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NOTES ON THE DEGENERATE HARMONIC NUMBERS AND POLYNOMIALS

BURAK KURT

Abstract. T. Kim and D. S. Kim ([8], [9], [10]) considered the degenerate harmonic numbers, the degenerate hyperharmonic numbers using the degenerate logarithm. Also, T. Kim and D. S. Kim [6] introduced the discrete Harmonic numbers and polynomials. They have investigated and gave some identities and explicit relations for these polynomials. Motivated by [6], we introduce the degenerate harmonic numbers and polynomials. We give explicit relations and some identities for these numbers and polynomials.

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1. Introduction and Notation

Dil et al. in ([4], [5]) introduced the harmonic numbers and investigated some relations for the related polynomials. Benjamin et al. in [1] gave combinatorial relations for the hyperharmonic numbers. Komatsu et al. in [11] and Wei et al. in [15] considered q-hyperharmonic numbers. Kim et al. in ([6]-[10]) introduced the discrete harmonic numbers and polynomials and gave some identities for these numbers and polynomials. Also, Rim et al. in [14] gave some relations and identities for the harmonic, hyperharmonic and Daehee numbers.

The n-th harmonic number is the n-th partial sum of the harmonic series

$$H_n := \sum_{k=1}^{n} \frac{1}{k}, \tag{1}$$

where $H_0 = 1$.

The generating function of the harmonic numbers are given by

$$\sum_{n=1}^{\infty} H_n \, t^n = -\frac{\log(1-t)}{1-t}, \tag{2}$$

([1], [4]-[14]).

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The Stirling numbers of the both kinds are defined by the following generating functions in ([6]-[14]) respectively

$$\sum_{k=m}^{\infty} S_1(k,m)\frac{t^k}{k!} = \frac{(\log(1+t))^m}{m!},\ k \geq m \tag{3}$$

and

$$\sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!} = \frac{\left(e^t-1\right)^k}{k!},\ n \geq k. \tag{4}$$

The well-known expansions of the logarithm is

$$\log(1+t) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{t^n}{n} \tag{5}$$

([1]-[13]).

Proposition 1.1. *The following equations satisfies for the harmonic numbers*

$$H_n - H_{n-1} = \frac{1}{n} \tag{6}$$

and

$$(-1)^n H_n\ n!\ S_2(m,n) = (-1)^m m. \tag{7}$$

Proof. Proof of (6);

From (2) and (5), for $n \geq 2$, we have (6) easily

$$\sum_{n=1}^{\infty} H_n\ t^n = -\frac{1}{1-t}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{t^n}{n}.$$

From the above equation, we have (6) easily.

Proof of (7);

By replacing t by $1-e^t$ in (2), we get

$$\sum_{n=1}^{\infty} H_n\ (1-e^t)^n = -te^{-t} = -t\sum_{l=0}^{\infty}\frac{(-1)^l\, t^l}{l!}.$$

Using (4) in above equation and comparing of the coefficients of t^n, we get (7). □

2. Degenerate Harmonic Numbers and Degenerate Harmonic Polynomials

In this section, firstly, we consider the degenerate harmonic numbers $H_{n,\lambda}$ and give two relations between the degenerate Harmonic numbers and the Stirling numbers of the both kinds. Motivated by the definition of the discrete Harmonic polynomials $H_{n,\lambda}^{(r)}(x)$ of order $r \in \mathbb{N}$ are defined by Kim and Kim in [6], we consider the new type degenerate harmonic polynomials $H_{n,\lambda}^{(r)}(x)$ of order r and give some explicit relations for the degenerate harmonic polynomials.

We define the generating function of the degenerate Harmonic number as

$$\sum_{n=1}^{\infty} H_{n,\lambda}\, t^n = \frac{-\ln\left(1-\ln(1+\lambda t)^{1/\lambda}\right)}{1-\ln(1+\lambda t)^{1/\lambda}}. \tag{8}$$

From (8), we get

$$\begin{aligned}\lim_{\lambda\to 0}\sum_{n=1}^{\infty} H_{n,\lambda}\, t^n &= \lim_{\lambda\to 0}\left\{\frac{-\ln\left(1-\ln(1+\lambda t)^{1/\lambda}\right)}{1-\ln(1+\lambda t)^{1/\lambda}}\right\}\\ &= \frac{-\ln(1-t)}{1-t} = \sum_{n=1}^{\infty} H_n\, t^n.\end{aligned}$$

Theorem 2.1. *The following relation holds true:*

$$H_{n,\lambda} = \sum_{m=1}^{n} H_{n-m,\lambda}\ \frac{(-1)^{m-1}}{m}\lambda^{m-1} + \frac{(m-1)!\lambda^{n-m}S_1(n,m)}{n!}. \tag{9}$$

Proof. From (3), (5) and (8), we write

$$\sum_{n=1}^{\infty} H_{n,\lambda}\, t^n - \lambda^{-1}\sum_{n=1}^{\infty} H_{n,\lambda}\, t^n \ln(1+\lambda t) = -\ln\left(1-\lambda^{-1}\ln(1+\lambda t)\right)$$

$$\begin{aligned}&\sum_{n=1}^{\infty} H_{n,\lambda}\, t^n - \lambda^{-1}\sum_{k=1}^{\infty} H_{k,\lambda}\, t^k \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\lambda^m t^m\\ &= -\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(-\lambda^{-1}\ln(1+\lambda t)\right)^n.\end{aligned} \tag{10}$$

L. H. S. of (10), we get

$$\sum_{n=1}^{\infty} H_{n,\lambda}\, t^n - \lambda^{-1}\sum_{n=2}^{\infty}\sum_{m=1}^{n} H_{n-m,\lambda}\ \frac{(-1)^{m-1}}{m}\lambda^m t^n. \tag{11}$$

R. H. S. of (10), we get

$$\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{(n-1)!\lambda^{k-n}\ S_1(k,n)}{k!}t^k. \tag{12}$$

For $n\geq 2$, comparing the coefficients of t^n on both sides of (11) and (12), we have (9). □

Theorem 2.2. *There is the following relations between the degenerate harmonic numbers and the Stirling numbers of the second kind*

$$H_k = \sum_{n=1}^{k-1} H_{n,\lambda}\ \lambda^{-n-k} n!\frac{S_2(k,n)}{k!}. \tag{13}$$

Proof. By replacing t by $\frac{e^{t/\lambda}-1}{\lambda}$ in (8) and using (4), we write

$$\sum_{n=1}^{\infty} H_{n,\lambda}\left(\frac{e^{t/\lambda}-1}{\lambda}\right)^n = \frac{-\ln(1-t)}{1-t} = \sum_{n=1}^{\infty} H_n\, t^n$$

$$\sum_{n=1}^{\infty} H_{n,\lambda}\,\lambda^{-n} n! \sum_{k=n}^{\infty} S_2(k,n)\,\frac{t^k}{k!}\lambda^{-k} = \sum_{n=1}^{\infty} H_n\, t^n$$
$$\sum_{k=1}^{\infty}\sum_{n=1}^{k-1} H_{n,\lambda}\,\lambda^{-n-k} n! S_2(k,n)\,\frac{t^k}{k!} = \sum_{k=1}^{\infty} H_k\, t^k.$$

Comparing the coefficients of t^l, we have (13). □

We consider the degenerate harmonic polynomials $H_{n,\lambda}^{(r)}(x)$ of order $r \in \mathbb{N}$ as

(14)
$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(r)}(x)\, t^n = \frac{\left(-\ln\left(1-\ln(1+\lambda t)^{1/\lambda}\right)\right)^{r+1}}{\ln(1+\lambda t)^{1/\lambda}\left(1-\ln(1+\lambda t)^{1/\lambda}\right)}\left(1-\ln(1+\lambda t)^{1/\lambda}\right)^{x}.$$

When $x=0$, $H_{n,\lambda}^{(r)}(0) = H_{n,\lambda}^{(r)}$ are called the degenerate harmonic number of order r.

For $x = r = 0$ in (14) and using (5), we get

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(0)}\, t^n = \frac{\left(-\ln\left(1-\ln(1+\lambda t)^{1/\lambda}\right)\right)}{\ln(1+\lambda t)^{1/\lambda}\left(1-\ln(1+\lambda t)^{1/\lambda}\right)}$$
$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(0)}\, t^n \ln(1+\lambda t)^{1/\lambda} = \frac{-\ln\left(1-\ln(1+\lambda t)^{1/\lambda}\right)}{1-\ln(1+\lambda t)^{1/\lambda}}$$
$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} H_{n-k,\lambda}^{(0)}\,\lambda^{-1}\frac{(-1)^{k-1}}{k} t^n \lambda^k = \sum_{n=1}^{\infty} H_{n,\lambda} t^n.$$

From the above equation, we have

$$\sum_{k=1}^{n} H_{n-k,\lambda}^{(0)}\,\lambda^{k-1}\frac{(-1)^{k-1}}{k} = H_{n,\lambda}.$$

From (14), we get

$$\begin{aligned}
&\lim_{\lambda\to 0}\sum_{n=0}^{\infty} H_{n,\lambda}^{(r)}(x) t^n \\
= &\lim_{\lambda\to 0}\frac{\left(-\ln\left(1-\ln(1+\lambda t)^{1/\lambda}\right)\right)^{r+1}}{\ln(1+\lambda t)^{1/\lambda}\left(1-\ln(1+\lambda t)^{1/\lambda}\right)}\left(1-\ln(1+\lambda t)^{1/\lambda}\right)^{x} \\
= &\frac{(-\ln(1-t))^{r+1}}{t(1-t)}(1-t)^{x}.
\end{aligned}$$

Theorem 2.3. *The following relation holds true*

$$\begin{aligned}
&\sum_{k=1}^{n} H_{n-k,\lambda}^{(1)}(x)\,\lambda^{k-1}\frac{(-1)^{k-1}}{k} \\
= &\sum_{k=1}^{n} H_{n-k,\lambda}^{(0)}(x)\sum_{l=1}^{k}\frac{(l-1)! S_1(k,l)\,\lambda^{k-l}}{k!}.
\end{aligned} \tag{15}$$

Proof. For $r=1$ in (14) and using (3), we write

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(1)}\ (x)\, t^n \frac{\ln(1+\lambda t)}{\lambda}$$

$$= \left(-\ln(1+\lambda t)^{1/\lambda}\right) \frac{\left(-\ln\left(1-\ln(1+\lambda t)^{1/\lambda}\right)\right)}{\left(1-\ln(1+\lambda t)^{1/\lambda}\right)} \left(1-\ln(1+\lambda t)^{1/\lambda}\right)^x$$

$$\sum_{m=0}^{\infty} H_{m,\lambda}^{(1)}\ (x)\, t^m \lambda^{-1} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\lambda^k t^k}{k}$$

$$(16) \qquad = \sum_{n=0}^{\infty} H_{n,\lambda}^{(0)}\ (x)\, t^n \left(-\ln\left(1-\ln(1+\lambda t)^{1/\lambda}\right)\right).$$

L. H. S. of (16), we get

$$(17) \qquad \sum_{n=1}^{\infty}\sum_{k=1}^{n} H_{n-k,\lambda}^{(1)}\ (x)\, \lambda^{k-1} \frac{(-1)^{k-1}}{k} t^n$$

and R. H. S. of (16), we get

$$-\sum_{m=1}^{\infty} H_{m,\lambda}^{(0)}\ (x)\, t^m \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} \left(\frac{-\ln(1+\lambda t)}{\lambda}\right)^l$$

$$= -\sum_{m=1}^{\infty} H_{m,\lambda}^{(0)}\ (x)\, t^m \sum_{k=1}^{\infty}\sum_{l=1}^{k} \frac{(l-1)! S_1(k,l)\, \lambda^{k-l}}{k!} t^k$$

$$(18) \qquad = \sum_{n=2}^{\infty}\sum_{k=1}^{n} H_{n-k,\lambda}^{(0)}\ (x) \sum_{l=1}^{k} \frac{(l-1)! S_1(k,l)\, \lambda^{k-l}}{k!} t^n.$$

For $n \geq 2$, comparing the coefficients of (17) and (18), we have (15). □

Theorem 2.4. *The following relation holds true:*

$$(19) \qquad H_{n,\lambda}^{(r)}\ (x) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k,\lambda}^{(r)} \sum_{l=0}^{k} (x)_l\ (-1)^l \frac{S_1(k,l)\, \lambda^{k-l}}{k!}.$$

Proof. From (14), we write

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(r)}\ (x)\, t^n = \sum_{m=0}^{\infty} H_{m,\lambda}^{(r)}\, t^m \sum_{l=0}^{\infty} \binom{x}{l} \left(-\ln(1+\lambda t)^{1/\lambda}\right)^l$$

$$= \sum_{m=0}^{\infty} H_{m,\lambda}^{(r)}\, t^m \sum_{l=0}^{\infty} (x)_l\ (-1)^l \lambda^{-l} \sum_{k=l}^{\infty} \frac{S_1(k,l)\, \lambda^k t^k}{k!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} H_{n-k,\lambda}^{(r)} \sum_{l=0}^{k} (x)_l\ (-1)^l \frac{S_1(k,l)\, \lambda^{k-l}}{k!} t^n,$$

where $(x)_l = x(x-1)\cdots(x-(l-1))$, $(x)_0 = 1$ and $l \geq 1$.

Comparing the coefficients of t^n on both sides, we have (19). □

Theorem 2.5. *The following relation between the degenerate harmonic polynomials and the Stirling numbers of the second kind holds true:*

$$H_{k+1} = \sum_{n=0}^{k} H_{n,\lambda}^{(0)} \ n!\frac{\lambda^{-n}}{k!}S_2(k,n). \tag{20}$$

Proof. For $x=r=0$ and by replacing t by $\frac{e^{\frac{t}{\lambda}}-1}{\lambda}$ in (14), we get

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(0)} \ (x)\left(\frac{e^{\frac{t}{\lambda}}-1}{\lambda}\right)^n = \frac{(-\ln(1-t))}{t(1-t)}$$

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(0)} \lambda^{-n} n! \sum_{n=l}^{\infty} S_2(l,n)\frac{t^l}{\lambda^l l!} = \frac{1}{t}\sum_{n=1}^{\infty} H_n \ t^n$$

$$\sum_{n=0}^{k} H_{n,\lambda}^{(0)} \frac{n!\lambda^{-n}}{k!} S_2(k,n) = H_{k+1}.$$

Comparing the coefficients of both sides, we have result. □

Proposition 2.6. *The following equations hold true:*

$$\begin{aligned} H_{n,\lambda}^{(r)} \ (x+y) &= \sum_{k=0}^{n} H_{n-k,\lambda}^{(r)} \ (x) \sum_{l=0}^{k} (y)_l \ \ (-1)^l S_1(k,l)\lambda^{k-l} \\ &= \sum_{k=0}^{n} H_{n-k,\lambda}^{(r)} \ (y) \sum_{l=0}^{k} (x)_l \ \ (-1)^l S_1(k,l)\lambda^{k-l}. \end{aligned}$$

The proof of this proposition is obtained easily from (14).

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Akdeniz University, Faculty of Educations, Mathematics of Department, Antalya TR-07058, Turkey
E-mail address: burakkurt@akdeniz.edu.tr