}^{n-1}\mathbf{V}_{m,\lambda}(n-1,k)\big(mx+\lambda)^k(mx+\lambda-\lambda m(n-1)-\lambda\big)\\
&=\sum_{k=0}^{n-1}\mathbf{V}_{m,\lambda}(n-1,k)(mx+\lambda)^{k+1}\end{aligned}$$

$$-\lambda\sum_{k=0}^{n-1}\mathbf{V}_{m,\lambda}(n-1,k)\big(m(n-1)+1\big)(mx+\lambda)^k$$

$$=\sum_{k=0}^{n}\mathbf{V}_{m,\lambda}(n-1,k-1)(mx+\lambda)^k$$

$$-\lambda\sum_{k=0}^{n-1}\mathbf{V}_{m,\lambda}(n-1,k)\big(m(n-1)+1\big)(mx+\lambda)^k$$

$$=\sum_{k=0}^{n}\Big(\mathbf{V}_{m,\lambda}(n-1,k-1)-\lambda\big(m(n-1)+1\big)\mathbf{V}_{m,\lambda}(n-1,k)\Big)(mx+\lambda)^k.$$

Therefore, by comparing the coefficients on both sides of (27), we obtain the following theorem.

Theorem 2.7. *For $n,\ k\ \geq 1$ with $n\geq k$, we have*

$$\mathbf{V}_{m,\lambda}(n,k)=\mathbf{V}_{m,\lambda}(n-1,k-1)-\lambda(m(n-1)+1)\mathbf{V}_{m,\lambda}(n-1,k).$$

From Theorem 6, we note that

(28)

$$\begin{aligned}\sum_{n=k}^{\infty}\mathbf{V}_{m,\lambda}(n,k)\frac{t^n}{n!}&=\frac{1}{\lambda^k m^k}\frac{1}{k!}(\log(1+m\lambda t))^k(1+m\lambda t)^{-\frac{1}{m}}\\&=\frac{1}{\lambda^k m^k}\frac{1}{k!}(\log(1+m\lambda t))^k e^{-\frac{1}{m}(\log(1+m\lambda t))}\\&=\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\frac{(\log(1+m\lambda t))^l}{m^l}\frac{1}{k!\lambda^k m^k}(\log(1+m\lambda t))^k\\&=\sum_{l=0}^{\infty}\frac{(-1)^l(k+l)!\lambda^l}{l!k!m^{k+l}}\frac{1}{(k+l)!}\left(\frac{\log(1+m\lambda t)}{\lambda}\right)^{k+l}\\&=\sum_{l=0}^{\infty}\binom{k+l}{k}(-1)^l\frac{\lambda^l}{m^{k+l}}\sum_{n=k+l}^{\infty}\mathbf{S}_{1,\lambda}(n,k+l)\frac{m^n t^n}{n!}\\&=\sum_{l=k}^{\infty}\binom{l}{k}(-1)^{l-k}\frac{\lambda^{l-k}}{m^l}\sum_{n=l}^{\infty}\mathbf{S}_{1,\lambda}(n,l)\frac{m^n}{n!}t^n\\&=\sum_{n=k}^{\infty}\left(\sum_{l=k}^{n}\binom{l}{k}(-1)^{l-k}m^{n-l}\lambda^{l-k}\mathbf{S}_{1,\lambda}(n,l)\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, by comparing the coefficients on both sides of (28), we obtain the following theorem.

Theorem 2.8. *For $n,\ k\geq 0$ with $n\geq k$, we have*

$$\mathbf{V}_{m,\lambda}(n,k)=\sum_{l=k}^{n}\binom{l}{k}(-1)^{l-k}m^{n-l}\lambda^{l-k}\mathbf{S}_{1,\lambda}(n,l).$$

Letting $\lambda \to 1$ in the identity of Theorem 8, we get

$$V_m(n,k) = \sum_{l=k}^{n} \binom{l}{k} (-1)^{l-k} S_1(n,l) m^{n-l}.$$

Now, we consider the λ-*Tanny-Dowling polynomials* given by

$$\mathcal{F}_{m,\lambda}(n,x) = \int_0^\infty \mathbf{D}_{m,\lambda}(n,yx) e^{-y} dy. \tag{29}$$

By (29), we get

$$\begin{aligned}\mathcal{F}_{m,\lambda}(n,x) &= \int_0^\infty \mathbf{D}_{m,\lambda}(n,yx) e^{-y} dy \\ &= \sum_{k=0}^{n} \mathbf{W}_{m,\lambda}(n,k) x^k \int_0^\infty y^k e^{-y} dy = \sum_{k=0}^{n} \mathbf{W}_{m,\lambda}(n,k) x^k k!.\end{aligned}$$

By Theorem 2, we get

$$\begin{aligned}\sum_{n=0}^\infty \mathcal{F}_{m,\lambda}(n,x) \frac{t^n}{n!} &= \sum_{n=0}^\infty \int_0^\infty \mathbf{D}_{m,\lambda}(n,yx) e^{-y} dy \frac{t^n}{n!} \\ &= \int_0^\infty e^{-y} \sum_{n=0}^\infty \mathbf{D}_{m,\lambda}(n,yx) \frac{t^n}{n!} dy \\ &= \int_0^\infty e^{-y} e^{\lambda t} e^{\frac{yx}{\lambda m}(e^{\lambda m t}-1)} dy \\ &= e^{\lambda t} \int_0^\infty e^{-y(1-\frac{x}{\lambda m}(e^{\lambda m t}-1))} dy \\ &= e^{\lambda t} \frac{1}{1-\frac{x}{\lambda m}(e^{\lambda m t}-1)}, \quad (1 > \frac{x}{\lambda m}(e^{\lambda m t}-1).\end{aligned} \tag{30}$$

Therefore, by (30), we obtain the following theorem.

Theorem 2.9. *For $m \in \mathbb{N}$, and for all t with $1 > \frac{x}{\lambda m}(e^{\lambda m t}-1)$, we have*

$$e^{\lambda t} \frac{1}{1-\frac{x}{\lambda m}(e^{\lambda m t}-1)} = \sum_{n=0}^\infty \mathcal{F}_{m,\lambda}(n,x) \frac{t^n}{n!}.$$

From Theorem 2, we note that

$$\begin{aligned}\sum_{n=0}^\infty \mathbf{D}_{m,\lambda}(n,x) \frac{t^n}{n!} &= e^{\lambda t} e^{\frac{x}{\lambda m}(e^{\lambda m t}-1)} \\ &= e^{-\frac{x}{m\lambda}} \sum_{k=0}^\infty \frac{x^k e^{\lambda(mk+1)t}}{\lambda^k m^k k!} \\ &= e^{-\frac{x}{m\lambda}} \sum_{k=0}^\infty \frac{x^k}{\lambda^k k! m^k} \sum_{n=0}^\infty \lambda^n (mk+1)^n \frac{t^n}{n!} \\ &= \sum_{n=0}^\infty \left(e^{-\frac{x}{m\lambda}} \lambda^n \sum_{k=0}^\infty \frac{x^k}{k! m^k \lambda^k} (mk+1)^n \right) \frac{t^n}{n!}.\end{aligned} \tag{31}$$

By comparing the coefficients on both sides of (31), we obtain the following theorem.

Theorem 2.10. *For $n \geq 0$, we have*

$$\mathbf{D}_{m,\lambda}(n,x) = e^{-\frac{x}{m\lambda}}\lambda^n \sum_{k=0}^{\infty} \frac{x^k}{k! m^k \lambda^k}(mk+1)^n.$$

In particular, for $x = 1$, we have

$$\mathbf{D}_{m,\lambda}(n) = e^{-\frac{1}{m\lambda}}\lambda^n \sum_{k=0}^{\infty} \frac{1}{\lambda^k m^k k!}(mk+1)^n.$$

By (18), we see that

$$\Phi_{n,\lambda}(x) = e^{-\frac{x}{\lambda}}\lambda^n \sum_{k=0}^{\infty} \frac{x^k}{k!\lambda^k} k^n. \tag{32}$$

From Theorem 9 with $m = 1$ and for $n \geq 0$, we have

$$\begin{aligned} \mathbf{D}_{1,\lambda}(n,x) &= e^{-\frac{x}{\lambda}}\lambda^n \sum_{k=0}^{\infty} \frac{x^k}{\lambda^k k!}(k+1)^n \\ &= e^{-\frac{x}{\lambda}}\lambda^n \sum_{k=1}^{\infty} \frac{x^{k-1}}{\lambda^{k-1}(k-1)!} k^n \\ &= e^{-\frac{x}{\lambda}}\lambda^n \frac{\lambda}{x} \sum_{k=1}^{\infty} \frac{x^k}{\lambda^k k!} k^{n+1} = e^{-\frac{x}{\lambda}}\lambda^n \frac{\lambda}{x} \sum_{k=0}^{\infty} \frac{x^k}{\lambda^k k!} k^{n+1}. \end{aligned} \tag{33}$$

Thus, by (33), we get

$$x\mathbf{D}_{1,\lambda}(n,x) = e^{-\frac{x}{\lambda}}\lambda^{n+1} \sum_{k=0}^{\infty} \frac{x^k}{\lambda^k k!} k^{n+1} = \Phi_{n+1,\lambda}(x). \tag{34}$$

Therefore, by (32) and (34), we obtain the following theorem.

Theorem 2.11. *For $n \geq 0$, we have*

$$x\mathbf{D}_{1,\lambda}(n,x) = \Phi_{n+1,\lambda}(x).$$

By (13), we get

$$\begin{aligned} \sum_{n=k}^{\infty} \mathbf{W}_{m,\lambda}(n,k)\frac{t^n}{n!} &= \frac{1}{\lambda^k k!}\left(\frac{e^{\lambda m t}-1}{m}\right)^k e^{\lambda t} \\ &= \frac{1}{k!\lambda^k m^k} \sum_{l=0}^{k} \binom{k}{l}(-1)^{k-l} e^{(lm+1)\lambda t} \\ &= \frac{1}{k! m^k \lambda^k} \sum_{l=0}^{k} \binom{k}{l}(-1)^{k-l} \sum_{n=0}^{\infty} (lm+1)^n \lambda^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\frac{\lambda^n}{k! m^k \lambda^k} \sum_{l=0}^{k} \binom{k}{l}(-1)^{k-l}(lm+1)^n \right) \frac{t^n}{n!}. \end{aligned} \tag{35}$$

Therefore, by (35), we obtain the following theorem.

Theorem 2.12. *For $n, k \geq 0$, we have*

$$\frac{\lambda^n}{k! m^k \lambda^k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} (lm+1)^n = \begin{cases} \mathbf{W}_{m,\lambda}(n,k), & \text{if } n \geq k, \\ 0, & \text{if } 0 \leq n < k. \end{cases}$$

From Theorems 1 and 11, we have

$$\frac{\lambda^n}{k! \lambda^k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} (l+1)^n = \mathbf{W}_{1,\lambda}(n,k) = \mathbf{S}_{2,\lambda}(n+1,k+1), \quad (n \geq k). \tag{36}$$

By (36), we get

$$\frac{\lambda^n}{k! \lambda^k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} (l+1)^n = \mathbf{S}_{2,\lambda}(n+1,k+1), \quad (n \geq k). \tag{37}$$

Now, we observe that

$$\begin{aligned} \mathbf{W}_{m,\lambda}(n,k) &= \frac{\lambda^n}{k! m^k \lambda^k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} (lm+1)^n \\ &= \frac{\lambda^n}{k! m^k \lambda^k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{l} ((k-l)m+1)^n \\ &= \frac{\lambda^n}{k! m^k \lambda^k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{l} \sum_{i=0}^{n} \binom{n}{i} (k-l)^i m^i \\ &= \sum_{i=0}^{n} \binom{n}{i} m^{i-k} \frac{\lambda^n}{k! \lambda^k} \sum_{l=0}^{k} \binom{k}{l} (-1)^l (k-l)^i, \quad (n \geq k). \end{aligned} \tag{38}$$

From (9), we note that

$$\begin{aligned} \sum_{n=k}^{\infty} \mathbf{S}_{2,\lambda}(n,k) \frac{t^n}{n!} &= \frac{1}{\lambda^k k!} (e^{\lambda t} - 1)^k = \frac{1}{\lambda^k k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^l e^{(k-l)\lambda t} \\ &= \sum_{n=0}^{\infty} \left(\frac{\lambda^n}{\lambda^k k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^l (k-l)^n \right) \frac{t^n}{n!}. \end{aligned} \tag{39}$$

By comparing the coefficients on both sides of (39), we get

$$\frac{\lambda^n}{\lambda^k k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^l (k-l)^n = \begin{cases} \mathbf{S}_{2,\lambda}(n,k), & \text{if } n \geq k, \\ 0, & \text{if } 0 \leq n < k. \end{cases} \tag{40}$$

From (38) and (40), we note that

$$\begin{aligned} \mathbf{W}_{m,\lambda}(n,k) &= \sum_{i=0}^{n} \binom{n}{i} m^{i-k} \lambda^{n-i} \frac{\lambda^i}{k! \lambda^k} \sum_{l=0}^{k} \binom{k}{l} (-1)^l (k-l)^i \\ &= \sum_{i=k}^{n} \binom{n}{i} m^{i-k} \lambda^{n-i} \mathbf{S}_{2,\lambda}(i,k), \quad (n \geq k \geq 0). \end{aligned} \tag{41}$$

Therefore, by (41), we obtain the following theorem.

Theorem 2.13. *For $n,k \geq 0$ with $n \geq k$, we have*

$$\mathbf{W}_{m,\lambda}(n,k) = \sum_{i=k}^{n} \binom{n}{i} m^{i-k}\lambda^{n-i}\mathbf{S}_{2,\lambda}(i,k).$$

Let $\triangle$ be the difference operator with $\triangle f(x) = f(x+1) - f(x)$. Then we have

$$\triangle^k f(x) = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} f(x+l), \quad (k \geq 0).$$

Taking $f(x) = \lambda^n (mx+1)^n$, we have

$$\begin{aligned} \lambda^n \triangle^k (mx+1)^n|_{x=0} &= \lambda^n \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} (lm+1)^n \\ &= k! m^k \lambda^k \mathbf{W}_{m,\lambda}(n,k), \quad (n \geq k \geq 0). \end{aligned} \tag{42}$$

Therefore, by (42), we obtain the following theorem.

Theorem 2.14. *For $n,k \geq 0$ with $n \geq k$, we have*

$$\mathbf{W}_{m,\lambda}(n,k) = \frac{\lambda^n}{k! m^k \lambda^k} \triangle^k (mx+1)^n|_{x=0}.$$

From (13), we note that

$$\begin{aligned} &\sum_{n=k-1}^{\infty} \mathbf{W}_{m,\lambda}(n+1,k)\frac{t^n}{n!} = \frac{d}{dt}\frac{1}{\lambda^k k!}\left(\frac{e^{\lambda m t}-1}{m}\right)^k e^{\lambda t} \\ &= \frac{\lambda e^{\lambda t}}{\lambda^k k!}\left(\frac{e^{\lambda m t}-1}{m}\right)^k + \frac{e^{\lambda t}}{k! \lambda^k} k \left(\frac{\lambda m e^{\lambda m t}}{m}\right)\left(\frac{e^{\lambda m t}-1}{m}\right)^{k-1} \\ &= \sum_{n=k-1}^{\infty}\left(\lambda \mathbf{W}_{m,\lambda}(n,k) + \sum_{i=k-1}^{n}\binom{n}{i}\mathbf{W}_{m,\lambda}(i,k-1)\lambda^{n-i}m^{n-i}\right)\frac{t^n}{n!}. \end{aligned} \tag{43}$$

Therefore, by (43), we obtain the following theorem.

Theorem 2.15. *For $0 \leq k \leq n$, we have*

$$\mathbf{W}_{m,\lambda}(n+1,k) = \lambda \mathbf{W}_{m,\lambda}(n,k) + \sum_{i=k-1}^{n}\binom{n}{i}\mathbf{W}_{m,\lambda}(i,k-1)\lambda^{n-i}m^{n-i}.$$

By (16), we get

$$
\begin{aligned}
\sum_{n=0}^{\infty}\mathbf{D}_{m,\lambda}(n+1)\frac{t^n}{n!} &= \frac{d}{dt}\left(e^{\lambda t}e^{\frac{1}{m\lambda}(e^{\lambda mt}-1)}\right)\\
&= \lambda e^{\lambda t}e^{\frac{1}{m\lambda}(e^{\lambda mt}-1)} + e^{\lambda t}\frac{\lambda m e^{\lambda mt}}{m\lambda}e^{\frac{1}{\lambda m}(e^{\lambda mt}-1)}\\
&= \lambda e^{\lambda t}e^{\frac{1}{m\lambda}(e^{\lambda mt}-1)} + e^{\lambda t}e^{\frac{1}{\lambda m}(e^{\lambda mt}-1)}e^{\lambda mt}\\
&= \lambda\sum_{n=0}^{\infty}\mathbf{D}_{m,\lambda}(n)\frac{t^n}{n!} + \sum_{i=0}^{\infty}\mathbf{D}_{m,\lambda}(i)\frac{t^i}{i!}\sum_{j=0}^{\infty}\frac{\lambda^j m^j}{j!}t^j\\
&= \lambda\sum_{n=0}^{\infty}\mathbf{D}_{m,\lambda}(n)\frac{t^n}{n!} + \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\mathbf{D}_{m,\lambda}(n-j)\lambda^j m^j\right)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\lambda\mathbf{D}_{m,\lambda}(n) + \sum_{j=0}^{n}\binom{n}{j}\mathbf{D}_{m,\lambda}(n-j)\lambda^j m^j\right)\frac{t^n}{n!}.
\end{aligned}\tag{44}
$$

By comparing the coefficients on both sides of (44), we get

$$
\mathbf{D}_{m,\lambda}(n+1) = \lambda\mathbf{D}_{m,\lambda}(n) + \sum_{j=0}^{n}\binom{n}{j}\mathbf{D}_{m,\lambda}(n-j)\lambda^j m^j, \quad (n\geq 0).
$$

From (10), we note that

$$
m^n\left(\frac{x-\lambda}{m}\right)_{n,\lambda} = \sum_{k=0}^{n}\mathbf{V}_{m,\lambda}(n,k)x^k. \tag{45}
$$

Thus, by (45), we get

$$
\begin{aligned}
\mathbf{V}_{m,\lambda}(n,0) &= m^n\left(\frac{-\lambda}{m}\right)_{n,\lambda}\\
&= m^n\left(\frac{-\lambda}{m}\right)\left(\frac{-\lambda}{m}-\lambda\right)\cdots\left(\frac{-\lambda}{m}-(n-1)\lambda\right)\\
&= (-1)^n\lambda^n(m+1)(2m+1)\cdots((n-1)m+1).
\end{aligned}\tag{46}
$$

3. Conclusion

Studying various degenerate versions of some special numbers and polynomials regained interests of many mathematicians, as we have witnessed in recent years. Explorations for degenerate versions began with the pioneering work of Carlitz on the degenerate Bernoulli polynomials and the degenerate Euler polynomials (see [2]). They have been done by employing such tools as combinatorial methods, generating functions, umbral calculus techniques, p-adic analysis, differential equations, special functions, probability theory, operator theory and analytic number theory.

While, in this paper, we considered λ-analogues of Whitney numbers and Dowling polynomials, namely the λ-Whitney numbers and the λ-Dowling polynomials and derived some properties, generating functions, recurrence relations, and explicit expressions in terms of the λ-analogues of Stirling numbers.

It is one of our future projects to continue to explore various degenerate versions and λ-analogues of many special polynomials and numbers by using aforementioned tools and to find applications to physics, science and engineering as well as to mathematics.

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