Advanced Studies in Contemporary Mathematics
33 (2023), No. 3, pp. 265 - 272
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2023.33.3.265

ON UPPER EDGE DOMINATION IN GRAPHS

S. V. DIVYA RASHMI1 AND L. DILEEP2

Abstract. In this paper we find exact value of upper edge domination number of a graph $G = (V, E)$ for some special classes of graphs .

2000 Mathematics Subject Classification. 05C69.

Keywords and phrases. Edge domination, edge domination number, upper edge domination number.

1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [4]

The open neighborhood of a vertex $v \in V$ is given by $N(v) = \{u \in V : uv \in E\}$ and its closed neighborhood is $N[v] = N(v) \cup \{v\}$. Given $S \subseteq V$ and $v \in S$, a vertex $u \in V$ is an S-private neighbor of v if $N(u) \cap S = \{v\}$. The set of all S-private neighbors of v is denoted by $PN(v, S)$. If further $u \in V \setminus S$, then u is called an S-external private neighbor (abbreviated S-epn) of v. The set of all S-epns of v is denoted by $EPN(v, S)$. A set $S \subseteq V$ is called a dominating set of G if every vertex in $V \setminus S$ is adjacent to a vertex in S. A dominating set S is called a minimal dominating set of G if $S \setminus \{v\}$ is not a dominating set for all $v \in S$. The minimum (maximum) cardinality of a minimal dominating set of G is called the domination number (upper domination number) of G and is denoted by $\gamma(G)$ ($\Gamma(G)$). Many authors have studied parameters like upper domination, upper paired domination, upper total domination and upper secure domination [10, 9, 8, 6].

A subset S of E is an edge dominating set of G if every edge in $E \setminus S$ is adjacent to an edge in S. The edge domination number $\gamma'(G)$ of G is the minimum cardinality taken over all edge dominating sets of G. If G has at least one edge, then $1 \leq \gamma'(G) \leq |E(G)|$. If G has no edges then $\gamma'(G) = 0$. $E(G)$ is the unique maximum edge dominating set of G. The concept of edge domination was introduced by Mitchel and Hedetniemi [12]. It has been further explored by S R Jayaram [11] and S Arumugam [3]. Moreover B Xu [8], B Senthil Kumar et.al.[14], D. K. Thakkar and B. M. Kakrecha[16] , R. Dutton and W. Klostermeyer[7], J. Sreedevi and B. Maheswari[15], B. Zelinka[19], S. Arumugam and S. Jerry[2], S. K. Vaidya and R. M. Pandit[17],Araya Chaemchan[1], and Cristina Bazgan et. al.[5] studied the concept of edge domination in various contexts.

S R Jayaram [11] obtained a characterization of minimal edge dominating sets.

Theorem 1.1. [11] *An edge dominating set S is minimal if and only if for each $e \in S$, one of the following two conditions holds:*

(1) $N(e) \cap S = \phi$
(2) *There exists an edge $f \in E - S$ such that $N(f) \cap S = \{e\}$.*

The following result was by Araya Chaemchan

Theorem 1.2. [1] *Let G be a connected graph of order n.*
Then $\Gamma'(G)$ =min$\{|X| : X$ is a maximal independent edge set of $G\}$.
In particular $\Gamma'(G) \leq \lfloor n/2 \rfloor$.

A characterization of graphs reaching upper bound was obtained by S. Arumugam and S. Velammal

Theorem 1.3. [3] *For any connected graph G of even order n, $\Gamma'(G) = \lfloor n/2 \rfloor$ if and only if G is isomorphic to K_n or $K_{n/2,n/2}$(complete bipartite graph on n vertices).*

Senthil Kumar et. al. characterised trees with domination number equal to twice edge domination number[14].

Monnot J. et. al. [13] introduced the concept of upper edge domination in graphs and established an upper bound for the same in terms of order of the graph and used it for results on approximability of upper edge dominating sets.

Definition 1.4. [13] *The maximum size of a minimal edge dominating set of G is called the upper edge domination number of G and is denoted by $\Gamma'(G)$. Any minimal edge dominating set S of G with $|S| = \Gamma'(G)$ is called a Γ'-set of G.*

Theorem 1.5. [13] *For any connected graph $G = (V, E)$ with $n \geq 3$ vertices, $\Gamma'(G) \leq n - 2$. Further the bound is tight if and only if*

(1) $n = 3$, *or*
(2) $n = 4$ *and G contains $2K_2$ (the complement of the cycle on four vertices) as a subgraph, or*
(3) $n \geq 5$ *and G contains $K_{2,n-2}$ (the complete bipartite graph on n vertices) as a subgraph.*

The authors [13] further considered the below given central optimization problem called **Upper EDS** on upper edge dominating set and showed that **Upper EDS** is APX$-$complete in bipartite graphs of maximum degree 4, and NP$-$hard in planar bipartite graphs of maximum degree 4.

Question: Upper EDS
Input: A graph $G = (V, E)$.
Solution: A minimal edge dominating set $S \subseteq E$ of maximum size.

Apart from the algorithmic aspects of upper edge dominating set, not much work is available in literature on upper edge domination number $\Gamma'(G)$. In this paper we find exact value of $\Gamma'(G)$ for some special classes of graphs and present several basic results. Investigation of upper edge domination number in various graph products is a promising direction for further research.

We need the following definition.

Definition 1.6. *The corona of two graphs G_1 and G_2, denoted by $G_1 \circ G_2$, is the graph obtained by taking $|V(G_1)|$ copies of G_2 and joining the vertex i of G_1 to every vertex in the i^{th} copy of G_2.*

In particular the graph $G_1 \circ K_1$ is the graph obtained from G_1 by adding one pendant vertex adjacent to each vertex of G_1.

2. Main Results

It follows immediately from the definition that for a graph G with at least one edge, $1 \leq \gamma'(G) \leq \Gamma'(G) \leq E(G)$. For the star $K_{1,n-1}$, we have $\Gamma'(K_{1,n-1}) = 1$. For the complete graph $G = K_2$, we have $\Gamma'(G) = 1 = |E(G)|$. Also for the disconnected graph G which is two copies of K_2, we have $\Gamma'(G) = 2 = |E(G)|$. Thus the above bounds are sharp. Further $G = K_2$ is the only connected graph with $\Gamma'(G) = |E(G)|$. In this paper our study confine to only connected graphs. The following theorem gives a characterization of all graphs G for which $\Gamma'(G) = 1$.

Theorem 2.1. *For a connected graph G of order n, we have $\Gamma'(G) = 1$ if and only if $G = K_{1,n-1}$ or K_3.*

Proof. Suppose $\Gamma'(G) = 1$. Let X be a Γ'-set of G. Let $X = \{e\}$ where $e = uv$. It follows from Theorem 1.1 that for each $e \in S$ one of the conditions (1) or (2) holds.Thus each edge in $E \setminus X$ must be adjacent to e.Then we have the following two cases: (i)Each edge in $G - e$ must be incident on the same end vertex of e, say u and no two edges in $G - e$ have a common end vertex $w(\neq u)$. For otherwise we can easily obtain a minimal edge dominating set of cardinality two by choosing one edge incident on only u and not v, and other edge incident on only v and not u, which is a contradiction. Thus $G = K_{1,n-1}$.(ii)All edges in $G - e$ must have a common end vertex, say $w, w \notin \{u, v\}$. Then $G = K_3$. Conversely, suppose $G = K_{1,n-1}$ or K_3. Then any edge e of G forms a minimal edge dominating set and hence $\Gamma'(G) = 1$. □

Bistar $B_{r,s}$ is the graph obtained by joining the centre (apex) vertices of two star graphs $K_{1,r}$ and $K_{1,s}$, $r, s \geq 1$ by an edge .

Remark 2.2. *For a bistar $B_{r,s}$, $\Gamma'(B_{r,s}) = 2$. Any two non-adjacent edges form a $\Gamma'-$ set for the graph.*

Theorem 2.3. *Let G_1 and G_2 be two connected graphs with $|V(G_1)| \geq 2, |V(G_2)| \geq 2$. Then*

$$\Gamma'(G_1 \circ G_2) = |V(G_1)|\Gamma'(G_2) + \Gamma'(G_1).$$

Proof. Let $G = G_1 \circ G_2$. Fix a Γ'-set S of G_2. Let S_i be the corresponding Γ'-set in the i^{th} copy of G_2 in $G_1 \circ G_2$. Now no edge of G_1 is adjacent to any edge in S_i. Consider a Γ'-set X of G_1. Then $S = \left(\bigcup_{i=1}^{|V(G_1)|} S_i \right) \cup X$ is a minimal edge dominating set of G. Hence

$$\Gamma'(G_1 \circ G_2) \geq |V(G_1)|\Gamma'(G_2) + \Gamma'(G_1).$$

Now let S be any Γ'-set of G. We claim that S can have no edge with one end vertex in G_1 and the other end vertex in any copy of G_2. For if e is any such edge in S then we can find a minimal edge dominating set S' of G with $|S'| \geq |S|$, which is a contradiction to our choice of S. Thus for any copy of G_2 in G, $S \cap E(G_2)$ is a minimal edge dominating set of G_2. Hence $|S \cap E(G_2)| \leq \Gamma'(G_2)$. Also $S \cap E(G_1)$ is a minimal edge dominating set of G_1. Hence $|S \cap E(G_1)| \leq \Gamma'(G_1)$. Furthermore no edge in G_1 is adjacent to any edge in any copy of G_2. Adding $|V(G_1)|$ inequalities for the copies of G_2 in G and inequality for G_1 in G, we get

$$\Gamma'(G_1 \circ G_2) = |S| \leq |V(G_1)|\Gamma'(G_2) + \Gamma'(G_1).$$

This completes the proof. □

Remark 2.4. *If G is a connected graph of order $n \geq 2$, then*

$$\Gamma'(G_1 \circ K_1) = |V(G_1)| = n.$$

Theorem 2.5. *For the complete bipartite graph $G = K_{m,n}$, with $m, n \geq 2$, we have $\Gamma'(G) = \max\{m, n\}$.*

Proof. Let X, Y be the bipartition of $G = K_{m,n}$ with $|X| = m, |Y| = n$ and let $m \leq n$. The set of all edges incident on a vertex in X is a minimal edge dominating set of G. Therefore $\Gamma'(K_{m,n}) \geq n$. Now let S be any Γ'- set of $K_{m,n}$. Then we have the following three cases.
Case(i): If S contains independent edges, then $|S| = m$.
Case(ii): If S is the set of all edges incident on any one vertex in X, then $|S| = n$.
Case(iii): If S is the set of all edges incident on any one vertex in Y, then $|S| = m$.
Thus in all cases we have $\Gamma'(K_{m,n}) = |S| \leq n$. Hence $\Gamma'(G) = \max\{m, n\}$. □

Corollary 2.6. *For the complete t-partite graph $G = K_{p_1,p_2,...,p_t}$ with $2 \leq p_1 \leq p_2 \leq \cdots \leq p_t, t \geq 3$, we have $\Gamma'(G) = p_2 + ... + p_t$.*

Proof. Since the set of all edges incident on a vertex of partite set X_1 with $|X_1| = p_1$ is a minimal edge dominating set of G, it follows that $\Gamma_s(G) \geq p_2 + ... + p_t$. By using an argument similar to that of Theorem 2.5 we can prove that $\Gamma_s(G) \leq p_2 + ... + p_t$. □

For the complete graph $K_n, n \geq 3$, let X be the set of all independent edges of K_n. Then $|X| = \lfloor \frac{n}{2} \rfloor$. The edges in X cover n vertices if n is even and $n - 1$ vertices if n is odd. Clearly X is a minimal edge dominating set of K_n. But we can always find another minimal edge dominating set S with $|S| \geq |X|$. In particular, for $n \geq 5$, we can find S with $|S| > |X|$. Hence X is not a Γ'- set of K_n for $n \geq 5$.

Theorem 2.7. $\Gamma'(K_3) = 1, \Gamma'(K_4) = 2$ *and for* $n \geq 5, \Gamma'(K_n) = n - 2$.

Proof. Any one edge of K_3 forms a minimal edge dominating set of K_3. Hence $\Gamma'(K_3) = 1$. Any one edge of K_4 is not an edge dominating set of K_4. Hence $\Gamma'(K_4) > 1$. Now set of any two edges form a minimal edge dominating set of K_4, hence $\Gamma'(K_4) = 2$. When $n = 5$, any three edges incident on a vertex form a minimal edge dominating set of K_5. Hence $\Gamma'(K_5) \geq 3$. Also any set of 4 edges is an edge dominating set, but is not minimal. Thus $\Gamma'(K_5) = 3$. For $n > 5$, let X be the set of all $n - 2$ edges incident on a vertex of K_n. Then X is a minimal edge dominating set of K_n. Hence $\Gamma'(K_n) \geq |X| = n-2$. We prove the reverse inequality by induction on n. The result is true for $n = 5$. Now let S be any Γ'- set of K_n. Then by induction hypothesis, $|S| \leq n-2$. Further if S_1 is any Γ'- set of K_{n+1}, then $S_1 = S \cup \{e\}$ where e is any edge in K_{n+1} but not in K_n. Hence $|S_1| \leq |S| + 1 \leq n-1$. This completes the proof. □

Theorem 2.8. *For the path P_n on n vertices we have* $\Gamma'(P_n) = \lfloor \frac{n}{2} \rfloor$.

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$ with edges $e_i = \{v_i, v_{i+1}\}, 1 \leq i \leq n-1$. Let $S = \{e_1, e_3, ..., e_k\}$ where $k = n-1$ if n is even, and $k = n-2$ if n is odd. Then S is an edge dominating set of P_n and $S - \{e\}$ is not an edge dominating set of P_n for all $e \in S$. Hence $\Gamma'(P_n) \geq |S| = \lfloor \frac{n}{2} \rfloor$. We now proceed to prove the reverse inequality by induction on n. The result is true if $n = 2, 3, 4$ or 5. Now let S be any Γ'- set of P_n. If $n = 2k-1$, by induction hypothesis $|S| \leq k-1$. Further if S_1 is any Γ'- set of P_{n+2}, then $S_1 = S \cup \{e_{n+1}\}$, where $e_{n+1} = \{v_{n+1}, v_{n+2}\}$. Hence $|S_1| \leq |S| + 1 \leq k$. If $n = 2k$, by induction hypothesis $|S| \leq k$. Further if S_1 is any Γ'- set of P_{n+2}, then $S_1 = S \cup \{e_{n+1}\}$. Hence $|S_1| \leq |S| + 1 \leq k+1$. This completes the proof. □

Corollary 2.9.

(1) *If n is even, then* $\Gamma'(P_{n+1}) = \Gamma'(P_n)$.
(2) *If n is odd, then* $\Gamma'(P_{n+1}) = \Gamma'(P_n) + 1$.

We now proceed to determine Γ' for cycle C_n.

Theorem 2.10. *For a cycle C_n on n vertices, $n \geq 4$, we have* $\Gamma'(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$. Let the edges be $e_1, e_2, ..., e_n$, where $e_i = \{v_i, v_{i+1}\}$ and $e_n = \{v_n, v_1\}$. Let $S = \{e_1, e_3, e_5, ..., e_k\}$ where

$$k = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n-2 & \text{if } n \text{ is odd.} \end{cases}$$

Then S is an edge dominating set of C_n and $S - \{e\}$ is not an edge dominating set of C_n for all $e \in S$. Hence $\Gamma'(C_n) \geq |S| = \lfloor \frac{n}{2} \rfloor$. We now prove the reverse inequality by induction on n. The result is obvious if $n = 3$ or 4. Suppose the result is true for all cycles of length less than n. Now, let S be any Γ'-set of C_n. Then $S = S_1 \cup \{e_n\}$ or $S_1 \cup \{e_{n-1}\}$ or $S_1 \cup \{e_{n-2}\}$ where S_1 is a Γ'-set of the cycle $C_{n-2} = (v_1, v_2, \ldots, v_{n-2}, v_1)$. By induction $|S_1| \leq \left\lfloor \frac{n-2}{2} \right\rfloor$ and hence $|S| \leq \left\lfloor \frac{n-2}{2} \right\rfloor + 1 = \left\lfloor \frac{n}{2} \right\rfloor$. Thus $\Gamma'(C_n) \leq \left\lfloor \frac{n}{2} \right\rfloor$. This completes the proof. □

Corollary 2.11.

(1) *If n is even, then* $\Gamma'(C_{n+1}) = \Gamma'(C_n)$.
(2) *If n is odd, then* $\Gamma'(C_{n+1}) = \Gamma'(C_n) + 1$.

Wheel graph W_n of order n ia an $(n-1)-$cycle with one additional vertex that is adjacent to each of the vertices in the cycle.

Theorem 2.12. *For the wheel $W_n, n \geq 4$ we have* $\Gamma'(W_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. Let $W_n = C_{n-1} + K_1$. Let $C_{n-1} = (v_1, v_2, ..., v_{n-1}, v_1)$ and $K_1 = \{v\}$. Let $E(G) = E(C_n) \cup \{e_1, e_2, ..., e_{n-1}\}$, where $e_i = \{v, v_i\}$. Let $S =$

$\{e_1, e_3, ..., e_{n-1}\}$ if n is even and $S = \{e_1, e_3, ..., e_{n-2}\}$ if n is odd. Then S is an edge dominating set of W_n. Also $S - \{e\}$ is not an edge dominating set of W_n for each $e \in S$. Hence $\Gamma'(W_n) \geq |S| = \left\lfloor \frac{n}{2} \right\rfloor$. We now prove the reverse inequality by induction on n. The result is obvious for $n = 4$ or 5. Suppose the result is true for all wheels of order less than n. Now let S be any Γ'- set of W_n. Then $S = S_1 \cup \{e_{n-2}\}$ or $S = S_1 \cup \{e_{n-1}\}$ or $S = S_1 \cup \{v_{n-2}, v_{n-1}\}$ is a Γ'- set of W_n, where S_1 is a Γ'- set of W_{n-2}. By induction, $|S_1| \leq \lfloor \frac{n-2}{2} \rfloor$. Hence $|S| = |S_1| + 1 \leq \lfloor \frac{n-2}{2} \rfloor + 1 = \lfloor \frac{n}{2} \rfloor$. This completes the proof. □

For any graph G, we have $\gamma'(G) \leq \Gamma'(G)$. Hence the following problems arise naturally.

Problem 2.13. *Characterize graphs G for which $\gamma'(G) = \Gamma'(G)$.*

Problem 2.14. *Given two positive integers a and b with $a \leq b$ does there exist a graph G with $\gamma'(G) = a$ and $\Gamma'(G) = b$?*

Since $\Gamma'(G) = 1$ if and only if $G = K_1, n-1$ and in this case $\gamma'(G) = 1$ we see that $G = K_1, n-1$ is the only graph G with $\gamma'(G) = 1$ and $\Gamma'(G) = 1$. Any bistar G has $\gamma'(G) = 1$ and $\Gamma'(G) = 2$. In all other cases we shall prove the existence of G in the following two theorems.

Theorem 2.15. *Let G be a graph with $\gamma'(G) = 1$. Then $\Gamma'(G) \leq 2$. Further $\Gamma'(G) = 1$ only when $G = K_{1,n-1}$.*

Proof. Let G be any graph with $\gamma'(G) = 1$. Let S be any γ'- set of G. Then $|S| = 1$. Let $S = \{e\}$. Then $e = uv$ is adjacent to every edge in G and any edge in G is incident on at least one of u or v. Hence any minimal edge dominating set of G can have at most two edges of G, thereby proving $\Gamma'(G) \leq 2$. Later part of the theorem follows from Theorem 2.1. □

Theorem 2.16. *Given two positive integers a and b with $1 < a \leq b$, there exists a graph G with $\gamma'(G) = a$ and $\Gamma'(G) = b$.*

Proof. For $m, n \geq 2$, we have $\gamma'(K_{m,n}) = min\{m, n\}$ and $\Gamma'(K_{m,n}) = max\{m, n\}$. Hence $G = K_{a,b}$ has $\gamma'(G) = a$ and $\Gamma'(G) = b$. □

Let $S(p, q)$ denote the spider consisting of p paths each of length q. Since $S(2, q)$ is the path P_{2q+1}, we have $\Gamma'(S(2, q)) = \left\lfloor \frac{2q+1}{2} \right\rfloor$. In the following theorem we determine $\Gamma'(S(p, q))$ when $p \geq 3$.

Theorem 2.17. *For the spider $G = S(p, q)$ with $p \geq 3$, we have*

$$\Gamma'(G) = \begin{cases} p\Gamma'(P_q) & \text{if } q \text{ is even} \\ p\Gamma'(P_q) + 1 & \text{if } q \text{ is odd.} \end{cases}$$

Proof. Let v_0 be the central vertex of G and let the p paths of G be given by $P_q^{(i)} = (v_{i1}, v_{i2}, \ldots, v_{iq}), 1 \leq i \leq p$. We assume that v_0 is adjacent to v_{iq}. Let $\{e_{i1}, e_{i2}, ..., e_{i(q-1)}\}, 1 \leq i \leq p$ be the edges in $P_q^{(i)}$. Let $e_1, e_2, ..., e_p$ be the edges joining v_{iq} to $v_0, 1 \leq i \leq p$ respectively. Let $X_i = \{e_{i1}, e_{i3}, ..., e_{ik}\}$, where $k = \begin{cases} q-1 & \text{if } q \text{ is even} \\ q-2 & \text{if } q \text{ is odd.} \end{cases}$

Let

$$X = \begin{cases} \bigcup_{i=1}^{p} X_i & \text{if } q \text{ is even} \\ \left(\bigcup_{i=1}^{p} X_i\right) \cup \{e_1\} & \text{if } q \text{ is odd} \end{cases}$$

Clearly X is an edge dominating set of G. Also $X - \{e\}$ is not an edge dominating set of G for any edge $e \in X$. Hence $\Gamma'(G) \geq |X|$. Therefore

$$\Gamma'(G) \geq \begin{cases} p\Gamma'(P_q) & \text{if } q \text{ is even} \\ p\Gamma'(P_q) + 1 & \text{if } q \text{ is odd.} \end{cases}$$

Now it follows from Corollary 2.9 that every Γ'-set X of G contains at least one of the edges incident on v_0 if and only if q is odd. Hence

$$\Gamma'(G) \leq \begin{cases} p\Gamma'(P_q) & \text{if } q \text{ is even} \\ p\Gamma'(P_q) + 1 & \text{if } q \text{ is odd.} \end{cases}$$

and the proof is complete. □

We now proceed to determine the value of Γ' for caterpillars. Two support vertices s_1, s_2 of a caterpillar T are said to be consecutive if all the internal vertices of the unique s_1-s_2 path are of degree 2.

Theorem 2.18. *Let T be a caterpillar with k support vertices $s_1, s_2, \ldots, s_k$ such that s_i and s_{i+1} are consecutive and $d(s_i, s_{i+1}) = a_i$. Then $\Gamma'(T) = k + \sum_i \Gamma'(P_{a_i-1})$ where the summation is taken over all i with $a_i \geq 3$.*

Proof. Let P_i be the s_i-s_{i+1} path in T where $a_i \geq 3$. Then $P_i' = P_i - \{s_i, s_{i+1}\}$ is a subpath of P_i with $a_i - 1$ vertices. Let X_i be a Γ'-set of P_i'. Let L denote the set of all edges consisting of exactly one pendant edge each incident on the k support vertices. Then $X = L \cup \left(\bigcup_i X_i\right)$ where the union is taken over all i with $a_i \geq 3$ is a minimal edge dominating set of T. Hence $\Gamma'(T) \geq |X| = k + \sum_i \Gamma'(P_{a_i-1})$.

Now, let X be any Γ'-set of T. Then for each edge incident on s_i, X contains k pendant edges, one pendant edge each incident on each s_i.Hence no other edges incident on any s_i is in X. Now $|X \cap E(P_{a_i-1})| \leq \Gamma'(P_{a_i-1})$ for all i with $a_i \geq 3$. Thus $|X| \leq k + \sum_i \Gamma'(P_{a_i-1})$ and hence the result follows. □

3. Conclusion and Scope

In this paper we have presented several results on upper edge domination number of a graph. The following are interesting problems for further investigation.

(1) Characterize the class of graphs G of size m for which $\Gamma' = m$.
(2) Characterize the class of graphs G of size m for which $\Gamma' = m - 1$.
(3) Obtain lower and upper bounds for $\Gamma'(G)$ in terms of other graph theoretic parameters.
(4) For any tree T determine $\Gamma'(T)$.

Acknowledgement. The authors are grateful to the referees of this article for their valuable comments and advice.

References

[1] Araya Chaemchan, *The edge domination number of connected graphs*, Australasian Journal of Combinatorics, 48(2010), 185-189.
[2] S. Arumugam and S. Jerry, *Fractional edge domination in graphs*, Appl. Anal. Discrete Math. 3 (2009), 359-370.
[3] S. Arumugam and S. Velammal, *Edge domination in graphs*, Taiwanese Journal of Mathematics, 2(2)(1998), 173-179.
[4] G. Chartrand and L. Lesniak, *Graphs* & *Digraphs* (Fourth Edition, Chapman & Hall/CRC, 2005).
[5] Cristina Bazgan, Ljiljana Brankovic, Katrin Casel, Henning Fernau, Klaus Jansen, Kim-Manuel Klein, Michael Lampis, Mathie Liedloff, Jerome Monnot, and Vangelis Th. Paschos, *Many facets of upper domination*, Theor. Comput. Sci. 717 (2018), 2-25.
[6] S.V. Divya Rashmi, A. Somasundaram and S. Arumugam, *Upper secure domination number of a graph*, Electronic Notes in Discrete Mathematics, 63 (2016), 297-306.
[7] R. Dutton and W. Clostermeyer, *Edge dominating sets and vertex covers*, Discussions Mathematicae 33 (2013), 437-456.
[8] P. Dorbec and M.A. Henning, *Upper paired domination in claw-free graphs*, J. Comb. Optim. 22 (2011), 235-251.
[9] P. Dorbec, M. A. Henning, and J. McCoy. *Upper total domination versus upper paired-domination*, Quaestiones Mathematicae, 30(1) (2007b), 1-12.
[10] O. Favaron , Henning MA , *Upper total domination in claw-free graphs*, J. Graph Theory, 44 (2003), 148-158.
[11] S. R. Jayaram , *Line domination in graphs*, Graphs and Combinatorics, 3 (1987), 357–363.
[12] S. Mitchell and S. T. Hedetniemi, *Edge domination in trees*, Graph Theory and Combinatorics-Puting, Congr. Numer.19 (1977), 489-509.
[13] J. Monnot, H. Fernau and D. Manlove, *Algorithmic aspects of upper edge domination*, Theoretical Computer Science, 877 (2021), 46-57.
[14] Senthil Kumar, Y. B. Venkatakrishnan, H. Naresh Kumar, *Domination and edge domination in trees*, Ural Mathematical Journal, 6(1) (2020), 147-152.
[15] J. Sreedevi and B. Maheswari, *Edge domination number of corona product graph of a cycle with a star*, International Journal of Computer Applications, 157 (8) (2017), 34-36.
[16] D. K. Thakkar and B. M. Kakrecha, *About the edge domination number of the graphs*, Advances in Theoretical and Applied Mathematics, 11(2)(2016), 93-98.
[17] S. K. Vaidya and R. M.Pandit, *Edge domination in various snake graphs*,International Journal of Mathematics and Soft Computing 7 (1) (2017), 43-50.
[18] B. Xu, *Note on edge domination numbers of graphs*, Discrete Mathematics, 294 (2005), 311-316.
[19] B. Zelinka, *Edge domination in graphs of cubes*, Czechoslovac Mathematical Journal, 52(2002), 875-879.

Affiliation1: Department of Mathematics, Vidyavardhaka College of Engineering, Mysuru 570002, Karnataka, India
Email address: rashmi.divya@gmail.com

Affiliation2: Department of Mathematics, Vidyavardhaka College of Engineering, Mysuru 570002, Karnataka, India
Email address: dileepl84@vvce.ac.in