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# CONTINUITY CRITERIA FOR LOCALLY BOUNDED ENDOMORPHISMS OF CENTRAL EXTENSIONS OF PERFECT LIE GROUPS

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Abstract.

We prove that every locally bounded endomorphism, of a linear connected Lie central extension of a connected perfect Lie group, taking the center of the group into the center is continuous if and only if it is continuous on the center. We also prove that, if $Z$ is a connected Abelian group without nontrivial compact subgroups, $H$ is a connected perfect Lie group and the short sequence of Lie groups $\{e\} \to Z \to G \to H \to \{e\}$ is exact, then every locally bounded endomorphism of $G$ taking the center of the group into the center is continuous if and only if it is continuous on the center of $G$.

## § 1. Introduction

As was proved in [1], every locally bounded endomorphism of a (real analytic) reductive Lie group is continuous if and only if it is continuous on the center of the group. In this paper, we prove that every locally bounded endomorphism of a linear connected Lie central extension of a connected perfect Lie group is continuous if and only if it is continuous on the center, which refines the main result of [1]. For not necessarily linear perfect Lie groups,

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## § 2. Preliminaries

Let us recall some information needed below.

A (not necessarily continuous) homomorphism $\pi$ of a topological group $G$ into a topological group $H$ is said to be *relatively compact* if there is a neighborhood $U = U_{e_G}$ of the identity element $e_G$ in $G$ whose image $\pi(U)$ has compact closure in $H$. Obviously, a homomorphism into a locally compact group is relatively compact if and only if it is *locally bounded*, i.e., there is a neighborhood $U_e$ whose image is contained in some element of the filter $\mathfrak{V}$ of neighborhoods of $e_V$ having compact closure.

Let us also recall the notion of discontinuity group of a homomorphism $\pi$ of a topological group $G$ into a topological group $H$, see [2] and [3]. Let $\mathfrak{U} = \mathfrak{U}_G$ be the filter of neighborhoods of $e_G$ in $G$. For every (not necessarily continuous) locally relatively compact homomorphism $\pi$ of $G$ into $H$, the set

$$\mathrm{DG}(\pi) = \bigcap_{U \in \mathfrak{U}} \overline{\pi(U)}$$

is called the discontinuity group of $\pi$. Here and further, the bar stands for the closure in the corresponding topology (here the closure is taken in the topology of $H$). (See Definition 1.1.1 of [2].)

The discontinuity group of a homomorphism has some important properties. Under the above conditions, the set $\mathrm{DG}(\pi)$ is a compact subgroup of the topological group $H$ and a compact normal subgroup of the closed subgroup $\overline{\pi(G)}$ of $H$. Moreover, the filter basis $\{\overline{\pi(U)} \mid U \in \mathfrak{U}\}$ converges to $\mathrm{DG}(\pi)$, and the homomorphism $\pi$ is continuous if and only if $\mathrm{DG}(\pi) = \{e_H\}$. (See Theorem 1.1.2 of [2].) If $G$ is a connected Lie group, then $\mathrm{DG}(\pi)$ is a compact connected subgroup of $H$. (See Lemma 1.1.6 of [2].) Let $G$ be a connected Lie group, let $N$ be a closed normal subgroup of $G$, and let $\pi$ be a locally bounded homomorphism of $G$ into a locally compact group $H$. Let $M$ be the discontinuity group of the restriction $\mathrm{DG}(\pi|_N)$. Then $M$ is a closed normal subgroup of the compact discontinuity group $\mathrm{DG}(\pi)$, and the corresponding quotient group $\mathrm{DG}(\pi)/M$

is isomorphic to the discontinuity group $\mathrm{DG}(\psi)$ of the homomorphism $\psi$ of $G$ obtained as the composition of the homomorphism $\pi$ and the canonical homomorphism $\overline{\pi(G)} \to \overline{\pi(G)}/M$. (See Lemma 1.1.7 of [2].)

## § 3. Main result

**Theorem 1.** *Every locally bounded endomorphism, of a linear connected Lie central extension of a connected perfect Lie group, taking the center of the group into the center is continuous if and only if it is continuous on the center of the group.*

**Proof.** Obviously, if an endomorphism of a group is continuous, then it is continuous on the center of the group, and thus it suffices to prove the converse assertion. Let $H$ be a perfect Lie group, let $Z$ be an Abelian Lie group, and let a linear connected Lie group $G$ enter a short exact sequence $\{e\} \to Z \xrightarrow{\iota} G \xrightarrow{\rho} H \to \{e\}$ with a continuous embedding $\iota$ and the canonical epimorphism $\rho$ of $G$ onto $H$, which is isomorphic to the quotient group $G/Z$. Then the commutator subgroup $G'$ of $G$ is closed (see Chap. 3, Exercise 41(e) of [2]) and is taken by $\rho$ onto the commutator subgroup $DH$ of $H$. Moreover, $G'$ is in a natural one-to-one correspondence with $DH$. Indeed, for every $z_1, z_2 \in Z$ and $b, c \in G$, we have $bz_1c_2(bz_1)^{-1}(cz_2)^{-1} = bzb^{-1}c^{-1} = [b, c]$ and, therefore, the commutator of $bZ$ and $cZ$ is $[b, c]Z$ for every $b, c \in G$. Thus, the commutator subgroup of $G$ is naturally isomorphic to that of $H$. However, $H$ is perfect, and hence $DH = H$. Hence the closed subgroup $G'$ of $G$ is naturally isomorphic to $H$, and every element of $G$ is a product of an element of $G'$ and an element of $Z$.

Let $\pi$ be a locally bounded endomorphism of $G$ taking the center of the group $G$ into the center. It naturally takes the commutator subgroup $G'$ into itself. The ($\iota$-image of the) group $Z$ (which we identify with $Z$) is central by the assumption concerning the extension, and hence the restriction of $\pi$ to $Z$ is continuous by the assumption concerning $\pi$. However, since $G$ is linear, it follows that $\pi$ can be regarded as a locally bounded finite-dimensional linear representation of $G$. By Theorem 1.3.2 of [2], the restriction of $\pi$ to $G'$ is continuous (see also [3] and [4]). Hence the representation $\pi$ is separately continuous with respect to the subgroups $Z$ and $G'$.

By the Namioka theorem [5], the representation $\pi$ has a point of joint continuity, and therefore is continuous. This completes the proof of Theorem 1.

For some Abelian normal subgroups, the condition that the extension is linear can be omitted.

**Theorem 2.** *Every locally bounded endomorphism, of a connected Lie extension $G$ by an Abelian Lie group $Z$ without nontrivial compact subgroups of a connected perfect Lie group $H$ ($\{e\} \to Z \to G \to H \to \{e\}$), taking the center of the group into the center, is continuous if and only if it is continuous on the center $Z$ of $G$.*

**Proof.** As above, it suffices to prove the "if" part. Let $\pi$ be a locally bounded endomorphism of $G$. Since $H$ is perfect, it follows from Lemma 1.1.7 of [2] that the discontinuity group $\mathrm{DG}(\pi)$ of $\pi$ can have no points outside $Z$. Thus, $\mathrm{DG}(\pi) \subset Z$, where $\mathrm{DG}(\pi)$ is a compact group. By the assumption concerning $Z$, this means that $\mathrm{DG}(\pi) = \{e\}$, and therefore $\pi$ is continuous. This completes the proof of Theorem 2.

## § 4. Comments

Note that, by Harish-Chandra's theorem [6], Theorem 1 holds if the connected perfect Lie group is linear and $Z$ is a vector group.

The phenomenon relating the continuity of a locally bounded endomorphism to the continuity of its restriction to the center is specific, see the example in [1].

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