Advanced Studies in Contemporary Mathematics
33 (2023), No. 3, pp. 199 - 202

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2023.33.3.199

ONE-DIMENSIONAL PSEUDOREPRESENTATIONS OF ALMOST CONNECTED LIE GROUPS

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Abstract. We describe the one-dimensional pseudorepresentations of almost connected Lie groups.

§ 1. Introduction

For the definitions, notation, and generalities concerning pseudocharacters, quasicharacters, pseudorepresentations, and quasirepresentations, see [1–4]. In particular, recall that a mapping π of a given group G into the field $\mathbb{C}$ of complex numbers is said to be a *one-dimensional quasirepresentation* of G on E if $\pi(e_G) = 1 \in \mathbb{C}$, where e_G stands for the identity element of G and if

$$|\pi(g_1g_2) - \pi(g_1)\pi(g_2)| \leq \varepsilon, \qquad g_1, g_2 \in G,$$

for some $\varepsilon \geq 0$, which is usually assumed to be sufficiently small, and the least upper bound of $|\pi(g_1g_2) - \pi(g_1)\pi(g_2)|$ for a one-dimensional quasirepresentation π is referred to as the *defect* of π; a one-dimensional quasirepresentation π of G is said to be a one-dimensional *pseudorepresentation* of G if $\pi(g^n) = \pi(g)^n$ for any $n \in \mathbb{Z}$ and $g \in G$.

We prove here that a one-dimensional pseudorepresentation of an almost connected Lie group is the product of a one-dimensional pseudorepresentation of the group naturally extending a one-dimensional pseudorepresentation of the connected component of the almost connected locally compact

2010 *Mathematics Subject Classification.* Primary 22A99, Secondary 22E99.
Submitted May 19, 2023.
Key words and phrases. almost connected Lie group, Guichardet–Wigner pseudocharacter, one-dimensional pseudorepresentation.

group, which is a product of an exponential of the Guichardet–Wigner character on this connected component and a central (ordinary) character of this connected component, and of an ordinary character, of a Lee supplementary subgroup of the group, which is trivial on the intersection of the Lee subgroup with the connected component.

§ 2. Preliminaries

Recall Dong Hoon Lee's supplement theorem (Theorem 2.13 of [5]): every almost connected locally compact group G with the connected component G_0 (i.e., a locally compact group G for which the quotient group G/G_0 is compact) admits a totally disconnected compact subgroup D such that $G = G_0D$.

Recall a result we need below.

Theorem. *A one-dimensional pseudorepresentation π with a defect $\varepsilon < q_0 = \sqrt{3}/5$ of an almost connected locally compact group G with the connected component G_0 is equal to an exponential of a pseudocharacter on G coinciding with π on G_0 and with a real character on a Lee's supplementary subgroup D of the group G if and only if π is trivial on D.*

For the proof, see [6].

§ 3. Main Result

Theorem 1. *Let G be an almost connected Lie group, let G_0 be the connected component of G, and let R be the radical (the maximal connected solvable normal subgroup) of G (it coincides with the radical of G_0). Let S be a (semisimple) Levi subgroup and let D be a (finite) Lee subgroup of G. Let π be a locally bounded one-dimensional pseudorepresentation of G. Then there are a unitary character ϕ of D which is trivial on $D \cap G_0$, a unitary one-dimensional pseudorepresentation θ of S, and an ordinary unitary central (with respect to G) unitary character ψ of R, where θ coincides with psi on $S \cap R$, such that the one-dimensional pseudorepresentation π of the almost connected Lie group G has the form*

$$\pi(g) = \phi(d)\theta(s)\psi(r), \qquad \text{for} \quad g = dsr, \quad \text{where} \quad d \in D, \quad s \in S, \quad r \in R.$$

Proof. Let G_0 be the connected component of G, and let R be the radical (the maximal connected solvable normal subgroup) of G. Since R is connected, it follows that $R \subset G_0$. Since G_0 is characteristic, it follows that R is the radical of G_0 as well. Let S be a (semisimple) Levi subgroup of G_0 and let D be a (finite) Lee subgroup.

Let π be a one-dimensional representation of G. As was shown in [6], the restriction of π to G_0 is an exponential of a pseudocharacter ρ on G_0. On p. 121 of [6], it is described how to extend the pseudocharacter ρ on G_0 with defect ε to a pseudocharacter σ on G with defect $4\ln(1+\varepsilon)$; σ vanishes automatically on D since D is finite. Then the appropriate exponential θ of σ coincides with π on G_0. Therefore, $\pi\cdot\sigma^{-1}$ is a one-dimensional pseudorepresentation ϕ of G that is trivial on G_0. Thus, this pseudorepresentation is defined by θ and an ordinary complex character ϕ of the group $D/D\cap G_0$ which is naturally isomorphic to G/G_0 and hence ϕ can be viewed as a character of G.

It remains to clarify the structure of ρ and hence of σ. The restriction of ρ to S is a Guichardet–Wigner pseudocharacter on S, and its restriction to the discrete group $S\cap R$ defines an ordinary real character χ of $S\cap R$. Since χ is defined by an ordinary real character on R which is the restriction of ρ to R, and thus the restriction of π to R is a unitary exponential ψ of χ (recall that unbounded one-dimensional pseudorepresentations are ordinary one-dimensional representations), it follows that this real character is central with respect to G. Then the mapping

$$g = dsr \mapsto \psi(r), \qquad d \in D, \quad s \in S, \quad r \in R,$$

is a central unitary character of G, and, finally,

$$\pi(g) = \phi(d)\theta(s)\psi(r), \quad \text{for } g = dsr, \text{ where } d \in D,\ s \in S,\ r \in R,$$

and the nonuniqueness of the representation of $g \in G$ in the form $g = dsr$ does not affect the result since ϕ is trivial on $D\cap G_0$ and the character ψ agrees with the one-dimensional pseudorepresentation θ on $S\cap R$.

This completes the proof of the theorem.

In particular, during the above proof, we have proved the following assertion.

Theorem 2. *Every pseudocharacter on an almost connected Lie group G is the sum of a Guichardet–Wigner pseudocharacter on G and an ordinary central real character on G agreeing with the Guichardet–Wigner pseudocharacter on the intersection of the radical and a Levi subgroup of the connected component of G.*

§ 4. Discussion

Recall that the construction of σ is nontrivial (see p. 121 of [6]).

Acknowledgments

I thank Professor Taekyun Kim for the invitation to publish this paper in the Advanced Studies of Contemporary Mathematics.

Funding

The research was supported by FSI SRISA RAS according to the project no. 0580-2021-007 (Reg. no. 121031300051-3).

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