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ON QUASIMULTIPERFECT NUMBERS WITH THREE DISTINCT PRIME FACTORS

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ABSTRACT. Min Tang and Meng Li called a positive integer n *quasimultiperfect* (QM) number if $\sigma(n) = kn + 1$ for some integer $k \geq 2$, where $\sigma(n)$ denotes the sum of the positive divisors of n. In particular, n is said to be *quasiperfect* (QP) if $k = 2$ and *quasitriperfect* (QT) if $k = 3$. Although no QM number is known several necessary conditions on such n, if exists, were proved. For instance, if a QM number with three distinct prime factors exists then it is of the form $n = 2^{\alpha} \cdot 3^{\beta} \cdot p^2$, where α and β are even integers; and p is an odd prime. In this paper we prove that for each even integer α there can be only a finite number of even integers β such that n of the above form may be QM. As an application, we show that if such n exists then $\alpha \geq 512$ and that $n > (2.6115) \times 10^{374}$, which improve the hitherto known lower bounds for both α and n.

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KEYWORDS AND PHRASES. Quasimultiperfect number, Quasitriperfect number.

1. INTRODUCTION

For a positive integer n, let $\sigma(n)$ denote the sum of its positive divisors and $\omega(n)$ denote the number of its distinct prime factors. In [4], a positive integer n is called a *quasimultiperfect* (QM) number if $\sigma(n) = kn + 1$ for some integer $k \geq 2$. In particular, n is said to be *quasiperfect* (QP) if $\sigma(n) = 2n + 1$ and *quasitriperfect* (QT) if $\sigma(n) = 3n + 1$. So far no QM number is known.

But certain necessary conditions on such n, if exists, are proved. First of all, Cattaneo [2] has initiated the study of QP numbers and several researchers have carried it forward, the details of which can be seen in the book by Sandor and Crstici [7, p.38-39]; and in recent papers [5] and [6] of two of the authors. As regards the QM numbers the results given below are known:

Lemma 1.1 ([4], Theorem 1). *If a QM number n exists then $\omega(n) \geq 7$ or 3 according as n is odd or even.*

Lemma 1.2 ([4], Theorem 2). *If n is QM number with $\omega(n) = 3$ then n is QT and is of the form*

$$n = 2^{\alpha}.3^{\beta}.p^2, \tag{1}$$

where α and β are even integers; and $p = p(\alpha, \beta)$ is an odd prime. Also $p = [F(\alpha, \beta)]$, where $F(\alpha, \beta) = 2^{\alpha+1}.3^{\beta+1}/(2^{\alpha+1} + 3^{\beta+1} - 1)$. (Here $[x]$, as *usual, denotes the greatest integer not exceeding the real number* x*). Further* $p > max\{\sqrt{2^{\alpha+1}}, \sqrt{3^{\beta+1}}\}$ *and*

$$F(\alpha,\beta) - \frac{1}{2} < p(\alpha,\beta) < F(\alpha,\beta) \tag{2}$$

As a consequence of Lemma 1.2, it has been proved:

Lemma 1.3 ([4], Corollary 3)**.** *If* n *is a QM number with* $\omega(n) = 3$ *then* $n > 10^{100}$*.*

Also it is shown that every even QT number is the square of an integer ([3], Lemma 2.3); and that any QM number n with $\omega(n) = 4$ is QT ([3], Theorem 1.1) so that such n is the square of an integer. Recently the authors have proved that a QM number with $\omega(n) = 4$ can not be the fourth power of an integer (see [1],Theorem B and Remark 1.3).

The purpose of this paper is to show that for any even integer α there can be only a finite number of even integers β such that the numbers of the form (1) may be QM (Theorem 2.3). As an application of this result, it is proved that for any QM number of the form (1) we must have $\alpha \geq 512$ (Corollary 2.5). Further the lower bound for n in Lemma 1.3 is improved to $(2.6115) \times 10^{374}$ (Corollary 2.6).

2. Main Results

In the rest of the paper, n stands for a number with $\omega(n) = 3$ and is of the form (1) so that α and β are always even integers; and p is an odd prime.

Then, by Lemma 1.2, we have:
n is QM only if $[F(\alpha,\beta)]$ is a prime and that

$$p = p(\alpha,\beta) = [F(\alpha,\beta)]. \tag{3}$$

The lemmas given below, proved by the authors earlier, are needed:

Lemma 2.1 ([1], Lemma 2.2)**.** *If* n *is QM then* $p \equiv 1 \pmod 4$.

Lemma 2.2 ([1], Lemma 2.3)**.** *If* n *is QM then* $p \equiv 2, 8$ or $5 \pmod 9$ *according to* $\alpha \equiv 0, 2$ or $4 \pmod 6$*.*

Now we prove:

Theorem 2.3. *For any* α*, write* $A_\alpha = \log_3\{(2^{\alpha+1}-1)(2^{\alpha+2}-1)\} - 1$*. If* $\beta > A_\alpha$ *then* n *cannot be QM. In otherwords, for any* α*,* n *may be QM only if* $\beta \leq A_\alpha$*.*

Proof. Note that

$$F(\alpha,\beta) = 2^{\alpha+1} - g(\alpha,\beta), \tag{4}$$

where

$$g(\alpha,\beta) = 2^{\alpha+1}\left(2^{\alpha+1}-1\right) / \left(2^{\alpha+1} + 3^{\beta+1} - 1\right). \tag{5}$$

Clearly $g(\alpha,\beta) > 0$ for any α and β. Also if $\beta > A_\alpha$ then $\beta + 1 > \log_3\{(2^{\alpha+1}-1)(2^{\alpha+2}-1)\}$ so that $3^{\beta+1} > (2^{\alpha+1}-1)(2^{\alpha+2}-1)$ and therefore $2^{\alpha+1}+3^{\beta+1}-1 > (2^{\alpha+1}-1)2^{\alpha+2}$, which shows that $g(\alpha,\beta) < \frac{1}{2}$ in this case. Therefore

$$0 < g(\alpha,\beta) < \frac{1}{2} \ for \ \beta > A_\alpha. \tag{6}$$

Using (6) in (4), we get

$$2^{\alpha+1} - \frac{1}{2} < F(\alpha,\beta) < 2^{\alpha+1} \ for \ \beta > A_\alpha, \tag{7}$$

which gives

$$[F(\alpha,\beta)] = 2^{\alpha+1} - 1 \ for \ \beta > A_\alpha. \tag{8}$$

Now (7) and (8) give

$$\frac{1}{2} < F(\alpha,\beta) - [F(\alpha,\beta)] < 1 \ for \ \beta > A_\alpha. \tag{9}$$

But if $[F(\alpha,\beta)] = p$ is a prime then, by (2), we must have

$$0 < F(\alpha,\beta) - [F(\alpha,\beta)] < \frac{1}{2}. \tag{10}$$

From (9) and (10), it follows that $[F(\alpha,\beta)]$ is composite for $\beta > A_\alpha$ and therefore the corresponding n is not in QM, by (3). □

Remark 2.4. *An alternative proof of the theorem is worth to note. By* (8)*, we have* $[F(\alpha,\beta)] \equiv 3 \pmod 4$ *for* $\beta > A_\alpha$*, so that the corresponding* n *cannot be QM, in view of* (3) *if* $[F(\alpha,\beta)]$ *is composite; and by Lemma 2.1 if* $[F(\alpha,\beta)]$ *is a prime.*

Corollary 2.5. *If* $2 \le \alpha \le 510$ *then there is no* β *for which* n *is QM. In otherwords, if* n *is QM then* $\alpha \ge 512$.

Proof. In view of Theorem 2.3, it suffices to show:

$$n \text{ is not QM if } 2 \le \alpha \le 510 \text{ and } 2 \le \beta \le A_\alpha \tag{11}$$

For convenience we prove (11) in three cases of (i) $2 \le \alpha \le 38$,
(ii) $40 \le \alpha \le 100$ and (iii) $102 \le \alpha \le 510$

To do this some simple computations are to be made. First, using Excel, one can find for each even integer α, the value of A_α. For instance, $A_2 = 3.23621$, $A_8 = 10.985$, $A_{28} = 36.22486$ and $A_{510} = 644.4411$. Then for each ordered pair (α,β) of even integers, again using Excel, one can find the values of $[F(\alpha,\beta)]$. For example, note $[F(2,4)] = 7$, $[F(6,14)] = 127$ and $[F(18,20)] = 524261$ are primes; and $[F(8,4)] = 165$ is composite while $[F(46,46)] = 140737487610389$.

Using these Excel tables so framed we deal the three cases.

Case (i). For $2 \le \alpha \le 38$ and $2 \le \beta \le A_\alpha$, in view of (3), ignoring the composite values of $[F(\alpha,\beta)]$ using the trivial tests for the divisibility by 2, 3 and 5; and then the list of primes upto 10^{12} available on the internet, we

find that there are only thirteen prime values of $[F(\alpha,\beta)] = p(\alpha,\beta)$ which are such that $p(2,4) = 7 \equiv 3 \pmod 4$; $p(8,8) = 499 \equiv 3 \pmod 4$;
$p(10,14) = 2047 \equiv 3 \pmod 4$; $p(14,10) = 27653 \equiv 5 \pmod 9$;
$p(16,16) = 130939 \equiv 3 \pmod 4$;
$\underline{P = p(18, 20) = 524261 \equiv 1 \pmod 4 \text{ and } P \equiv 2 \pmod 9}$;
$p(22,10) = 173483 \equiv 3 \pmod 4$; $p(24,24) = 33553103 \equiv 3 \pmod 4$;
$\underline{Q = p(30, 24) = 2142054533 \equiv 1 \pmod 4 \text{ and } Q \equiv 2 \pmod 9}$;
$p(32,12) = 1594027 \equiv 3 \pmod 4$; $p(34,12) = 1594249 \equiv 7 \pmod 9$; $p(34,30) = 34357827121 \equiv 7 \pmod 9$ and $p(38,40) = 549755805601 \equiv 1 \pmod 9$.

These $p(\alpha,\beta)'s$, except $P = p(18,20)$ and $Q = p(30,24)$, are failing to satisfy either Lemma 2.1 or Lemma 2.2 and therefore the corresponding n cannot be QM. Thus the two possible QM numbers with $2 \le \alpha \le 38$ and $2 \le \beta \le A_\alpha$ are $M = 2^{18} \cdot 3^{20} \cdot P^2$ and $N = 2^{30} \cdot 3^{24} \cdot Q^2$ which must be QT also. But we observe that

$$\sigma(M) \neq 3M + 1 \text{ and } \sigma(N) \neq 3N + 1 \tag{12}$$

In fact, note that $P \equiv 1 \pmod{10}$ so that $3M + 1 = 2^{18} \cdot 3^{21} \cdot P^2 + 1 \equiv (4.3.1) + 1 \pmod{10} \equiv 3 \pmod{10}$ and $\sigma(M) = \sigma(2^{18})\sigma(3^{20})\sigma(P^2) = (2^{19} - 1)(1 + 3 + 3^2 + ... + 3^{20})(1 + P + P^2) \equiv (7.1.3) \pmod{10} \equiv 1 \pmod{10}$ from which we get $\sigma(M) \neq 3M + 1$. Again since $Q \equiv 3 \pmod{10}$ we have $3N + 1 = 2^{30} \cdot 3^{25} \cdot Q^2 + 1 \equiv (4.3.9) + 1 \pmod{10} \equiv 9 \pmod{10}$; while $\sigma(N) = \sigma(2^{30})\sigma(3^{24})\sigma(Q^2) = (2^{31} - 1)(1 + 3 + 3^2 + ... + 3^{24})(1 + Q + Q^2) \equiv (7.1.3) \pmod{10} \equiv 1 \pmod{10}$ which gives $\sigma(N) \neq 3N + 1$. This proves (12) and therefore if $2 \le \alpha \le 38$ and $2 \le \beta \le A_\alpha$ then n cannot be QM.

Case (ii). For $40 \le \alpha \le 100$ and $2 \le \beta \le A_\alpha$, some of the values of $[F(\alpha,\beta)]$ are greater than 10^{12} and therefore their primality could not be identified using the list of primes upto 10^{12}. However the procedure adopted earlier can be used to decide whether such an $[F(\alpha,\beta)]$ yields a QM number or not. As before, ignoring those composite values of $[F(\alpha,\beta)]$ whose divisibility by 2, 3 and 5 can be verified easily, we find that there are only thirty seven **odd** values which are greater than 10^{12} and they satisfy the congruence properties given below:
$[F(40,28)] \equiv 4 \pmod 9$; $[F(40,48)] \equiv 3 \pmod 4$; $[F(40,50)] \equiv 2 \pmod 9$;
$[F(40,52)] \equiv 3 \pmod 4$; $[F(42,32)] \equiv 8 \pmod 9$; $[F(42,34)] \equiv 3 \pmod 4$;
$[F(42,54)] \equiv 1 \pmod 9$; $[F(44,22)] \equiv 3 \pmod 4$; $[F(44,24)] \equiv 2 \pmod 9$;
$[F(44,38)] \equiv 3 \pmod 4$; $[F(44,42)] \equiv 7 \pmod 9$; $[F(44,44)] \equiv 3 \pmod 4$;
$[F(44,56)] \equiv 3 \pmod 4$; $[F(46,26)] \equiv 3 \pmod 4$; $[F(46,34)] \equiv 2 \pmod 9$;
$[F(46,36)] \equiv 3 \pmod 4$; $[F(46,44)] \equiv 3 \pmod 4$;
$\underline{R = [F(46,46)] = 140737487610389 \equiv 1 \pmod 4 \text{ and } R \equiv 5 \pmod 9}$;
$[F(46,48)] \equiv 3 \pmod 4$; $[F(46,50)] \equiv 3 \pmod 4$; $[F(48,60)] \equiv 8 \pmod 9$;
$[F(52,16)] \equiv 7 \pmod 9$; $[F(54,18)] \equiv 7 \pmod 9$; $[F(58,20)] \equiv 8 \pmod 9$;
$[F(60,26)] \equiv 5 \pmod 9$; $[F(64,30)] \equiv 1 \pmod 9$; $[F(66,30)] \equiv 3 \pmod 4$;
$[F(68,26)] \equiv 4 \pmod 9$; $[F(70,26)] \equiv 3 \pmod 4$; $[F(78,24)] \equiv 7 \pmod 9$;
$[F(78,30)] \equiv 8 \pmod 9$; $[F(82,30)] \equiv 8 \pmod 9$; $[F(84,28)] \equiv 4 \pmod 9$;
$[F(90,28)] \equiv 7 \pmod 9$; $[F(90,30)] \equiv 8 \pmod 9$; $[F(94,30)] \equiv 8 \pmod 9$

and $[F(96,28)] \equiv 3 \pmod 4$.

If any of these $[F(\alpha,\beta)]$ is composite then it does not yeild a QM number, in view (3). Also if $[F(\alpha,\beta)] = p(\alpha,\beta)$ is a prime then, except the case of $R = [F(46,46)]$, the others fail to satisfy either Lemma 2.1 or Lemma 2.2, so that such $p(\alpha,\beta)$ does not give rise to a QM number. Further, if R is a prime then the possible QM number that arises will be $X = 2^{46} \cdot 3^{46} \cdot R^2$ which must be QT. One can show that $\sigma(X) \neq 3X + 1$, using the same technique to prove (12). Thus if $40 \leq \alpha \leq 100$ and $2 \leq \beta \leq A_\alpha$ then n cannot be QM.

Case (iii). In case $102 \leq \alpha \leq 510$, the table shows that $[F(\alpha,\beta)]$ is either composite or $[F(\alpha,\beta)] \equiv 3 \pmod 4$ if $2 \leq \beta \leq 30$; and that $[F(\alpha,\beta)]$ is composite if $32 \leq \beta \leq A_\alpha$. Such $[F(\alpha,\beta)]$ do not yeild QM numbers, by an argument similar to one given in Remark 2.4.

Thus (11) is proved completely. Hence the corollary. □

Corollary 2.6. *If n is QM with $\omega(n) = 3$ then $n > (2.6115) \times 10^{374}$.*

Proof. While proving the Lemma 1.3, it has been shown in [4] that $\alpha \leq 216$ and $\beta \leq 136$ cannot hold, if n of the form (1) is QM. Therefore, it follows that

$$\alpha \geq 218 \text{ and } \beta \geq 138 \tag{13}$$

Now, Corollary 2.5, (13) and Lemma 1.2 show that $n = 2^\alpha \cdot 3^\beta \cdot p^2$, where $\alpha \geq 512, \beta \geq 138$ and $p > max\{\sqrt{2^{513}}, \sqrt{3^{139}}\} = \sqrt{2^{513}}$. Therefore, we have

$$n > 2^{512} \cdot 3^{138} \cdot 2^{513} = 2^{1025} \cdot 3^{138}. \tag{14}$$

But

$$\begin{aligned} 2^{1025} &= \left(2^5\right)^{205} = (3.2)^{205} \cdot 10^{205} = (3.2)\,(10.24)^{102} \cdot 10^{205} \\ &= (3.2)\,(1.024)^{102} \cdot 10^{307} = (3.2)\,(1.024)^2\,\{(1.024)^4\}^{25} \cdot 10^{307} \\ &> (3.5184) \cdot (1.0995)^{25} \cdot 10^{307} \\ &> (3.5184) \cdot (1.0995) \cdot (1.2089)^{12} \cdot 10^{307} \\ &> (3.5184) \cdot (1.0995) \cdot (2.1357)^3 \cdot 10^{307} > (37.684) \cdot 10^{307} \\ &= (3.7684) \cdot 10^{308} \end{aligned} \tag{15}$$

and

$$3^{138} = \left(3^6\right)^{23} = (7.29)^{23} \cdot 10^{46} \tag{16}$$
$$> (7.29)\,(53.14)^{11} \cdot 10^{46}$$
$$= (7.29)\,(5.314)^{11} \cdot 10^{57} = (7.29)(5.314) \cdot \{(5.314)^2\}^5 \cdot 10^{57}$$
$$> (38.739)\,(28.2385)^5 \cdot 10^{57} = (3.8739)\,(2.82385)^5 \cdot 10^{63}$$
$$= (3.8739)(2.82385) \cdot \{(2.82385)^2\}^2 \cdot 10^{63}$$
$$> (10.9)\,(63.5871) \cdot 10^{63}$$
$$= (1.09)\,(6.35871) \cdot 10^{65} = (6.93) \cdot 10^{65}.$$

Using (15) and (16) in (14) we get

$$n > (3.7684)(6.93) \cdot 10^{373} > (26.115) \cdot 10^{373} = (2.6115) \cdot 10^{374},$$

proving the Corollary 2.6. □

Remark 2.7. *Observe that Corollary 2.5 improves the lower bound for α, given in* (13)*, while Corollary 2.6 increases the lower bound for n, given in Lemma 1.3.*

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