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SOME RESULTS ON VALUE DISTRIBUTION OF MEROMORPHIC FUNCTIONS CONCERNING DIFFERENCE-DIFFERENTIAL POLYNOMIALS

MEGHA M. MANAKAME AND HARINA P. WAGHAMORE

ABSTRACT. The purpose of the paper is to study with the notion of weighted sharing, the uniqueness problems of difference-differential polynomials of meromorphic functions sharing 1-points with weight l. The results in this paper extends some earlier existing results.

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1. INTRODUCTION, DEFINITIONS AND MAIN RESULTS

One of the numerous disciplines of complex analysis that has produced many excellent works is the Nevanlinna theory. It primarily focuses on analyzing how the zeros of the equation $f(z) = a$ in a disc $|z| \leq r$, where f is an entire or meromorphic function in the complex plane $\mathbb{C}$, $z \in \mathbb{C}$ and $a \in \mathbb{C} \cup \{\infty\}$. We employ basic findings and standard notation of Nevanlinna theory in our work(see [9, 15, 22]. For a meromorphic function $f(z)$, the quantity $m(r, f)$ typically denotes the proximity function of $f(z)$, whereas $N(r, f)$ generally denotes the counting function of poles $f(z)$ whose multiplicities are taken into consideration (respectively $\overline{N}(r, f)$ denotes the reduced counting function when multiplicities are ignored).

Recently, since Halburd and Korhonen [8] developed the difference analogous lemma of the logarithmic derivative, the study of shifts and differences of meromorphic functions obtained considerable space in the literature. There has been a new tendency to look into uniqueness results for derivative and difference containing difference-differential polynomials of meromorphic functions. The Nevanlinna characteristic function of a meromorphic function f plays a very important role in the value distribution theory and it is denoted by $T(r, f)$. We have $T(r, f) = m(r, f) + N(r, f)$, which clearly shows that $T(r, f)$ is non-negative. If $f(z) - a$ and $g(z) - a$ assumes the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share the value a CM (counting multiplicity) and we have $E(a, f) = E(a, g)$. Suppose, if $f(z) - a$ and $g(z) - a$ assumes the same zeros ignoring the multiplicities, then we say that $f(z)$ and $g(z)$ share the value a IM (ignoring multiplicity) and we will have $\overline{E}(a, f) = \overline{E}(a, g)$.

Definition 1.1. [13] *Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$ and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_k(a; f)$ the set of all $a-$ points of f where an $a-$ point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k. Clearly f, g share (a, k) then f, g share (a, p) for* *any integer* p, $0 \leq p \leq k$. *Also we note that* f, g *share a value* a *IM or CM if and only if* f, g *share* $(a, 0)$ *or* (a, ∞) *respectively.*

Definition 1.2. [13] *Let* $p \in \mathbb{N} \cup \{\infty\}$. *We denote by* $N_p(r, a; f)$ *the counting function of* $a-$ *points of* f, *where an* $a-$ *point of multiplicity* m *is counted* m *times if* $m \leq p$ *and* p *times if* $m > p$. *Then* $N_p(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f| \geq 2) + ... + \overline{N}(r, a; f| \geq p)$.

Definition 1.3. [13] *Let* $f(z)$ *and* $g(z)$ *be two meromorphic functions in the complex plane* $\mathbb{C}$. *If* $f(z)-a$ *and* $g(z)-a$ *assume the same zeros with the same multiplicities, then we say that* $f(z)$ *and* $g(z)$ *share the value* a *CM, and if we do not consider the multiplicity, then we say that* $f(z)$ *and* $g(z)$ *share the value* a *IM, where* a *is a complex number.*

Definition 1.4. [12] *Let* $a \in \mathbb{C} \cup \{\infty\}$. *We denote by* $N(r, a; f| = 1)$ *the counting function of simple* a *points of* f. *For* $p \in \mathbb{N}$ *we denote by* $N(r, a; f| \leq p)$ *the counting function of those a-points of* f *(counted with multiplicities) whose multiplicities are not greater than* p. *By* $\overline{N}(r, a; f| \leq p)$ *we denote the corresponding reduced counting function. In an analogous manner we can define* $N(r, a; f| \geq p)$ *and* $\overline{N}(r, a; f| \geq p)$.

Definition 1.5. [23] *Let* f *and* g *be two non-constant meromorphic functions such that* f *and* g *share* $(a, 0)$. *Let* z_0 *be an* $a-$ *point of* f *with multiplicity* p, *an* $a-$ *point of* g *with multiplicity* q. *We denote by* $\overline{N}_L(r, a; f)$ *the reduced counting function of those* $a-$ *points of* f *and* g *where* $p > q$, *by* $N_E^{1)}(r, a; f)$ *the counting function of those* $a-$ *points of* f *and* g *where* $p = q = 1$, *by* $\overline{N}_E^{(2}(r, a; f)$ *the reduced counting function of those* $a-$ *points of* f *and* g *where* $p = q \geq 2$. *In the same way we can define* $\overline{N}_L(r, a; g), N_E^{1)}(r, a; g), \overline{N}_E^{(2}(r, a; g)$. *In a similar manner we can define* $\overline{N}_L(r, a; f)$ *and* $\overline{N}_L(r, a; g)$ *for* $a \in \mathbb{C} \cup \{\infty\}$. *When* f *and* g *share* (a, m), $m \geq 1$, *then* $N_E^{1)}(r, a; f) = N(r, a; f| = 1)$.

Definition 1.6. [13, 14] *Let* f *and* g *share a value* $(a, 0)$. *We denote by* $\overline{N}_*(r, a; f, g)$ *the reduced counting function of those* $a-$ *points of* f *whose multiplicities differ from the multiplicities of the corresponding* $a-$ *points of* g. *Clearly* $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ *and* $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

In 2013, Qi-Yang [19] studied the analogous result considering $q-$ shift monomial as follows:

Theorem 1. [19] *Let* $f(z)$ *and* $g(z)$ *be two transcendental meromorphic functions of zero order. Suppose that* q *is a non-zero complex constant such that* $|q| \neq 1$ *and* r *is a positive integer satisfying* $r \geq 30$ *such that* $f(z)^r(f(z) - 1)f(qz)$ *and* $g(z)^r(g(z) - 1)g(qz)$ *share* $(1, 0)$, $f(z)$ *and* $g(z)$ *share* $(\infty, 0)$ *then* $f(z)^r(f(z) - 1)f(qz) = g(z)^r(g(z) - 1)g(qz)$.

In 2015, Zhao-Zhang [25] considered the derivative in the following manner.

Theorem 2. [25] *Let* $f(z)$ *and* $g(z)$ *be two transcendental entire functions of zero order. Suppose that* q, c *are two non-zero complex constants and* $n \in \mathbb{N}$ *is such that* $(f^n(z)f(qz + c))^{(k)}$ *and* $(g^n(z)g(qz + c))^{(k)}$ *share* $(1, l)$.

(1) *If* $l = \infty$ *and* $n > 2k + 5$;
(2) *If* $l = 0$ *and* $n > 5k + 11$.

then $f(z) \equiv tg(z)$ *or* $f(z)g(z) \equiv t$ *for some constants* t *that satisfy* $t^{n+1} = 1$.

Let $P(z) = a_m z^m + a_{m-1}z^{m-1} + ... + a_0$ is non-zero polynomial of degree m and $\Gamma_0 = m_1+m_2$ and $\Gamma_1 = m_1+2m_2$, where m_1, m_2 respectively be the number of simple and multiple zeros of $P(z)$ and $\lambda = \sum_{j=1}^{d} v_j$, where λ and v_j where $\{j = 1, 2, 3...d\}$ be positive integers.

Now it is natural to ask the following questions which are the motivation of the paper.

Question 1. What happen if we consider $f(z)^r(f(z)-1)f(qz)$ by $f^r(f^m(z)-1)^p \prod_{j=1}^{d} f(z+c_j)^{v_j}$ in Theorem 1.

Question 2. What happen if one consider the difference-differential polynomial of the form $(f^n(z)f(qz+c))^{(k)}$ by more generalized form $(P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j})^{(k)}$ in Theorem 2.

The major reason for writing the paper is to provide an affirmative response to the above questions and improve the Theorems 1 and 2 for the most generalised shift polynomial under the weighted sharing environment. We now present the following three theorems which are the main results of this paper.

Theorem 1.1. *Let* $f(z)$, $g(z)$ *be two transcendental meromorphic functions of zero order,* $c \in \mathbb{C}$. *Suppose that* $F = P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}$ *and* $G = P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j}$ *share* $(1, l)$. *Now*

(1) *if* $l \geq 2$ *and* $n > 2\Gamma_1 + 4 + 5\lambda$;
(2) *if* $l = 1$ *and* $n > 2\Gamma_1 + \frac{\Gamma_0}{2} + 5\lambda + d + \frac{9}{2}$;
(3) *if* $l = 0$ *and* $n > 2\Gamma_1 + 3\Gamma_0 + 5\lambda + 6d + 7$.

Then one of the following results holds:

(i) $P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}.P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j} \equiv 1$;
(ii) $P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j} \equiv P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j}$;
(iii) $f \equiv tg$, *for some constant* t *such that* $t^{n+\lambda} = 1$.

Theorem 1.2. *Let* $f(z)$, $g(z)$ *be transcendental meromorphic functions of zero order* $c \in \mathbb{C}$, *and* n *is an integer such that* $(P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j})^{(k)}$ *and* $(P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j})^{(k)}$ *share* $(1, l)$, *Now*

(1) *if* $l \geq 2$ *and* $n > (2m_2 + d + 1)k + 2\Gamma_1 + 4 + 5\lambda$;
(2) *if* $l = 1$ *and* $n > \left(\frac{5m_2}{2} + \frac{3}{2} + \frac{3d}{2}\right)k + 2\Gamma_1 + \frac{\Gamma_0}{2} + \frac{9}{2} + \frac{d}{2} + \frac{11\lambda}{2}$;
(3) *if* $l = 0$ *and* $n > (5m_2 + 4d + 4)k + 2\Gamma_1 + 3\Gamma_0 + 7 + 3d + 8\lambda$.

Then one of the following results holds:

(i) $(P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j})^{(k)}.(P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j})^{(k)} \equiv 1$;
(ii) $f \equiv tg$, *for some constant* t *such that* $t^l = 1$ *where* $l = GCD(\lambda+\gamma_0, \lambda+\gamma_1, .., \lambda+\gamma_n)$

$$\gamma_j = \begin{cases} j+1 & if \quad a_j \neq 0, \\ n+1 & if \quad a_j = 0, \end{cases}$$

(iii) f *and* g *satisfy algebraic equation* $R(f, g) = 0$, *where* $R(w_1, w_2) = P(w_1)\prod_{j=1}^{d} f(z+c_j)^{v_j} - P(w_2)\prod_{j=1}^{d} g(z+c_j)^{v_j}$.

Example 1. Let $P(z) = (z-1)^6(z+1)^6z^{11}$, $f(z) = sinz$, $g(z) = cosz$. Choose $d = 1, c = 2\pi, k = 0$ then $(P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j})^{(k)}$ and $(P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j})^{(k)}$

share $(1,l)$, Here f and g satisify the algebraic equation $R(f,g)=0$.
i.e

$$(P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j})(P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j}) \equiv 0$$

Theorem 1.3. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of zero order and r is an integer $f^r(f^m(z)-1)^p\prod_{j=1}^{d} f(z+c_j)^{v_j}$ and $g^r(g^m(z)-1)^p\prod_{j=1}^{d} g(z+c_j)^{v_j}$ share $(1,l)$, $f(z)$ and $g(z)$ share $(\infty,0)$. Now*

(1) *if $l\geq 2$ and $r>4m-mp+3d+11+\lambda$;*
(2) *if $l=1$ and $r>\frac{9m}{2}-mp+\lambda+4d+\frac{25}{2}$;*
(3) *if $l=0$ and $r>7m-mp+9d+20+\lambda$.*

Conclusion of Theorem 1.1 holds where $P(f)=f^r(f^m(z)-1)^p$ and r,m,p are positive integers.

2. Preliminary Lemmas

In this section, we state some Lemmas which will play key roles in proving the main results of the paper. For two non-constant meromorphic functions F and G, H represents the following function.

$$H=\Big(\frac{F''}{F'}-\frac{2F'}{F-1}\Big)-\Big(\frac{G''}{G'}-\frac{2G'}{G-1}\Big).$$

Lemma 2.1. [22] *Let f be a non-constant meromorphic function and $P(f)=a_nf^n+a_{n-1}f^{n-1}+...+a_0$, where $a_n(\neq 0),a_{n-1},...,a_0$ are complex constants. Then*

$$T(r,P(f))=nT(r,f)+S(r,f).$$

Lemma 2.2. [10] *Let f be a non-constant meromorphic function of finite order and $c\in\mathbb{C}$. Then*

$$N(r,\infty;f(z+c))\leq N(r,\infty;f(z))+S(r,f);\ N(r,0;f(z+c))\leq N(r,0;f)+S(r,f),$$

$$\overline{N}(r,\infty;f(z+c))\leq \overline{N}(r,\infty;f)+S(r,f);\ \overline{N}(r,0;f(z+c))\leq \overline{N}(r,0;f)+S(r,f).$$

Lemma 2.3. [6] *Let f be a meromorphic function finite order ρ and let $c\in\mathbb{C}\backslash\{0\}$. Then for each $\epsilon>0$, we have*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right)=O(r^{\rho-1+\epsilon}).$$

Lemma 2.4. [24] *Let F and G be two non-constant meromorphic functions sharing $(1,0)$ and $H\not\equiv 0$. Then*

$$N_E^{1)}(r,1;F)=N_E^{1)}(r,1;G)\leq N(r,H)+S(r,F)+S(r,G).$$

Lemma 2.5. [13] *If two non-constant meromorphic functions F and G share $(1,0)$ and $H\not\equiv 0$, then*

$$\overline{N}(r,\infty;H)\leq \overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}(r,\infty;F|\geq 2)+$$
$$\overline{N}(r,\infty;G|\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G'),$$

where $\overline{N}_0(r,0;F')$ we mean the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.

Lemma 2.6. [3] *Let f,g be two non-constant meromorphic functions sharing* $(1, l)$ *where* $0 \leq l \leq \infty$. *Then*

$$\overline{N}(r,1;f) + \overline{N}(r,1;g) - N_E^{1)}(r,1;f) + \left(l - \frac{1}{2}\right)\overline{N}_*(r,1;f,g)$$
$$\leq \frac{1}{2}[N(r,1;f) + N(r,1;g)].$$

Lemma 2.7. [2] *Let f and g be any two meromorphic function and suppose they share* $(1,l)$. *Then*
$\overline{N}_*(r,1;f,g) \leq \frac{1}{l+1}[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)]+S(r,f)+S(r,g).$

Lemma 2.8. [13] *Let f,g be two non-constant meromorphic functions sharing* $(1,2)$. *Then one of the following cases holds:*

(i) $T(r,f) \leq N_2(r,0;f)+N_2(r,0;g)+N_2(r,\infty;f)+N_2(r,\infty;g)+S(r,f)+S(r,g)$, *the same inequality holds for* $T(r,g)$;
(ii) $f = g$;
(iii) $f.g = 1$.

Lemma 2.9. [1] *Let f and g be two transcendental meromorphic functions sharing* $(1,1)$ *and* $H \not\equiv 0$, *then*

$$T(r,f) \leq N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;f) + N_2(r,\infty;g)$$
$$+ \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) + S(r,f) + S(r,g).$$

Lemma 2.10. [1] *Let f and g be two transcendental meromorphic functions sharing* $(1,0)$ *and* $H \not\equiv 0$, *then*

$$T(r,f) \leq N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;f) + N_2(r,\infty;g) + 2\overline{N}(r,0;f)$$
$$+ 2\overline{N}(r,\infty;f) + 2\overline{N}(r,0;g) + 2\overline{N}(r,\infty;g) + S(r,f) + S(r,g).$$

Lemma 2.11. [16] *Let f be a non-constant meromorphic function and let p, k be two positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right) \leq T(r,f^{(k)}) - T(r,f) + N_{p+k}(r,0;f) + S(r,f),$$
$$N_p\left(r,0;f^{(k)}\right) \leq k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f).$$

Lemma 2.12. [4] *Let F and G be two non-constant meromorphic functions sharing* $(1,2)$, $(\infty,0)$ *and* $H \not\equiv 0$. *Then*

$$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G)$$
$$+ \overline{N}_*(r,\infty;F,G) - m(r,1;G) - N_E^{(3}(r,1;F) - \overline{N}_L(r,1;G) + S(r,F) + S(r,G)$$

Similarly we can define for $T(r,G)$.

Lemma 2.13. [21] *Let F and G be two non-constant meromorphic functions sharing* $(1,1)$, $(\infty,0)$ *and* $H \not\equiv 0$. *Then*

$$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + \frac{3}{2}\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G)$$
$$+ \frac{1}{2}\overline{N}(r,0;F) + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G)$$

Similarly we can define for $T(r,G)$.

Lemma 2.14. [21] *Let F and G be two non-constant meromorphic functions sharing $(1,0)$, $(\infty,0)$ and $H \not\equiv 0$. Then*

$$T(r,F) \le N_2(r,0;F) + N_2(r,0;G) + 3\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G)$$

Similarly we can define for $T(r,G)$.

Lemma 2.15. *Let $f(z)$ be a transcendental meromorphic function of finite order.Let $F = P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}$. Then for $n > \lambda$ we have*

$$(n-\lambda)T(r,f) + S(r,f) \le T(r,F)$$

Proof. From first fundamental theorem, Lemmas 2.1 and 2.3 we obtain

$$\begin{aligned}(n+1)T(r,f) &= T(r,f(z)P(f)) + S(r,f)\\ &\le T\left(r,\frac{f(z)F}{\prod_{j=1}^{d} f(z+c_j)^{v_j}}\right) + S(r,f),\\ &\le T(r,F) + T\left(r,\frac{\prod_{j=1}^{d} f(z+c_j)^{v_j}}{f(z)}\right) + S(r,f),\\ &\le T(r,F) + m\left(r,\frac{\prod_{j=1}^{d} f(z+c_j)^{v_j}}{f(z)}\right) + N\left(r,\frac{\prod_{j=1}^{d} f(z+c_j)^{v_j}}{f(z)}\right) + S(r,f),\end{aligned}$$

Thus $(n-\lambda)T(r,f) \le T(r,F) + S(r,f)$.

Lemma 2.16. [18] *Let $f(z)$ be non-constant meromorphic function of finite order and $c_j (j = 1, 2..d)$ are integers. Let $F(z) = f^r(f^m(z)-1)^p \prod_{j=1}^{d} f(z+c_j)^{v_j}$, where r, m, p are positive integers. Then we have*

$$(r + mp + \lambda)T(r,f) \le T(r,F) \le (r + mp - \lambda)T(r,f)$$

3. Proof of the Theorems

Proof of Theorem 1.1. Let $F = P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}$ and $G = P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j}$. Then F and G share $(1,l)$.
Case 1. Let $H \not\equiv 0$. Using Lemmas 2.4 and 2.5 we have

$$\begin{aligned}N_E^{1)}(r,1;F) &\le N(r,H) + S(r,F) + S(r,G)\\ &\le \overline{N}(r,0;F|\ge 2) + \overline{N}(r,0;G|\ge 2) + \overline{N}(r,\infty;F|\ge 2) + \overline{N}(r,\infty;G|\ge 2)\\ &\quad + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + \overline{N}_*(r,1;F,G).\end{aligned} \tag{3.1}$$

By Second Fundamental Theorem [9], we get

$$T(r,F) \le \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) - \overline{N}_0(r,0;F') + S(r,f). \tag{3.2}$$

and

$$T(r,G) \le \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}(r,1;G) - \overline{N}_0(r,0;G') + S(r,g). \tag{3.3}$$

Combining (3.1),(3.2) and (3.3) with the help of Lemmas 2.6 and 2.7, we have

$$\begin{aligned}[T(r,F)+T(r,G)] &\le [\overline{N}(r,0;F)+\overline{N}(r,0;G)]+[\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)]+\overline{N}(r,1;F)\\ &\quad+\overline{N}(r,1;G)-[\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})]+S(r,f)+S(r,g)\\ &\le N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\\ &\quad+[\overline{N}(r,1;F)+\overline{N}(r,1;G)-N_E^{1)}(r,1;F)]+\overline{N}_*(r,1;F,G)\\ &\quad+S(r,f)+S(r,g)\\ &\le N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\\ &\quad+\frac{1}{2}[T(r,F)+T(r,G)]-\left(l-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\ &\le \frac{1}{2}[T(r,F)+T(r,G)]+N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)\\ &\quad+N_2(r,\infty;G)+\frac{(3-2l)}{2(l+1)}[\overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\\ &\quad\quad \overline{N}(r,\infty;G)]+S(r,f)+S(r,g). \end{aligned} \tag{3.4}$$

By (3.4) we get

$$\begin{aligned}[T(r,F)+T(r,G)] &\le 2[N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)]\\ &\quad+\frac{(3-2l)}{(l+1)}[\overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)]\\ &\quad +S(r,f)+S(r,g). \end{aligned} \tag{3.5}$$

Subcase 1.1. While $l\ge 2$, in view of Lemmas 2.2 and 2.15, from (3.5) we get

$$\begin{aligned}(n-\lambda)[T(r,f)+T(r,g)] &\le 2[(m_1+2m_2)T(r,f)+N(r,0;\prod_{j=1}^{d}f(z+c_j)^{v_j})+(m_1+2m_2)T(r,g)\\ &\quad+N(r,0;\prod_{j=1}^{d}g(z+c_j)^{v_j})+2\overline{N}(r,\infty;f)+N(r,\infty;\prod_{j=1}^{d}f(z+c_j)^{v_j})]\\ &\quad+2\overline{N}(r,\infty;g)+N(r,\infty;\prod_{j=1}^{d}g(z+c_j)^{v_j})]+S(r,f)+S(r,G)\\ &\le 2[m_1+2m_2+2+2\lambda]\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g). \end{aligned} \tag{3.6}$$

From (3.6) it follows that

$$(n-\lambda)[T(r,f)+T(r,g)]\le[2\Gamma_1+4+4\lambda]\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).$$

which contradicts for $n > 2\Gamma_1 + 4 + 5\lambda$.

Subcase 1.2. While $l = 1$, using Lemmas 2.2 and 2.15, from (3.5) we get

$$\begin{aligned}(n-\lambda)[T(r,f)+T(r,g)] \leq\ & 2[(m_1+2m_2)T(r,f) + N(r,0;\prod_{j=1}^{d} f(z+c_j)^{v_j}) + (m_1+2m_2)T(r,g) \\ & + N(r,0;\prod_{j=1}^{d} g(z+c_j)^{v_j}) + 2\overline{N}(r,\infty;f) + N(r,\infty;\prod_{j=1}^{d} f(z+c_j)^{v_j}) \\ & + 2\overline{N}(r,\infty;g) + N(r,\infty;\prod_{j=1}^{d} g(z+c_j)^{v_j})] + \left(\frac{1}{2}\right)[(m_1+m_2)T(r,f) \\ & + dN(r,0;\prod_{j=1}^{d} f(z+c_j)^{v_j}) + (d+1)\overline{N}(r,\infty;f) + (m_1+m_2)T(r,g) \\ & + dN(r,0;\prod_{j=1}^{d} g(z+c_j)^{v_j}) + (d+1)\overline{N}(r,\infty;g)] + S(r,f) + S(r,g) \\ \leq\ & \left[2(m_1+2m_2+2\lambda+2) + \left(\frac{1}{2}\right)(m_1+m_2+2d+1)\right]\{T(r,f)+T(r,g)\} \\ & +S(r,f)+S(r,g). \qquad (3.7)\end{aligned}$$

From (3.7) it follows that

$$(n-\lambda)[T(r,f)+T(r,g)] \leq [2\Gamma_1+4+4\lambda+\frac{1}{2}(\Gamma_0+2d+1)]\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).$$

which is a contradiction for $n > 2\Gamma_1 + \frac{\Gamma_0}{2} + 5\lambda + d + \frac{9}{2}$.

Subcase 1.3. While $l = 0$, using Lemmas 2.2 and 2.15, from (3.5) we get

$$\begin{aligned}(n-\lambda)[T(r,f)+T(r,g)] \leq\ & [2(m_1+2m_2+2\lambda+2+3(m_1+m_2+2d+1)]\,T(r,f)+T(r,g) \\ & +S(r,f)+S(r,g). \qquad (3.8)\end{aligned}$$

From (3.8) we get

$$(n-\lambda)[T(r,f)+T(r,g)] \leq [2\Gamma_1+3\Gamma_0+4\lambda+6d+7)]\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).$$

which is a contradiction for $n > 2\Gamma_1 + 3\Gamma_0 + 5\lambda + 6d + 7$.

Case 2. We now assume that $H \equiv 0$, then

$$\Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big) = 0.$$

Integrating both sides of the above equality twice we get,

$$\frac{1}{F-1} \equiv \frac{A}{G-1} + B,$$

$$\frac{1}{F-1} \equiv \frac{BG+A-B}{G-1}. \qquad (3.9)$$

where $A \neq 0$, B are constants. From (3.9) it is clear that F and G share $(1,\infty)$. We consider the following cases.

Subcase 2.1. Let $B \neq 0$ and $A \neq B$. If $B = -1$, then from (3.9) we have

$$F \equiv \frac{-A}{G-A-1}.$$

From Lemma 2.2, we see that

$$\overline{N}(r, A+1; G) = \overline{N}(r, \infty; F) \le (d+1)\overline{N}(r, \infty; f).$$

So in view of Lemmas 2.2 and 2.15 using the second fundamental theorem, we get

$$\begin{aligned}(n-\lambda)T(r,g) &\le \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}(r,A+1;G) + S(r,g)\\ &\le (m_1+m_2+2d+1)T(r,g) + (d+1)T(r,f) + S(r,g).\end{aligned}$$

In a similar way , we get

$$(n-\lambda)T(r,f) \le (m_1+m_2+2d+1)T(r,f) + (d+1)T(r,g) + S(r,f).$$

Combining above equations we get

$$(n-\lambda)\{T(r,f)+T(r,g)\} \le (\Gamma_0+3d+2)\{T(r,f)+T(r,g)\} + S(r,f) + S(r.g),$$

a contadiction for $n > 2\Gamma_1 + 4 + 5\lambda$. If $B \ne -1$, from (3.9) we obtain that

$$F - \left(1 + \frac{1}{B}\right) = \frac{-A}{B^2(G + \frac{A-B}{B})}$$

So,

$$\overline{N}\left(r, \frac{B-A}{B}; G\right) = \overline{N}(r, \infty; F).$$

Using Lemmas 2.2 and 2.15 and with the same argument as used in the case for $B = -1$ we can get a contradiction.

Subcase 2.2. Let $B \ne 0$ and $A = B$. If $B = -1$, then from (3.9) we have

$$FG \equiv 1,$$

i.e, $P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}.P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j} \equiv 1$. In particular, when $P(f) = f^n$, take $M(z) = f(z)g(z)$, When $M(z)$ is non-constant, we get from above

$$M^n(z) \equiv \frac{1}{\prod_{j=1}^{d} M(z+c_j)^{v_j}}$$

So, using first fundamental theorem and Lemma 2.3 we get
$nT(r,M) = T(r, (\prod_{j=1}^{d} M(z+c_j)^{v_j})) + O(1) = \lambda T(r,M) + S(r,M)$, a contradiction. So $M(z)$ must be a constant and so $M(z)^{n+\lambda} \equiv 1$ which implies $fg \equiv t$ where $t^{n+\lambda} = 1$.
If $B \ne -1$, from (3.9) we have

$$\frac{1}{F} = \frac{BG}{(1+B)G - 1}$$

Therefore $\overline{N}\left(r, \frac{1}{1+B}; G\right) = \overline{N}(r, 0; F)$.
So in view of Lemmas 2.2 and 2.15, using the second fundamental theorem, we have

$$\begin{aligned}(n-\lambda)T(r,g) &\le \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}(r, \frac{1}{1+B}; G) + S(r,g)\\ &\le (m_1+m_2+2d+1)T(r,g) + (m_1+m_2+d)T(r,f) + S(r,g).\end{aligned}$$

Similarly we can get $(n-\lambda)T(r,f) \le (m_1+m_2+2d+1)T(r,f) + (m_1+m_2+d)T(r,g) + S(r,f)$.
Combining the above equations
$(n-\lambda)\{T(r,f)+T(r,g)\} \le (2\Gamma_0+3d+1)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g),$

a contradiction for $n > 2\Gamma_1 + 4 + 5\lambda$
Subcase 2.3. Let $B = 0$, From (3.9) we obtain

$$F \equiv \frac{G + A - 1}{A} \tag{3.10}$$

If $A \neq 1$ then from (3.10) we obtain

$$\overline{N}(r, 1 - A; G) = \overline{N}(r, 0; F)$$

Now using a similar process as done in subcase 2.2, for $B \neq -1$, we can deduce a contradiction. Therefore $A = 1$ and from (3.10) we obtain $F \equiv G$, that is

$$P(f) \prod_{j=1}^{d} f(z + c_j)^{v_j} \equiv P(g) \prod_{j=1}^{d} g(z + c_j)^{v_j}$$

In particular, when $P(f) = f^n$, Let $H(z) = \frac{f(z)}{g(z)}$, next proceeding in the same manner when $B = -1$ in subcase 2.2, we can show that $H(z)$ must be constant and $f \equiv tg$, where $t^{n+\lambda} = 1$.
The proof is completed.
Proof of Theorem 1.2. Let $\Phi = (F(z))^{(k)} = (P(f) \prod_{j=1}^{d} f(z + c_j)^{v_j})^{(k)}$ and $\Psi = (G(z))^{(k)} = (P(g) \prod_{j=1}^{d} g(z + c_j)^{v_j})^{(k)}$. Then Φ and Ψ share $(1, l)$. Applying Lemmas 2.2, 2.3, 2.11, we have

$$\begin{aligned} N_2(r, 0; \Phi) = N_2(r, 0; F^{(k)}) &\leq N_{k+2}(r, 0; F) + k\overline{N}(r, \infty; F) + S(r, f) \\ &\leq [(m_1 + (k + 2)m_2 + \lambda + (1 + d)k]T(r, f) \\ &\leq ((m_2 + d + 1)k + \Gamma_1 + \lambda)T(r, f) + S(r, f). \end{aligned} \tag{3.11}$$

$$\begin{aligned} N_2(r, \infty; \Phi) = N_2(r, \infty; F^{(k)}) + S(r, f) &\leq N_2(r, \infty; F) + S(r, f) \\ &\leq [2 + \lambda]T(r, f) + S(r, f). \end{aligned} \tag{3.12}$$

$$\begin{aligned} \overline{N}(r, 0; \Phi) = \overline{N}(r, 0; F^{(k)})) + S(r, f) &\leq k\overline{N}(r, \infty; F) + N_{k+1}(r, 0; F) + S(r, f) \\ &\leq [(m_1 + (k + 1)m_2 + \lambda + (1 + d)k]T(r, f) \\ &\leq ((m_2 + d + 1)k + \Gamma_0 + \lambda)T(r, f) + S(r, f). \end{aligned} \tag{3.13}$$

$$\begin{aligned} \overline{N}(r, \infty; \Phi) = \overline{N}(r, \infty; F^{(k)})) + S(r, f) &\leq \overline{N}(r, \infty; F) + S(r, f) \\ &\leq (1 + d)T(r, f) + S(r, f). \end{aligned} \tag{3.14}$$

Now we consider the following cases.
Case 1. Let $H \not\equiv 0$, now by applying Lemma 2.11 we have

$$\begin{aligned} N_2(r, 0; \Phi) &\leq N_2(r, 0; F^{(k)}) + S(r, f) \leq T(r, F^{(k)}) - T(r, F) + N_{k+2}(r, 0; F) + S(r, f) \\ &\leq T(r, \Phi) - T(r, F) + N_{k+2}(r, 0; F) + S(r, f) \end{aligned}$$

That is

$$T(r, F) \leq T(r, \Phi) - N_2(r, 0; \Phi) + N_{k+2}(r, 0; F) + S(r, f) \tag{3.15}$$

Combining Lemma 2.15 and (3.15), we have

$$(n - \lambda)T(r, f) \leq T(r, F) \leq T(r, \Phi) - N_2(r, 0; \Phi) + N_{k+2}(r, 0; F) + S(r, f). \tag{3.16}$$

Subcase 1.1. While $l \geq 2$ in view of (i) of Lemma 2.8, using (3.11),(3.12) and (3.16) we have

$$\begin{aligned}(n-\lambda)T(r,f) &\leq N_2(r,0;\Psi)+N_2(r,\infty;\Phi)+N_2(r,\infty;\Psi)+N_{k+2}(r,0;F)+S(r,f)+S(r,g)\\ &\leq ((m_2+d+1)k+\Gamma_1+\lambda)T(r,g)+(2+\lambda)T(r,f)+(2+\lambda)T(r,g)\\ &\quad +(m_1+(k+2)m_2+\lambda)T(r,f)+S(r,f)+S(r,g).\end{aligned}$$

Similarly

$$\begin{aligned}(n-\lambda)T(r,g) &\leq ((m_2+d+1)k+\Gamma_1+\lambda)T(r,f)+(2+\lambda)T(r,g)+(2+\lambda)T(r,f)\\ &\quad +(m_1+(k+2)m_2+\lambda)T(r,g)+S(r,f)+S(r,g).\end{aligned}$$

Combining above two equations , we get

$$(n-\lambda)[T(r,f)+T(r,g)] \leq (2m_2+d+1)k+2\Gamma_1+4\lambda+4$$

which is a contradiction for $n > (2m_2+d+1)k+2\Gamma_1+4+5\lambda$.

Subcase 1.2. While $l=1$, in view of Lemma 2.9, using (3.11),(3.12),(3.13),(3.14) and (3.16), we have

$$\begin{aligned}(n-\lambda)(T(r,f) &\leq N_2(r,0;\Psi)+N_2(r,\infty;\Phi)+N_2(r,\infty;\Psi)+N_{k+2}(r,0;F)\\ &\quad +\frac{1}{2}\overline{N}(r,0;\Phi)+\frac{1}{2}\overline{N}(r,\infty;\Phi)+S(r,f)+S(r,g).\\ &\leq [(1+d+m_2)k+\Gamma_1+\lambda]T(r,g)+(2+\lambda)T(r,f)+(2+\lambda)T(r,g)\\ &\quad +(m_1+(k+2)m_2+\lambda)T(r,f)+\frac{1}{2}[(1+d+m_2)k+\Gamma_0+\lambda)]T(r,f)\\ &\quad +\frac{1}{2}[(1+d)]T(r,f)+S(r,f)+S(r,g).\end{aligned}$$

Similarly we can write for $T(r,g)$

$$\begin{aligned}(n-\lambda)(T(r,g) &\leq [(1+d+m_2)k+\Gamma_1+\lambda]T(r,f)+(2+\lambda)T(r,g)+(2+\lambda)T(r,f)\\ &\quad +(m_1+(k+2)m_2+\lambda)T(r,g)+\frac{1}{2}[(1+d+m_2)k+\Gamma_0+\lambda)]T(r,g)\\ &\quad +\frac{1}{2}[(1+d)]T(r,g)+S(r,f)+S(r,g).\end{aligned}$$

Combining above two equations, we have

$$(n-\lambda)[T(r,f)+T(r,g)] \leq \left(\frac{5m_2}{2}+\frac{3}{2}+\frac{3d}{2}\right)k+2\Gamma_1+\frac{\Gamma_0}{2}+\frac{9\lambda}{2}+\frac{9}{2}+\frac{d}{2}.$$

which is a contradiction for

$$n > \left(\frac{5m_2}{2}+\frac{3}{2}+\frac{3d}{2}\right)k+2\Gamma_1+\frac{\Gamma_0}{2}+\frac{9}{2}+\frac{d}{2}+\frac{11\lambda}{2}.$$

Subcase 1.3. While $l = 0$, in view of Lemma 2.10, using (3.11),(3.12),(3.13),(3.14) and (3.16), we have

$$\begin{aligned}(n-\lambda)(T(r,f) &\leq N_2(r,0;\Psi) + N_2(r,\infty;\Phi) + N_2(r,\infty;\Psi) + N_{k+2}(r,0;F) \\ &\quad + 2\overline{N}(r,0;\Phi) + 2\overline{N}(r,\infty;\Phi) + \overline{N}(r,0;\Psi) + \overline{N}(r,\infty;\Psi) + S(r,f) + S(r,g). \\ &\leq [(1+d+m_2)k + \Gamma_1 + \lambda]T(r,g) + (2+\lambda)T(r,f) + (2+\lambda)T(r,g) \\ &\quad + (m_1 + (k+2)m_2 + \lambda)T(r,f) + 2[(1+d+m_2)k + \Gamma_0 + \lambda)]T(r,f) \\ &\quad + 2[(1+d)]T(r,f) + [(1+d+m_2)k + \Gamma_0 + \lambda]T(r,g) + (1+d)T(r,g) \\ &\quad + S(r,f) + S(r,g).\end{aligned}$$

Similarly ,

$$\begin{aligned}(n-\lambda)T(r,g) &\leq [(1+d+m_2)k + \Gamma_1 + \lambda]T(r,f) + (2+\lambda)T(r,g) + (2+\lambda)T(r,f) \\ &\quad + (m_1 + (k+2)m_2 + \lambda)T(r,g) + 2[(1+d+m_2)k + \Gamma_0 + \lambda)]T(r,g) \\ &\quad + 2[(1+d)]T(r,g) + [(1+d+m_2)k + \Gamma_0 + \lambda]T(r,f) + (1+d)T(r,f) \\ &\quad + S(r,f) + S(r,g).\end{aligned}$$

Combining the above two equations we have

$$(n-\lambda)[T(r,f) + T(r,g)] \leq (5m_2 + 4d + 4)\,k + 2\Gamma_1 + 3\Gamma_0 + 7 + 3d + 7\lambda.$$

which is a contradiction for

$$n > (5m_2 + 4d + 4)\,k + 2\Gamma_1 + 3\Gamma_0 + 7 + 3d + 8\lambda.$$

Case 2. We now assume that $H \equiv 0$, then

$$\Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big) = 0.$$

Integrating both sides of the above equality twice we get,

$$\frac{1}{\Phi - 1} \equiv \frac{A}{\Psi - 1} + B,$$

$$\frac{1}{\Phi - 1} \equiv \frac{B\Psi + A - B}{\Psi - 1} \tag{3.17}$$

where $A \neq 0$, B are constants. From (3.17) it is clear that Φ and Ψ share $(1,\infty)$. We consider the following cases.

Subcase 2.1. Let $B \neq 0$ and $A \neq B$. If $B = -1$, then from (3.17) we have

$$\Phi \equiv \frac{-A}{\Psi - A - 1}.$$

From Lemma 2.2 and (3.14) , we see that

$$\overline{N}(r, A+1;\Psi) = \overline{N}(r,\infty;\Phi) \leq (d+1)\overline{N}(r,\infty;f).$$

So, using the second fundamental theorem, we get

$$\begin{aligned}T(r,\Psi) &\leq \overline{N}(r,0;\Psi) + \overline{N}(r,\infty;\Psi) + \overline{N}(r,A+1;\Psi) + S(r,g) \\ &\leq \overline{N}(r,0;\Psi) + \overline{N}(r,\infty;\Psi) + \overline{N}(r,\infty;\Phi) + S(r,f) + S(r,g).\end{aligned}$$

From Lemma 2.11 we see that

$$\overline{N}(r,0;\Psi) \leq T(r,\Psi) - T(r,G) + N_{k+1}(r,0;G) + S(r,g).$$

These two inequalities imply

$$T(r,G) \le \overline{N}(r,\infty;\Psi) + \overline{N}(r,\infty;\Phi) + N_{k+1}(r,0;G) + S(r,f) + S(r,g).$$

From the above equation, using (3.14) and Lemmas 2.2, 2.15 we have

$$\begin{aligned}(n-\lambda)T(r,g) &\le \overline{N}(r,\infty;\Psi) + \overline{N}(r,\infty;\Phi) + N_{k+1}(r,0;G) + S(r,f) + S(r,g).\\ &\le (1+d)T(r,g) + (1+d)T(r,f) + (m_1+(k+1)m_2+\lambda)T(r,g)\\ &\quad + S(r,f) + S(r,g)\\ &\le (1+d)T(r,f) + (m_1+(k+1)m_2+\lambda+1+d)T(r,g) + S(r,f) + S(r,g).\end{aligned}$$

In a similar way

$$(n-\lambda)T(r,f) \le (1+d)T(r,g) + (m_1+(k+1)m_2+\lambda+1+d)T(r,f) + S(r,f) + S(r,g).$$

Combining the above two equations we can get

$$(n-\lambda)[T(r,f)+T(r,g)] \le (\Gamma_1+km_2+\lambda+2+2d)[T(r,f)+T(r,g)] + S(r,f) + S(r,g),$$

a contradiction for $n > (2m_2+d+1)k + 2\Gamma_1 + 4 + 5\lambda$.
If $B \ne -1$, from (3.17) we obtain that

$$\Phi - \left(1+\frac{1}{B}\right) = \frac{-A}{B^2(\Psi + \frac{A-B}{B})}$$

So,

$$\overline{N}\left(r, \frac{B-A}{B}; \Psi\right) = \overline{N}(r,\infty;\Phi).$$

Using Lemmas 2.2, 2.11 and 2.15 and with the same argument as used in the case for $B \ne -1$ we get a contradiction.
Subcase 2.2. Let $B \ne 0$ and $A = B$. If $B = -1$, then from (3.17) we have

$$\Phi\Psi \equiv 1,$$

That is, $\left(P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}\right)^{(k)} . \left(P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j}\right)^{(k)} \equiv 1.$
If $B \ne -1$ from (3.17) we have

$$\frac{1}{\Phi} = \frac{B\Psi}{(1+B)\Psi - 1}$$

Therefore , $\overline{N}\left(r, \frac{1}{1+B}; \Psi\right) = \overline{N}(r,0;\Phi)$. So, using the second fundamental theorem, we get

$$\begin{aligned}T(r,\Psi) &\le \overline{N}(r,0;\Psi) + \overline{N}(r,\infty;\Psi) + \overline{N}(r, \frac{1}{1+B}; \Psi) + S(r,g)\\ &\le \overline{N}(r,0;\Psi) + \overline{N}(r,\infty;\Psi) + \overline{N}(r,0;\Phi) + S(r,f) + S(r,g).\end{aligned}$$

By Lemma 2.11 we see that

$$\overline{N}(r,0;\Psi) \le T(r,\Psi) - T(r,G) + N_{k+1}(r,0;G) + S(r,g).$$

These two equations imply

$$T(r,G) \le \overline{N}(r,\infty;\Psi) + \overline{N}(r,0;\Phi) + N_{k+1}(r,0;G) + S(r,f) + S(r,g).$$

From the above equation, using (3.13),(3.14) and Lemmas 2.2, 2.15, we have

$$\begin{aligned}(n-\lambda)T(r,g) &\leq \overline{N}(r,\infty;\Psi)+\overline{N}(r,0;\Phi)+N_{k+1}(r,0;G)+S(r,f)+S(r,g)\\ &\leq (1+d)T(r,g)+(1+d+m_2)k+\Gamma_0+\lambda)T(r,f)\\ &\quad +(m_1+(k+1)m_2+\lambda)T(r,g)+S(r,f)+S(r,g).\end{aligned}$$

In a similar manner , we can get

$$\begin{aligned}(n-\lambda)T(r,f) &\leq (1+d)T(r,f)+(1+d+m_2)k+\Gamma_0+\lambda)T(r,g)\\ &\quad +(m_1+(k+1)m_2+\lambda)T(r,f)+S(r,f)+S(r,g).\end{aligned}$$

Combining above two equations , we can get

$$\begin{aligned}(n-\lambda)\{T(r,f)+T(r,g)\} &\leq ((2m_2+d+1)k+\Gamma_1+\Gamma_0+d+1+2\lambda)\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g).\end{aligned}$$

a contradiction for $n>(2m_2+d+1)k+2\Gamma_1+4+5\lambda$.

Subcase 2.3. Let $B=0$ from (3.17) we obtain

$$\Phi=\frac{\Psi+a-1}{a} \tag{3.18}$$

If $A\neq 1$, then from (3.18), we obtain

$$\overline{N}(r,1-a;\Psi)=\overline{N}(r,0;\Phi).$$

So, using the same argument as done in Case 2, for $B\neq -1$, we can similarly deduce a contradiction. Therefore $a=1$ and from (3.18) we obtain $\Phi\equiv\Psi$ that is $(P(f)\prod_{j=1}^{d}f(z+c_j)^{v_j})^{(k)}\equiv(P(g)\prod_{j=1}^{d}g(z+c_j)^{v_j})^{(k)}$. On Integration we have $(P(f)\prod_{j=1}^{d}f(z+c_j)^{v_j})\equiv(P(g)\prod_{j=1}^{d}g(z+c_j)^{v_j})+p(z)$, where $p(z)$ is a polynomial of degree atmost $k-1$. If $p(z)\not\equiv 0$ then from the second main theorem for the small function and Lemma 2.15 we get

$$\begin{aligned}(n-\lambda)T(r,f) &\leq T(r,F)+S(r,f)\\ &\leq \overline{N}(r,F)+\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G})+S(r,f)\\ &\leq (\Gamma_0+2d+1)T(r,f)+(\Gamma_0+d)T(r,g)+S(r,f).\end{aligned}$$

Similarly $(n-\lambda)T(r,g)\leq(\Gamma_0+2d+1)T(r,g)+(\Gamma_0+d)T(r,f)+S(r,g)$.
Therefore
$(n-\lambda)[T(r,f)+T(r,g)]\leq(2\Gamma_0+3d+1)[T(r,f)+T(r,g)]+S(r,f)+S(r,g)$, which is a contradiction for $n>(2m_2+d+1)k+2\Gamma_1+4+5\lambda$.
Thus $p(z)\equiv 0$, which implies

$$P(f)\prod_{j=1}^{d}f(z+c_j)^{v_j}\equiv P(g)\prod_{j=1}^{d}g(z+c_j)^{v_j} \tag{3.19}$$

$(a_nf^n+a_{n-1}f^{n-1}...+a_1f+a_0)(\prod_{j=1}^{d}f(z+c_j)^{v_j})=(a_ng^n+a_{n-1}g^{n-1}...+a_1g+a_0)(\prod_{j=1}^{d}g(z+c_j)^{v_j})$
Let $h=fg$ then following two cases holds.

Case A. Suppose that $h(z)$ is constant , say h. Substituting $f=hg$ in (3.19) we obtain $(a_n(gh)^n+a_{n-1}(gh)^{n-1}+...+a_1(gh)+a_0)\left(\prod_{j=1}^{d}g(z+c_j)^{v_j}\prod_{j=1}^{d}h(z+c_j)^{v_j}\right)=(a_ng^n+a_{n-1}g^{n-1}...+a_1g+a_0)(\prod_{j=1}^{d}g(z+c_j)^{v_j})$,

$\left(\prod_{j=1}^{d} g(z+c_j)^{v_j}\right)[a_n g^n(h^{n+\lambda}-1)+a_{n-1}g^{n-1}(h^{n+\lambda-1}-1)+...a_0(h^{\lambda}-1)] = 0$, where a_n is non zero complex constant $(\prod_{j=1}^{d} g(z+c_j)^{v_j}) \not\equiv 0$, since $g(z)$ is a non constant meromorphic function then

$$[a_n g^n(h^{n+\lambda}-1)+a_{n-1}g^{n-1}(h^{n+\lambda-1}-1)+...a_0(h^{\lambda}-1)] = 0 \tag{3.20}$$

If $a_n(\not\equiv 0)$ and $a_{n-1} = a_{n-2} = ... = a_1 = a_0 = 0$ then from (3.20), and g is non-constant meromorphic function, we get $h^{n+\lambda}-1 = 0$ which implies $h^{n+\lambda} = 1$.If $a_n(\not\equiv 0)$ and there exist $a_i \neq 0$ where $i \in [0, 1, 2, ...n-1]$. Supose that $h^{n+\lambda} \neq 1$, from (3.20) we have $T(r, g) = S(r, g)$, which is contradiction with transcendental function g. Then $h^{n+\lambda} = 1$, similar to this discussion we can see that $h^{n+\lambda} = 1$, where $a_j \not\equiv 0$, for some $j = 0, 1, 2...n$. Thus we have $f(z) = tg(z)$, for a constant t such that $t^l = 1$, where $l = GCD(\lambda+\gamma_0, \lambda+\gamma_1, .., \lambda+\gamma_n)$

$$\gamma_j = \begin{cases} j+1 & if \quad a_j \neq 0, \\ n+1 & if \quad a_j = 0. \end{cases}$$

Case B. Suppose $h(z)$ is not a constant, then $f(z)$ and $g(z)$ satisifies the algebraic equation $R(f, g) = 0$, where $R(w_1, w_2) = P(w_1)\prod_{j=1}^{d} f(z+c_j)^{v_j} - P(w_2)\prod_{j=1}^{d} g(z+c_j)^{v_j}$.
The proof is completed.
Proof of Theorem 1.3. In this case we have to proceed in the same manner as done in Theorem 1.1, we put $\Gamma_0 = m+1$ and $\Gamma_1 = 2(m+1)$ and using Lemmas 2.12, 2.13, 2.14 and 2.16 respectively.

4. Conclusions

Using the Nevanlinna theory, we have investigated the value distribution and uniqueness of difference-differential polynomial of the type $P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}$ and $P(g)\prod_{j=1}^{d} g(z+c_j)^{v_j}$ transcendental meromorphic function f of zero order respectively. Our findings extends some previous results of Theorems 1 and 2 and also by considering weighted sharing concept introduced by Indrajit Lahiri.

5. Open Problems

In this paper, we give two open questions for further research.
1)Can we relax the lower bound of n in Theorems 1.1 to 1.3 ?
2)Can the difference polynomials in Theorems 1.1 to 1.3 is replaced by the difference polynomials of form $P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}\prod_{j=1}^{s} f^{(i)}(z)$?

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Megha M. Manakame, Department of Mathematics, Jnanabharathi Campus, Bangalore University, Bengaluru,560-056,Karnataka,India.

Email address: megha.manakame80@gmail.com

Harina P. Waghamore, Department of Mathematics, Jnanabharathi Campus, Bangalore University, Bengaluru,560-056,Karnataka,India.

Email address: harinapw@gmail.com