Advanced Studies in Contemporary Mathematics
33 (2023), No. 3, pp. 221 - 241

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2023.33.3.221

A Method for Generating Mersenne Primes and the Extent of the Sequence of the Even Perfect Numbers

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Abstract. A condition is obtained for the generation of new Mersenne primes from a combination of Mersenne numbers with prime indices. It is verified that all known Mersenne prime indices greater than 19 have the form $p_1 + p_2 - 1$, where $2^{p_1} - 1$ is prime and $2^{p_2} - 1$ is composite. Arithmetical sequences for the exponents of composite Mersenne numbers are obtained from partitions into consecutive integers and congruence relations for products of two Mersenne numbers. A congruence condition for a Mersenne number to be composite is derived. A necessary requirement of a Mersenne prime is the absence of prime divisors congruent to 1 moduli 8. The characteristic function for Mersenne primes is described, and the sum over the positive integers is found to be divergent.

MSC: 11N32, 11N80, 11P83

Keywords: *Mersenne primes, perfect numbers, composite Mersenne numbers, polynomial approximation*

1. Introduction

The perfect numbers, defined by a condition on the sum of the divisors, are represented by a sequence of integers conjectuted to have connections with the ideal characteristics of physical systems. After a geometrical proof by Euclid that integers of the form $2^{p-1}(2^p-1)$, with 2^p-1 being prime, would be perfect numbers [1], it was hypothesized later that all perfect numbers were even, every even perfect number equals $2^{p-1}(2^p-1)$ for some prime 2^p-1, annd there are infinitely many perfect numbers [2]. The next perfect numbers were discovered in the early thirteenth century [3]. After a series of perfect numbers of perfect numbers verified only until $2^{18}(2^{19-1)}$ [4], a systematic investigation b^n-1 began with the letter of Fermat to Mersenne [5] and the following theorems: 2^n-1 is composite if n is composite; if n is prime and p is a prime divisor of 2^n-1, then $p-1$ is a multiple of n. Further primes of the kind 2^p-1 were suggested, and it was demonstrated by Euler that $2^{30}(2^{31}-1)$ was a perfect number [6] and the unique form $2^{p-1}(2^p-1)$ of every even perfect number [7]. No new perfect numbers were found until $2^{60}(2^{61}-1)$ [8], whereas it had been shown that $2^{67}-1$ was not a prime and $2^{127}-1$ was a Mersenne prime [9]. Lucas also proved that every perfect number greater than 6 must end in the digits 16, 28, 36, 56, 76 or 96 [10]. The last result led to the conjecture of Catalan that the sequence $(2^p-1,\ 2^{2^p-1}-1,\ ...)$, consists of primes for $p=2$ [11]. The powers increase very rapidly, and computer tests beyond the fourth term are exceedingly complex. The first four Catalan-Mersenne numbers for $p=2$, $\{2^2-1,\ 2^3-1,\ 2^7-1,\ 2^{127}-1,...\}$ are known to be Mersenne primes. The lower limit for the prime divisors of the the fifth number, $2^{2^{127}-1}-1$ is currently set at 5×10^{51} [12]. However, the sequences for other primes have composite numbers generally and the probability of higher terms being prime is exceedingly infinitesimal and tend to zero. While $2^{88}(2^{89}-1)$ was verified as a perfect number in 1911 [13], the use of computer was found to be necessary for the larger of the known Mersenne primes and the extent of the sequence remains to be established.

The Lucas-Lehmer test, together with several characteristics of the prime divisors of Mersenne numbers that are valid for all Lucas sequences, can be used to determine theoretically whether the integer 2^p-1 is prime. The congruence $s_{p-2}\equiv 0\ (mod\ 2^p-1)$, $s_n=s_{n-1}^2-2$ [10][14], is satisfied by the known Mersenne primes, although the difficulty of the computation increases for large values of n, Further properties of perfect numbers include the form of the prime index p being $1+T_n$, where T_n is a triangular number [15], the equality of x^3+1 and a perfect number only for the integer 28 [16], the integrality of the harmonic mean of the divisors [17] and the proportionality of the number of divisors of a perfect number N to $ln\ ln\ N$ [18]. The conjecture of the existence of infinitely many Mersenne primes and the problem of establishing the infinite extent of the sequence of composite Mersenne numbers with prime indices [19][20][21] may be considered without reference to specific values of the exponent. The generality of the statements is evident in proofs based on conditions on integers of the same order as the exponents. It is is shown in §2, for example, that all known Mersenne primes greater than $2^{19}-1$ have exponents of the form p_1+p_2-1, where p_1 is a Mersenne prime index and p_2 is a composite Mersenne number index.

It was hypothesized by Euler [22] and proven by Lagrange [23] that 2^p-1 is a composite Mersenne number if the prime p has the form $4k+3$ and $2p+1$ is a prime. An infinite number of Sophie Germain primes congruent to 3 modulo 4 would imply the existence of an infinite number of composite Mersenne numbers with prime exponents. The Mersenne numbers also have a geometrical representation which may be used to derive congruence relations for compositeness based on the partition of the array representing 2^n-1. The solutions to the congruence relations yield arithmetical sequences for the exponents. However, the greatest common denominator of the initial term and the difference can be equated to $ord_{2^m-k}(k)$ for some m, k, such that none of the integers in the sequence are prime. Nevertheless, this allows a characterization of the set of exponents greater than 6 of composite Mersenne numbers [25]. The congruence relations for $2^{p_1+p_2-1}$ provide a further indication of the existence of infinitely many composite Mersenne numbers with prime exponents.

The conditions for a prime factorization of a Mersenne number 2^p-1 is examined in §3. By comparing the coefficient of the prime p derived from the congruence $2^p-1\equiv 1\ (mod\ p)$ with that resulting from the product of the prime divisors, a congruence relation that includes the Fermat quotient $F_2(p)$ is found. The conditions for a Mersenne prime then may be deduced. These include the absence of a prime factor congruent to 1 modulo 8.

The existence of a finite number of prime solutions to $a^{f(n)}-b^{f(n)}\equiv 0\ (mod\ n)$, when $f(x)$ does not have a zero at $x=1$, may be used to develop an algorithm for locating the next Mersenne prime based on the intersections of polynomials at prime arguments. A theoretical foundation for the investigation of the extent of the sequence of even perfect numbers is given in §4. A characteristic function can be defined for the function 2^y-1, and the sum over the integers is proven to be infinite.

2. The Exponents of Mersenne Numbers and Arithmetical Progressions

There are two infinite sequences, of Mersenne primes of odd index, and primes, in the arithmetical progression $6n+1$, $n \in \mathbb{Z}^+$, and the coincidences of these two sequence determine whether the set of even perfect numbers continues indefinitely.

Since $6n+1$ can be factorized only if n has the form $6xy \pm (x+y)$, with $x,\ y \in \mathbb{Z}^+$ [25], the Mersenne number $2^p - 1$ is prime only if it equals $6n+1$, $n = 6xy \pm (x+y) + z$, $z \neq 0$, with $6xy \pm (x+y) + z \neq 6x' \pm (x'+y')$ for any integers $x',\ y'$. Given the condition $2^p - 1 = (6x \pm 1)(6y \pm 1)$, consider two prime p_1 and p_2 such that

$$\begin{aligned} &2^{p_1} - (6z_1 + 1) = (6x_1 \pm 1)(6y_1 \pm 1) = 6h_1 + 1 \qquad && h_1 = 6x_1y_1 \pm (x_1 + y_1) \\ &2^{p_2} - (6z_2 + 1) = (6x_2 \pm 1)(6y_2 \pm 1) = 6h_2 + 1 \qquad && h_2 = 6x_2y_2 \pm (x_2 + y_2) \end{aligned} \tag{2.1}$$

Multiplication of these two integers gives

$$2^{p_1+p_2} - (6z_1+1)(6z_2+1) - (6z_1+1)(6h_1+1) - (6z_2+1)(6h_1+1) = (6h_1+1)(6h_2+1) \tag{2.2}$$

or equivalently

$$2p_1 + p_2 - 1 = [3(h_1 + z_1) + 1][6(h_2 + z_2) + 2] \tag{2.3}$$

If $z_1 \neq 0$, and $2^{p_1} - 1$ is prime, while z_2 is set equal to zero, $2^{p_2} - 1$ is allowed to be composite,

$$2^{p_1+p_2-1} - [6(1-\gamma_1)z_1 + (1-\gamma_2)h_1 + (1-\gamma_3)h_2) + 1] = 6(3(h_1+z_1)h_2 + \gamma_1 z_1 + \gamma_2 h_1 + \gamma_3 h_2) + 1 \tag{2.4}$$

for some fractions $\gamma_1,\ \gamma_2,\ \gamma_3$ with $3(h_1 + z_1)h_2 + \gamma_1 z_1 + \gamma_2 h_1 + \gamma_3 h_2 = 6x'y' \pm (x' + y')$, $x',\ y' \in \mathbb{Z}$. The Mersenne numberr $2^{p_1+p_2-1}$ is prime if there is no solution to Eq.(2.4) with $\gamma_1 = \gamma_2 = \gamma_3 = 1$. If p_1 is a given Mersenne prime index, it can be conjectured that $p_1 - 1$ may be expressed as the difference between two primes p and p_2, since an even integer equals $p - p_2$ if $2(N + p_2)$ is given by the sum of the two primes p, p_2. The estimated number of prime pairs $(p, p+2N)$ with $p \le x$ [26] is conjectured to be

$$\pi_{2N}(x) \sim 2C_2 \frac{x}{(log\ x)^2} \prod_{\substack{p>2 \\ p|N}} \frac{p-1}{p-2}, \tag{2.5}$$

where C_2 is the twin-prime constant $\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)$ [27][39], and

$$\pi_2(x) \le 6.836\ C_2 \frac{x}{(log\ x)^2}\left[1 + \mathcal{O}\left(\frac{log\ log\ x}{log\ x}\right)\right]. \tag{2.6}$$

The conjecture holds for primes of the form $4k-1$, since there would exist a prime $4k'+1$ with the difference $2(2k''+1)$ being an even number.

Theorem 1. Every finite even positive integer $2N$ can be expressed as the difference between two primes if the Goldbach conjecture is valid.

Proof. By the Goldbach conjecture, every even integer $2N$, $N \geq 3$, equals $q_1 + q_2$, where q_1 and q_2 are odd primes [27][28]. The integer 4 clearly also equals the difference between two primes. The magnitudes will be selected such that $q_1 > q_2$. Then $2\bar{N} = q_1 - q_2$, with $\bar{N} = N - q_2$. This property for the integers $1 \leq \tilde{N} \leq \bar{N}$ may be demonstrated. By the existence of a Goldbach partition for even integers greater than or equal to 4, $2N - 2\ell = q_{1\ell} + q_{2\ell}$ for two primes $q_{1\ell}$, $q_{2\ell}$ and for each integer ℓ less than or equal to $N-2$ and

$$2N - 2(\ell + q_{2\ell}) = q_{1\ell} - q_{2\ell}. \tag{2.7}$$

This equality between a positive even integer and the difference of two primes requires $\ell + q_{2\ell}$ to be less than $N-1$. The set of integers of the form $\ell + q_{2\ell}$ can be enumerated as ℓ ranges over all integers between 1 and $N-1$. Since $q_{2\ell} \geq 2$, $\ell + q_{2\ell} \geq 3$. Fixing N, the set of primes $q_{2\ell}$ would be limited and $\ell + q_{2\ell}$ would cover only a restricted subset of integers between 3 and $N-2$. However, beginning with a set of primes $\{q_{2\ell}\}$ less than $N-1$, it is possible to add other primes to equal even integers less than $2N-2$. As $q_{2\ell}$ is selected to be one of the addends in a Goldbach partition, it follows that $\ell + q_{2\ell}$ must range over values between 3 and $N-1$. The set of pairs $\{q_{2\ell}, \ell + 2q_{2\ell}\}$ covers the entire range between for this addend from 2 to $N-2$.

Given that $\bar{q}_2 \geq 3$, $\bar{N} \leq N-3$, and every integer $2\tilde{N}$, with $1 \leq \tilde{N} \leq \bar{N}$ may be represented as the difference between two primes. Furthermore, this condition can be hypothesized for a proof by induction.

Now consider the even integer $2\bar{N}+2$. Since

$$\begin{aligned} 2\bar{N} + 2 &= \bar{q}_1 - \bar{q}_2 + 2 = 2N + 2 - 2\bar{q}_2 \\ &= q_3 + q_4 - 2\bar{q}_2 = (q_3 - \bar{q}_2 + k) - (\bar{q}_2 - q_4 + k) \end{aligned} \tag{2.8}$$

when $q_3 \geq \bar{q}_1$ and $q_4 \leq \bar{q}_2 + 2$ or $\bar{q}_1 > q_3 > \bar{q}_2$, $q_4 > \bar{q}_4 + 2$. When $q_3 = \bar{q}_1$ and $q_4 = \bar{q}_2 - 2$, $2\bar{N} - 2 = q_3 - q_4$. Otherwise, it remains to be shown that there is an integer k such that $q_3 - \bar{q}_2 + k$ and $\bar{q}_2 - q_4 + k$ are primes. Then all even integers less than or equal to $2\bar{N}$ would be the differences between two primes.

Suppose that the first set of inequalities holds. Adding an extra variable to adjust the subtahends, let

$$2\bar{N} + 2 = (q_3 - \bar{q}_2 + k + k') - (\bar{q}_2 - q_4 + k + k'). \tag{2.9}$$

The second subtrahend equals

$$\bar{q}_2 - q_4 + k + k' = \bar{q}_2 + k_1 + k' - (q_4 - k_2) \tag{2.10}$$

where $k_1 + k_2 = k$. The integers k_1 and k_2 may be altered while preserving the sum. The integer k' also is a free parameter. Then k_1, k_2 and k' may be selected such that

$$q_4 - k_2 = \bar{q}_2 + k_1 + k' - \bar{p}_2 \tag{2.11}$$

with $\bar{q}_2 + k_1 + k'$ and $\bar{p}_2$ being prime, since $q_4 - k_2 < 2\bar{N}$. The difference between the two primes fixes $k_1 + k'$ only, there is enough freedom in k' to adjust $q_3 - \bar{q}_2 + k + k'$ to be prime. Then Eq.(2.4) is equivalent to

$$2\bar{N} + 2 = (q_3 - \bar{q}_2 + k + k') - \bar{p}_2. \tag{2.12}$$

which is the difference between two primes.

When the second set of inequalities is valid,

$$2\bar{N}{+}_2 = \bar{q}_1 - \bar{q}_2 + 2 = (\bar{q}_1 + q_4 + k + k' + 2) - (q_4 - \bar{q}_2 + k + k'). \tag{2.14}$$

The lesser subtrahend equals

$$q_4 - \bar{q}_2 + k + k' = (q_4 + k_1 + k') - (\bar{q}_2 - k_2). \tag{2.15}$$

Since $\bar{q}_2 - k_2$ is an even integer less than $2\bar{N}$ for odd k_2,

$$\bar{q}_2 - k_2 = q_4 + k - 1 + k' - \bar{p}_3, \tag{2.16}$$

where $k_1 + k'$ is adjusted for both $q_4 + k_1 + k'$ and $\bar{p}_3$ to be prime. The integer k' then may be chosen such that $\bar{q}_1 + q_4 + k + k' + 2$ is prime. Again $2\bar{N} + 2$ is the difference between two primes. By induction, every positive integer is the difference between two primes given existence of a prime partition of every integer greater than or equal to 4.

■

Given a prime p_1, the pair (p, p_2), with $p - p_2 = p_1 - 1$ would exist. It is not feasible to consider a Mersenne prime index equal to $p_1 + p_2 - p_3$, for an odd prime p_3, because the relation (2.3) yields fractional terms. Therefore, p has the form $p_1 + p_2 - 1$, This property can be verified for the following pairs of prime indices (p_1, p_2)

$(3, 11); (7, 11); (3, 29); (19, 43); (41.59); (61, 47); (61.67); (89; 433); (61; 547); ,$

$(607, 673); (607, 1597); (2203, 79); (2281; 937); (2281, 1973); (2203, 2221); (4253; 5447);$

$(4254, 6961); (2281; 17657); (89.21613); (2281, 20929); (3217, 41281); (9941, 76303);$

$(607, 109897) : (44492, 87553); (23209, 192883); (132049; 624791); (19937; 839497);$

$(132049, 1125739); (86243, 1312027); (86243, 2889979); (21701, 3049677);$

$(3071377, 3901217); (216091, 13250827); (110503, 20885509); (1257787, 22778797);$

$(110503, 253874449); (3217, 303999241); (3021377, 29561281)$

The prime pairs (p_1, p_2) with $2^[p_1 - 1$, $2^{p_2} - 1$ and $2^{p_1+p_2-1}$ prime, $\{(2, 2); (2, 3);$ $(3, 5); (7, 7); (5, .13); (3; 17)l(7; 13); (13, 19); (31, 31)\}$ complement the larger set when $p_1 +$ $p_2 - 1 = 3,\ 5m\ 7,\ 13,\ 17,\ 19,\ 31,\ 61$.

One subset of the composte Mersenne numbers of the form $2^{p_1+p_2-1} - 1$ can be constructed from the integer solutions to the following sets of equations

$$\begin{aligned} h_1 &= 6x_1y_1 + (x_1 + y_1) \qquad h_2 - 6x_2y_2 + (x_{2y2}) \\ w_1 + w_2 &= 3(x_1 + y_1 + z_1)(x_2 + y_2) + (x_1 + y_1 + z_1) + (x_2 + y_2) \\ w_1w_2 &= 18x_1y_1x_2y_2 + 3x_1y_1(x_2 + y_2) + 3x_2y_2(x_1 + y_1 + z_1) + (x_1y_1 + x_2y_2) \end{aligned} \tag{2.17}$$

$$\begin{aligned} h_1 &= 6x_1y_1 - (x_1 + y_1) \qquad h_2 = 6x_2y_2 + (x_2 + y_2) \\ w_1 + w_2 &= -3(x_1 + y_1 - z_1)(x_2 + y_2)_6(x_1y_1 + x_2y_2) - (x_1 + y_1 - z_1) + (x_2 + y_2) \\ w_1w_2 &= 18x_1y_1x_2y_2 + 3x_1y_1(x_2 + y_2) - 3x_2y_2(x_1 + y_1 - z_1) + (x_1y_1 + x_2y_2) \end{aligned} \tag{2.18}$$

$$\begin{aligned} h_1 &= 6x_1y_1 + (x_1 + y_1) \qquad h_2 = 6x_2y_2 - (x_2 + y_2) \\ w_1 + w_2 &= -3(x_1 + y_1 + z_1)(x_2 + y_2) + 6(x_1y_1 + x_2y_2) + (x_1 + y_1 + z_1) - (x_2 + y_2) \\ w_1w_2 &= 18x_1y_1x_2y_2 - 3x_1y_1(x_2 + y_2) + 3x_2y_2(x_1 + y_1 + z_1) + (x_1y_1 + x_2y_2) \end{aligned} \tag{2.19}$$

$$\begin{aligned} h_1 &= 6x_1y_1 - (x_1 + y_1) \qquad h_2 = 6x_2y_2 - (x_2 + y_2) \\ w_1 + w_2 &= 3(x_1 + y_1 - z_1)(x_2 + y_2) - (x_1 + y_1 + z_1) - (x_2 + y_2) \\ w_1w_2 &= 18x_1y_1x_2y_2 - 3x_1y_1(x_2 + y_2) - 3x_2y_2(x_1 + y_1 - z_1) + (x_1y_1 + x_2y_2). \end{aligned} \tag{2.20}$$

Consider the equations determined by the equations $u + v = h_1 + z_1 + h_2$ and $uv = \frac{1}{2}(h_1 + z_1)h_2$. these two conditions imply

$$2u^2 = 2(h_1 + z_1 + h_2)u + (h_1 + z_1)h_2 = 0. \tag{2.21}$$

and

$$u = \frac{1}{2}\Bigg[h_1 + z_1 + h_2 \pm \sqrt{(h_1 + z_1 + h_2)^2 - 2(h_1 + z_1)h_2}\Bigg]. \tag{2.22}$$

Then u is integer only if $(h_1 + z_1)^2 + h_2^2$ is the square of an integer. Since the Pythagorean triples are multiples of the triples $(3 + 2n, 4 + 6n + 2n^2, 5 + 6n + 2n^2)$, there is and solution for $h_1 + z_1$ and h_2 as both integers must be odd.

More generally,

$$\begin{aligned} u + v &= \kappa_1(h_1 + z_1)h_2 + \kappa_2(h_1 + z_1 + h_2) \\ uv &= \kappa_3(h_1 + z_1)h_2 + \kappa_4(h_1 + z_1 + h_2) \end{aligned} \tag{2.24}$$

with

$$\begin{aligned} \kappa_1 + 6\kappa_3 &= 3 \\ \kappa_2 + 6\kappa_4 &= 1 \\ \kappa_1,\ \kappa_2,\ \kappa_3,\ \kappa_4 &\in \mathbb{Q}. \end{aligned} \tag{2.25}$$

Integrality of u and v requires that $\kappa_1(h_1+z_1)h_2+\kappa-2(h_1+z_1+h_2)$ and $\frac{3-\kappa_1}{6}(h_1+z_1)h_2+\frac{1-\kappa_1}{6}(h_1 + z_1 + h_2)$ are integer, while $[\kappa_1(h_1 + z_1)h_2 + \kappa_2(h_1 + z_1 + h_2)]^2 - 4\Bigg[\frac{3-\kappa_1}{6}(h_1 + z_1)h_2 + \frac{1-\kappa_1}{6}(h_1 + z_1 + h_2)\Bigg]$ is the square of an integer.

Additional constraints can be placed on $h_1 + z_1$, h_2 as the equality of $6[3(h_1+ z_1)h_2 +(h_1 + z_1) + h_2] + 1$ and $2^{p_1+p_2-1} - 1$ would yield the congruence conditions. First, after division by 2, either $h_1 + z_1 \equiv 0\ (mod\ 4),\ h_2 \equiv 5\ (mod\ 8);\ h_1 + z_1 \equiv 5\ (mod\ 8),\ h_2 \equiv 0`(mod`4);\ h_1 + z_1 \equiv 1\ (mod\ 4),\ h_2 \equiv 3\ (mod\ 4);\ h_1 + z_1 \equiv 2\ (mod\ 4),\ h_2 \equiv 1\ (mod\ 4)$. Since $3(h_1+z_1)h_2+(h_1+z_1)+h_2 \equiv \frac{2^n-`}{3}\ (mod\ 8),\ n\ even$ and $3(h_1+z_1)h_2+(h_1+z_1)+h_2 \equiv \frac{2^{n+1}-1}{3}\ (mod\ 2^n),\ n\ odd.$

Based on the pairwise relations between Mersenne prime indices, a sum extended over a set of these integers may be derived. Since $p'_n = p_{kc}+p'_\ell-1$ for some composite Mersenne number index p_{kc}, $\ell < n$ and $p_{kc} - 1$ can be expressed either as the sum of two integers that are differences beween a Mersenne prime index and a composite Mersenne index, or the sum of two primes by the Goldbach conjecture, the process can be iterated until all of the addends are either Mersenne prime indices, with either sign, or twice the previous index or ± 1.

The relations between Mersenne prime indices have a form similar to the equations for the sequence of primes [29][30]

$$
\begin{aligned}
p_{2n} &= 1 \pm p_1 \pm p_2 \pm ... \pm p_{2n-2} + p_{2n-1} \\
p_{2n+1} &= 1 \pm p_1 \pm p_2 \pm ... \pm p_{2n-1} + 2p_2 n
\end{aligned}
\tag{2.26}
$$

The equations are consistent with the estimate of the number of Mersenne primes with indices between x and $2x$ [31][32]. It is possible to extend the sequence of Mersenne prime indices by forming combinations having the form in Eq.(2.26) and using any of the various tests to verify that $2^p - 1$ is prime.

3. Fermat Quotients

When p is a prime,

$$2^p - 1 \equiv 1 \ (mod\ p) \tag{3.1}$$

and

$$
\begin{aligned}
2^p - 1 &= Ap + 1 \\
A &= 2F_2(p),
\end{aligned}
\tag{3.2}
$$

where $F_2(p) = \frac{2^{p-1}-1}{p}$ is the Fermat quotient. Suppose that $2^p - 1$ is not prime and has the prime factorization

$$2^p - 1 = \prod_i (2k_{0i}p + +1)^{\gamma_i} \prod_j (2k_{1j}p + 1)^{\delta_j}, \tag{3.3}$$

where $k_{0i} \equiv 0 \ (mod\ 4)$, $k_{1ij} \equiv -p \ (mod\ 4)$ for all $i,\ j$ [33][34]. Since $F_2(p)$ is an odd integer, compatibility of the congruences of the coefficients modulo p requires $\sum_j \delta_j$ to be odd. Furthermore,

$$\prod_i (2k_{0i}p + 1)^{\gamma_i} \prod_j (2k_{1j}p + 1)^{\delta_j} = 1 + 2\left(\sum_i \gamma_i k_{0i} + \sum_j \delta_j k_{1j}\right) p + \mathcal{O}(p^2) \tag{3.4}$$

and

$$\sum_i \gamma_i k_{0i} + \sum_j \delta_i k_{1j} \equiv F_2(p) \qquad (mod\ p). \tag{3.5}$$

When there are no Wieferich primes [35][36] in the hypothetical factorization, this condition reduces to

$$\sum_i k_{0i} + \sum_j k_{1j} \equiv F_2(p) \qquad (mod\ p), \tag{3.6}$$

When k_{0i} $(mod\ p)$ is odd for all i, k_{1j} $(mod\ p)$ is even for all j and $F_2(p)$ $(mod\ p)$ is odd, there must be a minimum of one prime factor of the type $8\overline{k_{0i}}p+1$, where $\overline{k_{0i}} = \frac{k_{0i}}{4}$. More generally, let

$$\begin{aligned} N_0^o &= \#\ odd\ k_{0i}\ (mod\ p) \\ N_0^e &= \#\ even\ k_{0i}\ (mod\ p) \\ N_1^o &= \#\ odd\ k_{1j}\ (mod\ p) \\ N_1^e &= \#\ even\ k_{1j}\ (mod\ p). \end{aligned} \tag{3.8}$$

Then $2^p - 1$ is composite for

$$\begin{aligned} F_2(p)\ (mod\ p)\ is\ odd;\ & N_0^o + N_1^o\ is\ odd\ if\ 2np < \sum_i k_{0i} + \sum_j k_{1j} < (2n+1)p; \\ & N_0^o + N_1^o\ is\ even\ if\ (2n+1)p < \sum_i k_{0i} + \sum_j k_{1j} < (2n+2)p \\ & \qquad n \in \mathbb{Z} \\ F_2(p)\ (mod\ p)\ is\ even;\ & N_0^o + N_1^o\ is\ even\ if\ 2np < \sum_i k_{0i} + \sum_j k_{1j} < (2n+1)p; \\ & N_0^o + N_1^o\ is\ odd\ if\ (2n+1)p < \sum_k k_{0i} + \sum_j k_{1j} < (2n+2)p \\ & \qquad n \in \mathbb{Z} \end{aligned} \tag{3.9}$$

given that there are no Wieferich prime factors. Since there are only two known Wieferich primes less than 6.7×10^{15} [38], the condition may be straightforwardly tested. From the table of factors of Mersenne numbers, all composite Mersenne numbers with prime exponents, $2^p - 1$, with $p < 100$, have a prime divisor congruent to 1 modulo 8, except for $p = 43$ and 79, where

$$\begin{aligned} 2^{43} - 1 &= 431 \cdot 9719 \cdot 2099863 \\ k_{11} &= 5,\ k_{12} = 113,\ k_{13} = 24417 \\ F_2(43) &\equiv 25\ (mod\ 43) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} 2^{79} - 1 &= 2687 \cdot 202029703 \cdot 1113491139767 \\ k_{11} &= 17,\ k_{12} = 1278669,\ k_{13} = 7047412277 \\ F_2(79) &\equiv 19\ (mod\ 79). \end{aligned} \tag{3.11}$$

It may be verified that Eq.(3.6) is valid for these two Mersenne numbers.

The Catalan sequence is defined to be

$$\{2^p - 1,\ 2^{2^p-1} - 1,\ 2^{2^{2^p-1}-1},\ ...\}. \tag{3.12}$$

When $p = 2$, it is

$$\{3,\ 7,\text{`}127,\ 2^{127}-1,\ ...\}. \tag{3.13}$$

The sequences for other values of p are

$$\begin{aligned}
&\{5,\ 31,\ 2147483647,\ 2^{2147483647}-1,\ ...\}\\
&\{13.\ 8191,\ 2^{8191}-1,\ 2^{2^{8191}-1}-1,\ ...\}\\
&\{17,\ 131071,\ 2^{131071}-1,\ 2^{2^{131071}-1}-1,\ ...\}\\
&\{19,\ 524287,\ 2^{584287}-1,\ 2^{2^{584287}-1}-1,\ ...\}\\
&\ \vdots
\end{aligned} \tag{3.14}$$

There are composite numbers by the second or third terms of these sequences.

It will be sufficient to investigate the extent of the set of primes for $p = 2$. Suppose that $2^p - 1$ is a prime. Then

$$2^{2^p-1} - 1 \equiv 1\ (mod\ 2^p - 1). \tag{3.15}$$

It follows that

$$\begin{aligned}
2^{2^p-1} - 1 &= \tilde{A}(2^p - 1) + 1\\
\tilde{A} &= 2^{2^p-p-1} + 2^{2^p-2p-1} + ... + 2^{p+1} + 2
\end{aligned} \tag{3.16}$$

The number $2^{2^p-1}-1$ may be tested for primality by postulating a factorization into prime divisors. The product of divisors congruent to 1 modulo $2(2^p - 1)$ equals

$$(2K(2^p-1)+1)(2K'(2^p-1)+1) = (4KK'(2^p-1)+2K+2K')(2^p-1)+1 \tag{3.17}$$

The coefficients of $2^p - 1$ modulo p are

$$\begin{aligned}
\tilde{A} &\equiv 2(2^{F_2(p)-1} - 1)\ (mod\ p)\\
4KK'(2^p-1) + 2K + 2K' &\equiv 2[2KK' + K + K']\ (mod\ p)
\end{aligned} \tag{3.18}$$

which are compatible when

$$2KK' + K + K' \equiv 2^{F_2(p)} - 1 \qquad (mod\ p) \tag{3.19}$$

iF $F_2(p) \equiv 0\ (mod\ p)$, $2KK' + K + K' \equiv 1\ (mod\ p)$, which has solutions for odd $K + K'$. Given the infinitesimal probability of the higher Catalan-Mersenne numbers being prime, there would be factorizations, nearly always, yielding solutions to this congruence relation.

4. The New Mersenne Prime Conjecture

By the new Mersenne prime conjecture, if $p = 2^k \pm 1$ or $4^k \pm 3$, and $\frac{2^p+1}{3}$ is prime, then $2^p - 1$ is prime [38]. When $p \equiv 1$ $(mod\ 4)$, the factors of $2^p - 1$ are congruent to 1 or $6p + 1$ $(mod\ 8p)$, while the factors of $\frac{2^p+1}{3}$ are congruent to 1 or $2p + 1$ $(mod\ 8p)$. If $p \equiv 3$ $(mod\ 4)$, the factors of $2^p - 1$ are congruent to 1 or $2p + 1$ modulo $8p$, while the divisors of $\frac{2^p+1}{3}$ are congruent to 1 or $6p + 1$ $(mod\ 8p)$.

If $\frac{2^p+1}{3}$ is prime,

$$2^{\frac{2^p+1}{3}} - 1 \equiv 1 \ \left(mod\ \frac{2^p + 1}{3} \right) \tag{4.1}$$

Then

$$2^{2^p - 1} = \left(2^{\frac{2^p+1}{3}} \right)^3 \cdot 2^{-2} \equiv 2^3 \cdot 2^2 \ \equiv 2 \ \left(mod\ \frac{2^p + 1}{3} \right). \tag{4.2}$$

and

$$2^{2^p - 1} - 1 \equiv 1 \ \left(mod\ \frac{2^p + 1}{3} \right). \tag{4.3}$$

This congruence condition is equivalent to

$$2^{2^p - 1} - 1 = \hat{A} \left(\frac{2^p + 1}{3} \right) + 1 \tag{4.4}$$

Given that

The congruence condition for $p \equiv 3$ $(mod\ 4)$ to be a Mersenne prime index is

$$\left(\frac{1 + \sqrt{5}}{2} \right)^{\frac{p+1}{2}} + \left(\frac{1 - \sqrt{5}}{2} \right)^{\frac{p+1}{2}} \equiv 0 \qquad (mod\ p) \tag{4.7}$$

When $p = 4^k + 3$. $\frac{p+1}{2} = 2^{2k-1+2}$, and this relation becomes

$$\left(\frac{1 + \sqrt{5}}{2} \right)^{2^{2k-1}+2} + \left(\frac{1 - \sqrt{5}}{2} \right)^{2;2k-1+2} \equiv 0 \qquad \left(mod\ 4^k + 3 \right) \tag{4.8}$$

and

$$\left(\frac{1 + \sqrt{5}}{2} \right)^{4^k+4} + \left(\frac{1 - \sqrt{5}}{2} \right)^{3^k+4} + 2 \left(\frac{1 - 5}{2} \right)^{2^{2k-1}+2 \equiv 0} \qquad (mod\ 4^k + +3).(4.9)$$

Since

$$\left(\frac{1+\sqrt{5}}{2}\right)^{p+1} = \frac{1}{2^{p+1}}\left\{1+\binom{p+1}{1}\sqrt{5}+\binom{p+1}{2}5+...+\binom{p+1}{p}5^{\frac{p}{2}}+5^{\frac{p+1}{2}}\right\}$$
$$\left(\frac{1-\sqrt{5}}{2}\right)^{p+1} = \frac{1}{2^{p+1}}\left\{1-\binom{p+1}{1}\sqrt{5}+\binom{p+1}{2}5+...-\binom{p+1}{p}5^{p}+5^{\frac{p+1}{2}}\right\} \tag{4.10}$$

$$\left(\frac{1+\sqrt{5}}{2}\right)^{p+1}+\left(\frac{1-\sqrt{5}}{2}\right)^{p+1} = \frac{1}{2^{p}}\left\{1+\binom{p+1}{2}\cdot 5+...+5^{\frac{p+1}{2}}\right\}. \tag{4.11}$$

Therefore,

$$\left(\frac{1+\sqrt{5}}{2}\right)^{p+1}+\left(\frac{1-\sqrt{5}}{2}\right)^{p+1} \equiv \frac{1}{2}\left(1+5^{\frac{p+1}{2}}\right) \qquad (mod\ p). \tag{4.12}$$

By the quadratic reciprocity law,

$$\binom{5}{p}\binom{p}{5} = (-1)^{\frac{5-1}{2}}\cdot\frac{p-1}{2} = 1$$

and $\binom{5}{p} = \binom{p}{5}$. it follows that

$$\binom{5}{p} = \begin{cases} 1 & if\ p\equiv 1,\ 4 \qquad (mod\ 5) \\ -1 & if\ p\equiv 2,\ 3 \qquad (mod\ 5) \end{cases} \tag{4.13}$$

and

$$\frac{1}{2}\left(1+5^{\frac{p+1}{2}}\right)\ (mod\ p) \equiv \frac{1}{2}\left(1+5\binom{5}{p}\right)\ \ (mod\ p) = \begin{cases} 3 & if\ p\equiv 1,\ 4\ (mod\ 5) \\ -1 & if\ p\equiv 2,\ 3\ (mod\ 5) \end{cases} \tag{4.14}$$

Two congruences

$$\begin{aligned} 2^{2^{2k-1}+3}+3 &\equiv 0 \qquad (mod\ 4^k+3)\ if\ 4^k+3\equiv 1,\ 4\ (mod\ 5) \\ 2^{2^{2k-1}+3}-2 &\equiv 0 \qquad (mod\ 4^k+3)\ if\ 4^k+3\equiv 2,\ 3\ (mod\ 5) \end{aligned} \tag{4.15}$$

It may be verified that $4^k+3\equiv 4\ (mod\ 5)$ when k is even and $4^k+2\equiv 2\ (mod\ 5)$ if k is odd, and

$$\begin{aligned} 2^{2^{2k-1}+3}+3 &\equiv 0 \qquad (mod\ 4^k+3)\ k\ is\ even \\ 2^{2^{2k-1}+3}-2 &\equiv 0 \qquad (mod\ 4^k+3)\ if\ 4^k+3\ if\ k\ odd \end{aligned} \tag{4.16}$$

The second congruence yields

$$2^{2^{2k-1}+2} \equiv 1 \qquad (mod\ 4^k+3)\ if\ 4^k+3\ if\ k\ is\ odd. \tag{4.17}$$

Adapting the method of §3, the congruences for the numbers 2^p-1 and $\frac{2^p+1}{3}$ may be related.

Theorem 2. If $p=2^k-1$ and $\frac{2^p+1}{3}$ is prime, 2^p-1 is prime.

Proof. From Eq.(3.16), let

$$2^{2^p-1}-1 \equiv (2^{2^p-p-1}+...+2^{p+1}+2)(2^p-1)+1. \tag{4.18}$$

Since

$$\begin{aligned} 2^p-1 &\equiv 2^p+1-2 \equiv -2 \left(mod\ \frac{2^p+1}{3}\right) \\ 2^{p+1} &\equiv 2\cdot 2^p \equiv -2 \\ 2^{2p+1} &\equiv 2 \\ &\vdots \end{aligned} \tag{4.19}$$

A complete cancellation amongst the terms in $2^{2^p-p-1}+...+2^{p+1}+2$ occurs as a result of an an odd number of multiples of the prime p present in the sequence from p to 2^p-p-1 inclusively. Therefore, if 2^p-1 is prime, $2^{2^p-1}-1 \equiv 1\ \left(mod\ \frac{2^p+1}{3}\right)$. It follows that

$$\begin{aligned} \left(2^{\frac{2^p+1}{3}}\right)^3 &\equiv 2^{2^p+1} \equiv 2^{2^p-1}\cdot 2^2 \equiv 2\cdot 2^2 \equiv 2^3 \quad \left(mod\ \frac{2^p+1}{3}\right) \\ 2^{\frac{2^p+1}{3}} &\equiv 2 \quad \left(mod\ \frac{2^p+1}{3}\right), \end{aligned} \tag{4.20}$$

which is a necessary condition for $\frac{2^p+1}{3}$ to be prime. Then, if 2^p-1 is prime, $\frac{2^p+1}{3}$ is a prime or a pseudoprime. Conversely, reversing the direction of the congruences, when $\frac{2^p+1}{3}$ is prime, 2^p-1 is a prime or pseudoprime. When n is a pseudoprime, 2^n-1 is a pseudoprime [39], and, if 2^n-1 is a prime number, n also must be prime. Given that $\frac{2^p+1}{3}$ is prime, a certain set of primes p may be selected such that 2^p-1 is prime. It is evident that the primes of the form 2^k-1 are such that $2^{2^p-1}\equiv 1\ (mod\ \frac{2^p+1}{3}$. Since

$$2^{2^k-1} \equiv -1 \quad \left(mod\ \frac{2^{2^k-1}+1}{3}\right), \tag{4.21}$$

$$2^{2^{2^k-1}-1} \equiv 2^{3\cdot\frac{2^{2^k-1}+1}{3}-2} \equiv 2^{3-2} \equiv 2 \quad \left(mod\ \frac{2^{2^k-1}+1}{3}\right) \tag{4.22}$$

or

$$2^{2^{2^k-1}-1} - 1 \equiv \left(mod \ \frac{2^{2^k-1}+1}{3} \right). \tag{4.23}$$

Furthermore, since $F_2(p) \equiv 1 \ (mod \ p)$, there must be a prime factor of $F_2(p)$ of the form $8\ell p + 1$ if $2^p - 1$ is composite. The factorization of double Mersenne number $2^{2^k-1} - 1$ will include divisors congruent to 1 and 7 modulo 8. Suppose that

$$2^{2^k-1} - 1 = (8\ell(2^k-1)+7)(8\ell'(2^k-1)+1). \tag{4.24}$$

Then

$$\begin{aligned}\frac{2^{2^k-1}+1}{3} = \frac{(8\ell+7)(8\ell'(2^k-1)+1)+2}{3} &= \frac{64\ell\ell'(2^k-1)^2+8(\ell+7\ell')(2^k-1))+9}{3} \\ &= \frac{64\ell\ell'(2^k-1)+8(\ell+7\ell')}{3}(2^k-1)+3\end{aligned} \tag{4.25}$$

which requires $3|(-16\ell\ell' + \ell + 7\ell')$ or $\ell \equiv \ell' \equiv 0, \ 2 \ (mod \ 3)$. Setting $\ell = 3x + 2$ and $\ell' = 3x' + 2$, $\frac{8\ell\ell'+\ell+7\ell'}{3}$ will be divisible by 3 if $x + x' \equiv 0 \ (mod \ 3)$. It is evident from the product in (4.24) that $x + x'$ must be congruent to 0 modulo 3. Then $\{p|p = 2^k - 1\}$ will be amongst the set of primes which yield integers $2^p - 1$ that are prime when $\frac{2^p+1}{3}$ is prime.

■

5. The Characteristic Function of the Mersenne Numbers

The classification of polynomial and exponential functions will confirm the conjectured density of Mersenne primes. Let

$$\begin{aligned} s_n &: I^1 \to \{0,1\} \\ s_n(x) &= \chi_P(\lfloor f(n+x) \rfloor) \\ S_N^{\psi} &: I^1 \to \mathbb{R}_+ \\ S_N(x) &= \sum_{n=1}^{N} \frac{s_n(x)}{\psi(n)} \\ D_{\psi_1}^{\psi_2}(P,f;N) &= \frac{\sum_{1 \le i \le N} \frac{\chi_P(\lfloor f(x+n) \rfloor)}{\psi_1(n)}}{\sum_{1 \le i \le N} \frac{\chi_P(n)}{\psi_2(n)}} \end{aligned} \tag{5.1}$$

where ψ, ψ_1, ψ_2 are weighting functions and χ_P is the characteristic function for the set of prime numbers. It will be seen that the correct choice for ψ is usually

$$\psi(y) = \begin{cases} max\{log\ 2,\ log\ y\} & f \in \mathcal{F}^*_{pol} \\ y & f \in \mathcal{F}^*_{exp} \end{cases}, \tag{5.2}$$

where

$$\begin{aligned} \mathcal{F}^*_{pol} &= \{ay^k + \sum_{i=1}^{m} a_i y^{k_i};\ a > 0,\ k > k_1 > ... > k_m \geq 0 |\ f(0) \geq 0,\ f'(0) > 0,\ f'(0) > 0 \\ &\qquad\qquad f''\ is\ monotonically\ increasing\} \\ \mathcal{F}^*_{exp} &= \{e^{ky+\ell} + f(y);\ k > 0,\ f \in \mathcal{F}^*_{pol} \cup \{0\}\}. \end{aligned} \tag{5.3}$$

with a convention, defined previously [40], except for a change in the lower bounds for $f(0)$ and $f'(0)$ in Eq.(4.3). Then, given that $2^y \in \mathcal{F}^*_{exp}$,

$$S^{\psi}_N = \sum_{1 \leq nleN} \frac{\chi_P(\lfloor [f(x+n)] \rfloor)}{n} \tag{5.4}$$

and

$$S_N = S^{\psi}_N(0) = \sum_{1 \leq n \leq N} \frac{\chi_P(\lfloor f(n) \rfloor)}{n}. \tag{5.5}$$

If $D^{log}_{har}(P, 2^y - 1; N)$ is defined with the weighting functions $\psi_1(y) = max\{log\ 2,\ log\ y\}$ and $\psi_2(y) = y$,

$$\begin{aligned} D^*_N &= (log\ 2) D^{\psi_2}{}_{\psi 1}(P, f; N)(0) = (log\ 2) \left(\sum_{1 \leq n < N} \frac{\chi_P(n)}{n} \right)^{-1} \sum_{1 \leq n \leq N} \frac{\chi_P(\lfloor f(n) \rfloor)}{n} \\ &= (log\ 2) \frac{S_N}{\sum_{1 \leq n < N} \frac{\chi_P(n)}{n}} \end{aligned} \tag{5.6}$$

Since $\sum_{\substack{p \leq N \\ p\ prime}} \frac{1}{p} = log\ log\ N + o(1)$ [41], and the probabilistic value of the density of the Mersenne prime indices [32] is

$$\frac{S_N}{N}\ N \to \infty\ \frac{e^{\gamma}}{ln\ 2} \frac{ln\ N}{N} \tag{5.7}$$

D^*_N is approximately $e^{\gamma}\ \frac{ln\ N}{ln\ ln\ N}$ for known large Mersenne prime indices [39].

A proof of $\left| \frac{D^*_N ln\ ln\ N}{ln\ N} - e^{\gamma} \right| < \epsilon$ would provide the actual asymptotic value of the density of the Mersenne primes. If an irreducible rational-coefficient polynomial is chosen to approximate $2^y - 1$ over a certain interval, the weighting factor should be replaced by $\psi(y) = y$, yielding

$$lim_{N\to\infty}\frac{S_N(P,f)}{ln\ N} = \frac{1}{det\ f}\prod_p \frac{p-\rho(p)}{p-1}. \tag{5.8}$$

where $\rho(p)$ is the number of solutions to $f(n) \equiv 0\ (mod\ p)$ [40]. When p increases, the number of solutions to $f(n) \equiv 0\ (mod\ p)$, $1 \le n \le N$, decreases rapidly and, as many of the terms in the product in Eq.(5.8) have the form $\frac{p}{p-1}$, approximate equality of $\frac{1}{deg\ f}\prod_p \frac{p-\rho(p)}{p-1}$ with the coefficient $\frac{e^\gamma}{ln\ 2}$ can be deduced. The function $2^y - 1$ may be approximated by rational-coefficient polynomials of given order only over an interval the known Mersenne primes, whereas a method of intersecting polynomials at the next Mersenne prime would be required for an approximation of the function throughout a larger range including this integer.

To determine the prime distribution of a function $\lfloor f(x+n)\rfloor$, the following intervals shall be to defined to be

$$\begin{aligned} I_{p,n} = |f^{-1}(p) - n,\ f^{-1}(p+1) - n) \qquad & p \in \mathcal{P} \\ & {I^{a,b}}_{p,n} I_{p,n} \cap [a,b]. \end{aligned} \tag{5.9}$$

noindent Since

$$\begin{aligned} \mu_t(I_{p,n}) &\sim |(f^{-1}(p-1) - n) - (f^{-1}(p) - n)| = |f^{-1}(p_1) - f^{-1}(p)| \\ &= \left|\left[f^{-1}(p) + (f^{-1})'(p)[(p+1) - p] + \frac{1}{2!}(f^{-1})''(p)[(p+1)-p]^2 + ...\right] - f^{-1}(p)\right| \\ &= \left|(f^{-1})'(p) + \frac{1}{2!}(f^{-1})''(p) + ...\right| \end{aligned} \tag{5.10}$$

the Lebesgue measure of $I^{a,b}_{p,n}$, which is not empty if $\mathcal{P}_n(a,b) = \{p \in \mathcal{P} | f(n+a) \le p \le f(n+b) - 1\}$ contains a prime, is given by

$$\begin{aligned} \mu(I^{a,b}_{p,n}) = \frac{1}{f'(n+\Theta_p)} \qquad a \le \Theta_p \le b \qquad \in \mathcal{P}^*_n(a,b) \\ \mathcal{P}^*_n(a,b) = \{p \in \mathcal{P} | f(n+a) \le p \le f(n+b) - 1\}. \end{aligned} \tag{5.11}$$

Using the natural weighting, with $\psi(y) = 1$, it may be shown that

$$\begin{aligned} \int_0^1 S_N^{nat}(\mathcal{P};x)\ dx &= \sum_{n=1}^{n}\int_0^1 s_n(x)\ dx = \sum_{n=1}^{N}\sum_{p\in\mathcal{P}_n(0,1)} \mu(I^{0,1})_{p,n}) \\ &= \sum_{\substack{p\in\mathcal{P} \\ f(0)\le p\le f(N)}} [(f^{-1})'(p+\Theta_p)] + \mathcal{O}(1) = \sum_{\substack{p\in\mathcal{P} \\ f(0)\le p\le f(N)}} [(f^{-1})'(p)] + \mathcal{O}(1) \end{aligned} \tag{5.12}$$

Suppose that $f(y) = 2^y - 1$. Then

$$\sum_n (f^{-1})'(p_n) = \sum_{n\geq 2} \frac{1}{f'(n+\Theta_{p_n})} = \sum_{n\geq 2} \frac{1}{f'(f^{-1}(p_n)))} = \sum_p \frac{1}{ln\ 2}. \tag{5.13}$$

It would appear that the index range of the sum is a presumption of the infinite extent of the sequence of Mersenne primes. However, the sufficiently fine subdivision of the unit interval, the infinitude of primes, the existence of a prime between $f(n+a)$ and $f(n+b)-1$ for $a < b$ and sufficiently large n, given that $f(n+1) = 2f(n)+1$, and the overlapping of the subintervals $[a, b]$ with the inverse images under f^{-1} of the primes and Mersenne numbers leads to the conclusion that the infinite sum (4.18) is direct evidence of the extent of the Mersenne prime sequence. It would follow that $lim_{N\to\infty} D_N^* \to e^\gamma \frac{log\ N}{log\ log\ N}$ continuously and monotonically, and this limit would be verification of the density of Mersenne primes.

This discussion clearly does not extend to integers of the form $a^y - 1$, $a \geq 3$, because these expressions can be trivially factored and the characteristic function $s_n = \chi_{\mathcal{P}}(\lfloor f(n + x)\rfloor)$ then vanishes.

Acknowledgements. Several of the properties of Mersenn prime indices in §2 were found in work on number theory at Universität Potsdam. A geometrical representation of Mersenne numbers and the resulting congruence conditions for the exponents were developed in research completed at the University of Sydney.

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