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ON $\gamma-$DOMINATION ZAGREB INDICES OF GRAPHS AND SOME GRAPH OPERATIONS

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Abstract. In graph theory, topological indices and domination parameters are essential topics. A dominating set for a graph $G = (V(G), E(G))$ is a subset D of $V(G)$ such that every vertex not in D is adjacent to at least one vertex of D. Hanan Ahmed et al. introduced novel topological indices known as $\gamma-$domination topological indices. In this research work, we find exact values for $\gamma-$domination Zagreb indices of some families of graphs including the join and corona product. Some bounds for these new topological indices are also found.

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1. Introduction

For a connected simple graph $G = (V(G), E(G))$, $V(G)$ and $E(G)$ denote the set of all vertices and edges, respectively. The degree of a vertex $u \in G$, $d_G(u)$ or $d(u)$ is the number of edges that are incident to u in G. $\overline{G}$ is the complement of G, having the same vertex set of G so that two vertices in $\overline{G}$ are neighboring if and only if they are not neighboring in G. If for any two vertices $u, v \in V$, there exists a (u, v)-path in G, then G is connected, otherwise, G is said to be disconnected. A molecular graph [13, 14] is a connected simple graph such that the vertices and edges are supposed to be atoms and chemical bonds respectively. In mathematical chemistry, molecular descriptors have a pioneering role in the field of the quantum structure-property relationship and the quantum structure-activity relationship. Among them, a salient area has been preserved for known topological indices. Topological indices are numerical parameters of the graph, such that these parameters are the same for the isomorphic graphs. Among the topological indices defined in the initial phase, Zagreb indices are related to the most common molecular descriptors, first introduced by Gutman and Trinajstić [15]. The Zagreb indices $M_1(G)$ and $M_2(G)$ are defined as follows:

$$M_1(G) = \sum_{v \in V(G)} d^2(v) = \sum_{uv \in E(G)} (d(u) + d(v))$$

$$M_2(G) = \sum_{uv \in E(G)} d(u)d(v).$$

A set $D \subseteq V$ is said to be a dominating set of G, if for any vertex $v \in V \setminus D$ there exists a vertex $u \in D$ such that u and v are adjacent. A dominating set is said to be minimal dominating set if no proper subset of it is a dominating set [11]. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a minimal dominating set in G. For more details on domination in graphs, see [7, 8, 10, 17, 19], for some applications of dominating sets in networks see [9, 18]. A dominating set of G of minimum cardinality is said to be a minimum dominating set.

In [1, 16] H. Ahmed et al., have introduced new degree-based topological indices called domination and $\gamma-$domination topological indices which are based on the domination degree and domination value, respectively. Observing the importance of topological descriptors and dominating sets we have introduced the first, second and modified $\gamma-$domination indices in Definition 1.2.

Definition 1.1. [12] *For each vertex $v \in V(G)$, the domination value of v denoted by $d_\gamma(v)$ or $d_{\gamma G}(v)$ is defined as:*
$d_\gamma(v) = |\{D \subseteq V(G) : D \text{ is a minimum dominating set and } v \in D\}|$.

Definition 1.2. [1] *Let G be any graph. Then first, second and modified first $\gamma-$domination Zagreb indices are defined as:*

$$\gamma M_1(G) = \sum_{v \in V(G)} d_\gamma^2(v) \quad , \quad \gamma M_2(G) = \sum_{uv \in E(G)} d_\gamma(u) d_\gamma(v) \ ,$$

$$\gamma M_1^*(G) = \sum_{uv \in E(G)} (d_\gamma(u) + d_\gamma(v)).$$

The reader is encouraged to refer to the papers [1, 2] for some applications and [3, 4] for the mathematical properties of domination and $\gamma-$domination topological indices. We use $T_{m\gamma}(G)$ to denote the total number of minimum dominating sets of G. In this paper, we study the above indices for some graphs and graph operations.

2. $\gamma-$Domination Zagreb indices of graphs and graph operations

In this section, we study $\gamma-$domination Zagreb indices for some standard graphs, and some graph families. Also, we will determine the exact value of $\gamma-$domination Zagreb indices for some graph operations.

Lemma 2.1.

(1) *For star graph S_{r+1},* $d_\gamma(v) = \begin{cases} 1, & \text{if } v \text{ is the center vertex;} \\ 0, & \text{otherwise.} \end{cases}$

(2) *If $G \cong \overline{S_{r+1}}$, then* $d_\gamma(v) = \begin{cases} r, & \text{if } v \text{ is the center vertex of } S_{r+1}; \\ 1, & \text{otherwise.} \end{cases}$

(3) *For complete graph K_n,*
$d_\gamma(v) = 1$ *for every* $v \in V(K_n)$.

(4) *For complete bipartite graph $K_{r,s}$, with $r, s \geq 3$ we have, $d_\gamma(v) = d(v)$.*

(5) *For double star graph $S_{r+1,s+1}$, with $r \geq 2$ and $s \geq 2$*
$d_\gamma(v) = \begin{cases} 1, & \text{if } v \text{ is a central vertex;} \\ 0, & \text{otherwise.} \end{cases}$

Proposition 2.2.

(1) *For star graph*
S_{r+1}, $\gamma M_1(S_{r+1}) = 1,\ \gamma M_2(S_{r+1}) = 0,\ \gamma M_1^*(S_{r+1}) = r.$

(2) *If $G \cong \overline{S_{r+1}}$, then*
$\gamma M_1(G) = r(r+1),\ \gamma M_2(G) = \frac{r(r-1)}{2},\ \gamma M_1^*(G) = r(r-1).$

(3) *For complete graph K_n*
$\gamma M_1(K_n) = n,\ \gamma M_2(K_n) = \frac{n(n-1)}{2},\ \gamma M_1^*(K_n) = n(n-1).$

(4) *For complete bipartite graph $K_{r,s}$*
$\gamma M_1(K_{r,s}) = \gamma M_1^*(K_{r,s}) = M_1(K_{r,s}), \gamma M_2(K_{r,s}) = M_2(K_{r,s}).$

(5) *For double star graph $S_{r+1,s+1}$, with $r \geq 2$ and $s \geq 2$*
$\gamma M_1(S_{r+1,s+1}) = 2,\ \gamma M_2(S_{r+1,s+1}) = 1,\ \gamma M_1^*(S_{r+1,s+1}) = 2 + r + s.$

A firefly graph [5] $F_{r,s,t}$ is a graph on $n = 2r + s + 2t + 1$ vertices that consists of r triangles, t pendant paths of length 2 and s pendant edges, sharing a common vertex.

Let w be the common vertex and let $v_1, v_2, \cdots, v_{2r}$ be the vertices of degree 2 of r triangles, $u_1, u_2, \cdots, u_s$ be the pendant vertices of s pendant edges and $x_1, x_2, \cdots, x_t, x_{t+1}, x_{t+2}, \cdots, x_{2t}$ be the vertices of pendant path of length 2.

Lemma 2.3.

(1) *Let* $G \cong F_{r,s,t}$, *with* r, s *not both zero and* $t \neq 0$. *Then*
$$d_\gamma(v) = \begin{cases} 2, & \text{if } v{=}w; \\ 1, & \text{if } v = x_i; \\ 0, & \text{otherwise.} \end{cases}$$

(2) *If* $G \cong F_{r,s,0}$, *with* r, s *not both zero, then*
$$d_\gamma(v) = \begin{cases} 1, & \text{if } v{=}w; \\ 0, & \text{otherwise.} \end{cases}$$

(3) *If* $G \cong F_{0,0,t}$, *then*
$$d_\gamma(v) = \begin{cases} 1, & \text{if } d(x_i) = 2; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.4.

(1) *If* $G \cong F_{r,s,t}$, *with* r, s *not both zero and* $t \neq 0$, *then* $\gamma M_1(G) = 4 + 2t,\ \gamma M_2(G) = 3t,\ \gamma M_1^*(G) = 2s + 4r + 5t.$

(2) *Let* $G \cong F_{r,s,0}$, *with* r, s *not both zero. Then* $\gamma M_1(G) = 1,\ \gamma M_2(G) = 0,\ \gamma M_1^*(G) = s + 2r.$

(3) *If* $G \cong F_{0,0,t}$, *then* $\gamma M_1(G) = t,\ \gamma M_2(G) = 0,\ \gamma M_1^*(G) = 2t.$

Proof. (1) Let $G \cong F_{r,s,t}$, with r, s not both zero and $t \neq 0$. Consider the common vertex w, the set $\{x_1, x_2, \cdots, x_t\}$ of vertices of degree two in the t pendant paths, and the set $\{x_{t+1}, x_{t+2}, \cdots, x_{2t}\}$, of vertices of degree one in the t pendant paths. There are two minimum dominating sets $D_1 = \{w, x_1, x_2, \cdots, x_t\}$, $D_2 = \{w, x_{t+1}, x_{t+2}, \cdots, x_{2t}\}$ so that, $T_{m\gamma}(G) = 2$. By Lemma 2.3 (1) and Definition 1.2, we get $\gamma M_1(G) = 4 + 2t,\ \gamma M_2(G) = 3t,\ \gamma M_1^*(G) = 2s + 4r + 5t.$

(2) Let $G \cong F_{r,s,0}$, with r, s not both zero. In this case, G has only one minimum dominating set and it contains only the common vertex. Hence, applying Lemma 2.3 (2) and Definition 1.2, we get the required results.

(3) If $G \cong F_{0,0,t}$, then G has only one minimum dominating set and it contains all noncentral vertices of degree two. Applying Lemma 2.3 (3) and Definition 1.2, we get,
$\gamma M_1(G) = t,\ \gamma M_2(G) = 0,\ \gamma M_1^*(G) = 2t.$

□

Observation 2.5. *For any graph* G *with domination number* $\gamma(G)$, *and total number of minimum dominating sets* $T_{m\gamma}(G)$, *we have*
$$\sum_{u \in V(G)} d_{\gamma G}(u) = \gamma(G) T_{m\gamma}(G).$$

Lemma 2.6. *For any graph* G, *let* $H \cong G \circ K_n$, *where* K_n *is a complete graph. Then* $T_{m\gamma}(H) = (n+1)^{|V(G)|}$, *and* $d_{\gamma H}(v) = (n+1)^{|V(G)|-1}$.

Theorem 2.7. *Let* G *be any graph. Then*

(1) $\gamma M_1(G \circ K_n) = |V(G)|(n+1)^{2|V(G)|-1}$.

(2) $\gamma M_2(G \circ K_n) = \frac{1}{2}(n+1)^{2(|V(G)|-1)}\Bigg(2|E(G)| + n(n+1)|V(G)|\Bigg).$

(3) $\gamma M_1^*(G \circ K_n) = (n+1)^{|V(G)|-1}\Bigg(2|E(G)| + n(n+1)|V(G)|\Bigg).$

Proof. Note that $|V(G\circ K_n)| = |V(G)|(1+n)$. Hence, by the definition of first $\gamma-$domination Zagreb index and Lemma 2.6, we get

$$\gamma M_1(G\circ K_n) = |V(G)|\big(n+1\big)^{2|V(G)|-1},$$

We can partition the edges of $G\circ K_n$ as follows: $E(G)$, $E(K_n)$ and E_1, the set of all edges connecting vertices of G and K_n. So, we have

$$\begin{aligned}
\gamma M_2(G\circ K_n) &= \sum_{uv\in E(G\circ K_n)} d_\gamma(u)d_\gamma(v)\\
&= \sum_{uv\in E(G)} d_\gamma(u)d_\gamma(v) + \sum_{uv\in E(K_n)} d_\gamma(u)d_\gamma(v) + \sum_{uv\in E_1} d_\gamma(u)d_\gamma(v)\\
&= |E(G)|(n+1)^{2(|V(G)|-1)} + (n+1)^{2(|V(G)|-1)}|E(K_n)||V(G)|\\
&\quad + n|V(G)|(n+1)^{2(|V(G)|-1)}\\
&= \frac{1}{2}(n+1)^{2(|V(G)|-1)}\Big(2|E(G)| + n(n+1)|V(G)|\Big).
\end{aligned}$$

$$\begin{aligned}
\gamma M_1^*(G\circ K_n) &= \sum_{uv\in E(G\circ K_n)} [d_\gamma(u)+d_\gamma(v)]\\
&= \sum_{uv\in E(G)} 2(n+1)^{|V(G)|-1} + \sum_{uv\in E(K_n)} 2(n+1)^{|V(G)|-1} + \sum_{uv\in E_1} 2(n+1)^{|V(G)|-1}\\
&= (n+1)^{|V(G)|-1}\Big(2|E(G)| + n(n+1)|V(G)|\Big).
\end{aligned}$$

□

Theorem 2.8. *If G is any graph, and $H\cong G\circ\overline{K}_n$ with $n>1$, then* $\gamma M_1(H) = |V(G)|$, $\gamma M_2(H) = |E(G)|$, $\gamma M_1^*(H) = 2|E(G)| + n|V(G)|$.

Proof. Let $H\cong G\circ\overline{K}_n$ with $n>1$. It is easy to see that only the set of vertices of G form a minimum dominating set of H so, $T_{m\gamma}(H)=1$. Hence

$$d_{\gamma H}(v) = \begin{cases} 1, & \text{if } v\in V(G);\\ 0, & \text{otherwise.}\end{cases}$$

Applying Definition 1.2, we get the required results. □

A join [6] of two graphs G and H denoted by $G+H$, with disjoint vertex sets $V(G)$ and $V(H)$, is the graph on the vertex set $V(G)\cup V(H)$ and the edge set $E(G)\cup E(H)\cup\{u_1u_2 : u_1\in V(G), u_2\in V(H)\}$.

Lemma 2.9. *For any two graphs G and H, with $\gamma(G)=2$ and $\gamma(H)>2$* $T_{m\gamma}(G+H) = T_{m\gamma}(G) + |V(G)||V(H)|$ *and*

$$d_{\gamma G+H}(v) = \begin{cases} d_{\gamma G}(v) + |V(H)|, & \text{if } v\in V(G);\\ |V(G)|, & \text{if } v\in V(H).\end{cases}$$

Theorem 2.10. *For any two graphs G and H, with $\gamma(G)=2$ and $\gamma(H)>2$*

(1) $\gamma M_1(G+H) = \gamma M_1(G) + 2|V(H)|\gamma(G)T_{m\gamma}(G) + |V(G)||V(H)|^2 + |V(G)|^2|V(H)|$.

(2)

$$\begin{aligned}
\gamma M_2(G+H) &= \gamma M_2(G) + |V(H)|\gamma M_1^*(G) + |V(H)|^2|E(G)| + |V(G)|^2|E(H)|\\
&\quad + |V(G)||V(H)|\Big(\gamma(G)T_{m\gamma}(G) + |V(G)||V(H)|\Big).
\end{aligned}$$

(3)

$$\gamma M_1^*(G+H) = \gamma M_1^*(G) + 2\Big(|V(H)||E(G)| + |V(G)||E(H)|\Big) + |V(H)|\gamma(G)T_{m\gamma}(G) + |V(G)||V(H)|^2 + |V(G)|^2|V(H)|.$$

Proof.

$$\begin{aligned}\gamma M_1(G+H) &= \sum_{v\in V(G+H)} d^2_{\gamma G+H}(v) = \sum_{v\in V(G)} (d_{\gamma G}(v) + |V(H)|)^2 + \sum_{v\in V(H)} |V(G)|^2 \\ &= \gamma M_1(G) + 2|V(H)| \sum_{v\in V(G)} d_{\gamma G}(v) + |V(G)||V(H)|^2 + |V(G)|^2|V(H)| \\ &= \gamma M_1(G) + 2|V(H)|\gamma(G)T_{m\gamma}(G) + |V(G)||V(H)|^2 + |V(G)|^2|V(H)|.\end{aligned}$$

$$\begin{aligned}\gamma M_2(G+H) &= \sum_{uv\in E(G+H)} d_{\gamma G+H}(u)d_{\gamma G+H}(v) \\ &= \sum_{uv\in E(G)} d_{\gamma G+H}(u)d_{\gamma G+H}(v) + \sum_{uv\in E(H)} d_{\gamma G+H}(u)d_{\gamma G+H}(v) \\ &+ \sum_{u\in V(G), v\in V(H)} d_{\gamma G+H}(u)d_{\gamma G+H}(v)\end{aligned}$$

We will find every part independently,

$$\begin{aligned}\sum_{uv\in E(G)} d_{\gamma G+H}(u)d_{\gamma G+H}(v) &= \sum_{uv\in E(G)} (d_{\gamma G}(u) + |V(H)|)(d_{\gamma G}(v) + |V(H)|) \\ &= \gamma M_2(G) + |V(H)|\gamma M_1^*(G) + |V(H)|^2|E(G)|\end{aligned}$$

$$\sum_{uv\in E(H)} d_{\gamma G+H}(u)d_{\gamma G+H}(v) = \sum_{uv\in E(H)} |V(G)||V(G)| = |V(G)|^2|E(H)|$$

$$\begin{aligned}\sum_{u\in V(G), v\in V(H)} d_{\gamma G+H}(u)d_{\gamma G+H}(v) &= \underbrace{(d_{\gamma G}(u_1) + |V(H)|)|V(G)| + \cdots + (d_{\gamma G}(u_1) + |V(H)|)|V(G)|}_{|V(H)| times} \\ &+ \underbrace{(d_{\gamma G}(u_2) + |V(H)|)|V(G)| + \cdots + (d_{\gamma G}(u_2) + |V(H)|)|V(G)|}_{|V(H)| times} + \cdots \\ &+ \underbrace{(d_{\gamma G}(u_{|V(G)|}) + |V(H)|)|V(G)| + \cdots + (d_{\gamma G}(u_{|V(G)|}) + |V(H)|)|V(G)|}_{|V(H)| times} \\ &= |V(G)||V(H)| \\ &\underbrace{\big(d_{\gamma G}(u_1) + |V(H)| + d_{\gamma G}(u_2) + |V(H)| + \cdots + d_{\gamma G}(u_{|V(G)|}) + |V(H)|\big)}_{|V(G)| times} \\ &= |V(G)||V(H)|\Big(\sum_{u\in V(G)} d_{\gamma G}(u) + |V(G)||V(H)|\Big) \\ &= |V(G)||V(H)|\Big(\gamma(G)T_{m\gamma}(G) + |V(G)||V(H)|\Big)\end{aligned}$$

Hence,

$$\gamma M_2(G+H) = \gamma M_2(G) + |V(H)|\gamma M_1^*(G) + |V(H)|^2|E(G)| + |V(G)|^2|E(H)| + |V(G)||V(H)|\Big(\gamma(G)T_{m\gamma}(G) + |V(G)||V(H)|\Big).$$

Similarly, one can get

$$\gamma M_1^*(G+H) = \gamma M_1^*(G) + 2\Big(|V(H)||E(G)| + |V(G)||E(H)|\Big) + |V(H)|(\gamma(G)T_{m\gamma}(G)) + |V(G)||V(H)|^2 + |V(G)|^2|V(H)|.$$

□

Lemma 2.11. *Let G and H be any two graphs with $\gamma(G) > 2$ and $\gamma(H) = 2$. Then $T_{m\gamma}(G+H) = T_{m\gamma}(H) + |V(G)||V(H)|$ and*
$$d_{\gamma G+H}(v) = \begin{cases} |V(H)|, & \text{if } v \in V(G); \\ d_{\gamma H}(v) + |V(G)|, & \text{if } v \in V(H). \end{cases}$$

Using a similar method, one can prove the following theorem:

Theorem 2.12. *If G and H are any two graphs with $\gamma(G) > 2$ and $\gamma(H) = 2$, then*

(1) $\gamma M_1(G+H) = \gamma M_1(H) + 2|V(G)|\gamma(H)T_{m\gamma}(H) + |V(G)||V(H)|^2 + |V(G)|^2|V(H)|$.

(2)
$$\gamma M_2(G+H) = \gamma M_2(H) + |V(G)|\gamma M_1^*(H) + |V(H)|^2|E(G)| + |V(G)|^2|E(H)| + |V(G)||V(H)|\Big(\gamma(H)T_{m\gamma}(H) + |V(G)||V(H)|\Big).$$

(3)
$$\gamma M_1^*(G+H) = \gamma M_1^*(H) + 2\Big(|V(G)||E(H)| + |V(H)||E(G)|\Big) + |V(G)|\gamma(H)T_{m\gamma}(H) + |V(G)||V(H)|^2 + |V(G)|^2|V(H)|.$$

Lemma 2.13. *Let G and H be any two graphs. If $\gamma(G) = 2$ and $\gamma(H) = 2$, then $T_{m\gamma}(G+H) = T_{m\gamma}(G) + T_{m\gamma}(H) + |V(G)||V(H)|$ and*
$$d_{\gamma G+H}(v) = \begin{cases} d_{\gamma G}(v) + |V(H)|, & \text{if } v \in V(G); \\ d_{\gamma H}(v) + |V(G)|, & \text{if } v \in V(H). \end{cases}$$

Theorem 2.14. *Let G and H be any two graphs. If $\gamma(G) = 2$ and $\gamma(H) = 2$, then*

(1)
$$\gamma M_1(G+H) = \gamma M_1(G) + \gamma M_1(H) + 2|V(H)|\gamma(G)T_{m\gamma}(G) + 2|V(G)|\gamma(H)T_{m\gamma}(H) + |V(G)|\Big(|V(H)|^2 + |V(G)||V(H)|\Big).$$

(2)
$$\gamma M_2(G+H) = \gamma M_2(G) + \gamma M_2(H) + |V(H)|\gamma M_1^*(G) + |V(G)|\gamma M_1^*(H) + |E(G)||V(H)|^2 + |E(H)||V(G)|^2 + \Big(|V(G)||V(H)| + \gamma(G)T_{m\gamma}(G)\Big)\Big(|V(G)||V(H)| + \gamma(H)T_{m\gamma}(H)\Big).$$

(3)
$$\gamma M_1^*(G+H) = \gamma M_1^*(G) + \gamma M_1^*(H) + |V(H)|\Big(\gamma(G)T_{m\gamma}(G) + 2|E(G)| + |V(G)|^2\Big) + |V(G)|\Big(\gamma(H)T_{m\gamma}(H) + 2|E(H)| + |V(H)|^2\Big).$$

Proof.

$$\begin{aligned}\gamma M_1(G+H) &= \sum_{v\in V(G+H)} d^2_{\gamma\, G+H}(v)\\ &= \sum_{v\in V(G)} (d_{\gamma\, G}(v)+|V(H)|)^2 + \sum_{v\in V(H)} (d_{\gamma\, H}(v)+|V(G)|)^2\\ &= \gamma M_1(G)+\gamma M_1(H)+2|V(H)|\sum_{v\in V(G)} d_{\gamma G}(v)\\ &+2|V(G)|\sum_{v\in V(H)} d_{\gamma H}(v)+|V(G)|(|V(H)|^2+|V(G)||V(H)|)\\ &= \gamma M_1(G)+\gamma M_1(H)+2|V(H)|\gamma(G)T_{m\gamma}(G)\\ &+2|V(G)|\gamma(H)T_{m\gamma}(H)+|V(G)|\Big(|V(H)|^2+|V(G)||V(H)|\Big).\end{aligned}$$

$$\begin{aligned}\gamma M_2(G+H) &= \sum_{uv\in E(G+H)} d_{\gamma\, G+H}(u)d_{\gamma\, G+H}(v)\\ &= \sum_{uv\in E(G)} d_{\gamma\, G+H}(u)d_{\gamma\, G+H}(v)+\sum_{uv\in E(H)} d_{\gamma\, G+H}(u)d_{\gamma\, G+H}(v)\\ &+\sum_{u\in V(G),v\in V(H)} d_{\gamma\, G+H}(u)d_{\gamma\, G+H}(v)\end{aligned}$$

We will find every part independently,

$$\begin{aligned}\sum_{uv\in E(G)} d_{\gamma\, G+H}(u)d_{\gamma\, G+H}(v) &= \sum_{uv\in E(G)} (d_{\gamma\, G}(u)+|V(H)|)(d_{\gamma\, G}(v)+|V(H)|)\\ &= \gamma M_2(G)+|V(H)|\gamma M_1^*(G)+|E(G)||V(H)|^2\end{aligned}$$

$$\begin{aligned}\sum_{uv\in E(H)} d_{\gamma\, G+H}(u)d_{\gamma\, G+H}(v) &= \sum_{uv\in E(H)} (d_{\gamma\, H}(u)+|V(G)|)(d_{\gamma\, H}(v)+|V(G)|)\\ &= \gamma M_2(H)+|V(G)|\gamma M_1^*(H)+|E(H)||V(G)|^2\end{aligned}$$

$$\begin{aligned}\sum_{u\in V(G),v\in V(H)} d_{\gamma\, G+H}(u)d_{\gamma\, G+H}(v) &= (d_{\gamma\, G}(u_1)+|V(H)|)(d_{\gamma\, H}(v_1)+|V(G)|)+\cdots\\ &+(d_{\gamma\, G}(u_1)+|V(H)|)(d_{\gamma\, H}(v_{|V(H)|})+|V(G)|)\\ &+(d_{\gamma\, G}(u_2)+|V(H)|)(d_{\gamma\, H}(v_1)+|V(G)|)+\cdots\\ &+(d_{\gamma\, G}(u_2)+|V(H)|)(d_{\gamma\, H}(v_{|V(H)|})+|V(G)|)\\ &+\cdots\\ &+(d_{\gamma\, G}(u_{|V(G)|})+|V(H)|)(d_{\gamma\, H}(v_1)+|V(G)|)+\cdots\\ &+(d_{\gamma\, G}(u_{|V(G)|})+|V(H)|)(d_{\gamma\, H}(v_{|V(H)|})+|V(G)|)\\ &= \Big(|V(G)||V(H)|+\sum_{u\in V(G)} d_{\gamma G}(u)\Big)\\ &\Big(|V(G)||V(H)|+\sum_{v\in V(H)} d_{\gamma H}(v)\Big)\\ &= \Big(|V(G)||V(H)|+\gamma(G)T_{m\gamma}(G)\Big)\\ &\Big(|V(G)||V(H)|+\gamma(H)T_{m\gamma}(H)\Big).\end{aligned}$$

Hence,

$$\gamma M_2(G+H) = \gamma M_2(G) + \gamma M_2(H) + |V(H)|\gamma M_1^*(G) + |V(G)|\gamma M_1^*(H) + |E(G)||V(H)|^2 + |E(H)||V(G)|^2 + \Big(|V(G)||V(H)| + \gamma(G)T_{m\gamma}(G)\Big)\Big(|V(G)||V(H)| + \gamma(H)T_{m\gamma}(H)\Big).$$

$$\begin{aligned}\gamma M_1^*(G+H) &= \sum_{uv\in E(G+H)} [d_{\gamma G+H}(u) + d_{\gamma G+H}(v)] \\ &= \sum_{uv\in E(G)} [d_{\gamma\, G+H}(u) + d_{\gamma\, G+H}(v)] + \sum_{uv\in E(H)} [d_{\gamma\, G+H}(u) + d_{\gamma\, G+H}(v)] \\ &+ \sum_{u\in V(G), v\in V(H)} [d_{\gamma\, G+H}(u) + d_{\gamma\, G+H}(v)].\end{aligned}$$

We will find every part independently,

$$\begin{aligned}\sum_{uv\in E(G)} [d_{\gamma\, G+H}(u) + d_{\gamma\, G+H}(v)] &= \sum_{uv\in E(G)} (d_{\gamma\, G}(u) + |V(H)|) + (d_{\gamma\, G}(v) + |V(H)|) \\ &= \gamma M_1^*(G) + 2|E(G)||V(H)|,\end{aligned}$$

$$\begin{aligned}\sum_{uv\in E(H)} [d_{\gamma\, G+H}(u) + d_{\gamma\, G+H}(v)] &= \sum_{uv\in E(H)} (d_{\gamma\, H}(u) + |V(G)|) + (d_{\gamma\, H}(v) + |V(G)|) \\ &= \gamma M_1^*(H) + 2|V(G)||E(H)|,\end{aligned}$$

$$\begin{aligned}\sum_{u\in V(G), v\in V(H)} [d_{\gamma\, G+H}(u) + d_{\gamma\, G+H}(v)] &= (d_{\gamma\, G}(u_1) + |V(H)|) + (d_{\gamma\, H}(v_1) + |V(G)|) + \cdots \\ &+ (d_{\gamma\, G}(u_1) + |V(H)|) + (d_{\gamma\, H}(v_{|V(H)|}) + |V(G)|) \\ &+ (d_{\gamma\, G}(u_2) + |V(H)|) + (d_{\gamma\, H}(v_1) + |V(G)|) + \cdots \\ &+ (d_{\gamma\, G}(u_2) + |V(H)|) + (d_{\gamma\, H}(v_{|V(H)|}) + |V(G)|) \\ &+ \cdots \\ &+ (d_{\gamma\, G}(u_{|V(G)|}) + |V(H)|) + (d_{\gamma\, H}(v_1) + |V(G)|) + \cdots \\ &+ (d_{\gamma\, G}(u_{|V(G)|}) + |V(H)|) + (d_{\gamma\, H}(v_{|V(H)|}) + |V(G)|) \\ &= |V(H)|(\sum_{u\in V(G)} d_{\gamma G}(u)) + |V(G)|(\sum_{v\in V(H)} d_{\gamma H}(v)) \\ &+ |V(G)||V(H)|^2 + |V(G)|^2|V(H)| \\ &= |V(H)|\gamma(G)T_{m\gamma}(G) + |V(G)|\gamma(H)T_{m\gamma}(H) \\ &+ |V(G)||V(H)|^2 + |V(G)|^2|V(H)|.\end{aligned}$$

Hence,

$$\gamma M_1^*(G+H) = \gamma M_1^*(G) + \gamma M_1^*(H) + |V(H)|\Big(\gamma(G)T_{m\gamma}(G) + 2|E(G)| + |V(G)|^2\Big) + |V(G)|\Big(\gamma(H)T_{m\gamma}(H) + 2|E(H)| + |V(H)|^2\Big).$$

□

Theorem 2.15. *If G and H are both complete graphs, then*
$\gamma M_1(G+H) = |V(G)| + |V(H)|$,
$\gamma M_2(G+H) = |E(G)| + |E(H)| + |V(G)||V(H)|$,
$\gamma M_1^*(G+H) = 2(|E(G)| + |E(H)| + |V(G)||V(H)|)$.

Proof. If G and H are both complete graphs, then $T_{m\gamma}(G+H)=|V(G)|+|V(H)|$ and $d_{\gamma G+H}(v)=1$, for all $v\in V(G+H)$. Applying Definition 1.2, we get the required results. $\square$

Lemma 2.16. *Let G and H be any two graphs with $\gamma(G)>2$ and $\gamma(H)>2$. Then* $T_{m\gamma}(G+H)=|V(G)||V(H)|$ *and* $d_{\gamma G+H}(v)=\begin{cases}|V(H)|, & \text{if } v\in V(G);\\ |V(G)|, & \text{if } v\in V(H).\end{cases}$

Theorem 2.17. *Let G and H be any two graphs with $\gamma(G)>2$ and $\gamma(H)>2$. Then*

(1) $\gamma M_1(G+H)=|V(G)||V(H)|^2+|V(G)|^2|V(H)|$.
(2) $\gamma M_2(G+H)=|V(H)|^2|E(G)|+|V(G)|^2|E(H)|+|V(G)|^2|V(H)|^2$.
(3) $\gamma M_1^*(G+H)=2(|V(H)||E(G)|+|V(G)||E(H)|)+|V(G)||V(H)|^2+|V(G)|^2|V(H)|$.

Proof. If G and H are any two graphs with $\gamma(G)>2$ and $\gamma(H)>2$, then applying Lemma 2.16, we get

$$\gamma M_1(G+H)=\sum_{v\in V(G+H)} d^2_{\gamma G+H}(v)=|V(G)||V(H)|^2+|V(G)|^2|V(H)|.$$

$$\begin{aligned}\gamma M_2(G+H)&=\sum_{uv\in E(G+H)} d_{\gamma G+H}(u)d_{\gamma G+H}(v)=\sum_{uv\in E(G)} d_{\gamma G+H}(u)d_{\gamma G+H}(v)\\
&+\sum_{uv\in E(H)} d_{\gamma G+H}(u)d_{\gamma G+H}(v)+\sum_{u\in V(G),v\in V(H)} d_{\gamma G+H}(u)d_{\gamma G+H}(v)\\
&=|V(H)|^2|E(G)|+|V(G)|^2|E(H)|+\underbrace{|V(H)||V(G)|+\cdots+|V(H)||V(G)|}_{|V(H)|times}\\
&+\underbrace{|V(H)||V(G)|+\cdots+|V(H)||V(G)|}_{|V(H)|times}+\cdots+\underbrace{|V(H)||V(G)|+\cdots+|V(H)||V(G)|}_{|V(H)|times}\\
&=|V(H)|^2|E(G)|+|V(G)|^2|E(H)|+\underbrace{|V(H)|(|V(G)||V(H)|)+\cdots+|V(H)|(|V(G)||V(H)|)}_{|V(G)|times}\\
&=|V(H)|^2|E(G)|+|V(G)|^2|E(H)|+|V(G)|^2|V(H)|^2.\end{aligned}$$

Similarly, one can get

$$\gamma M_1^*(G+H)=2\Big(|V(H)||E(G)|+|V(G)||E(H)|\Big)+|V(G)||V(H)|^2+|V(G)|^2|V(H)|.$$

$\square$

3. Some bounds for $\gamma-$domination Zagreb indices

Theorem 3.1. *Let G be any graph. Then*

$$\gamma M_1(G)\geq \frac{1}{|V(G)|}\gamma^2(G)T^2_{m\gamma}.$$

Equality holds if and only if G is a complete graph.

Proof. We have, $\gamma M_1(G)=\sum_{i=1}^{|V(G)|} d^2_{\gamma G}(v_i)=d^2_{\gamma G}(v_1)+d^2_{\gamma G}(v_2)+\cdots+d^2_{\gamma G}(v_{|V(G)|})$. By using Cauchy schwartz inequality on vectors $(d_{\gamma G}(v_1),d_{\gamma G}(v_2),\cdots,d_{\gamma G}(v_{|V(G)|}))$ and $(1,1,\cdots,1)$ we get,

$$\begin{aligned}\gamma M_1(G)|V(G)| &= (d^2_{\gamma G}(v_1), d^2_{\gamma G}(v_2), \cdots, d^2_{\gamma G}(v_{|V(G)|}))(1^2, 1^2, \cdots, 1^2)\\ &\geq \underbrace{\left(d_{\gamma G}(v_1)1 + d_{\gamma G}(v_2)1 + \cdots + d_{\gamma G}(v_{|V(G)|})1\right)^2}_{|V(G)| times}\\ &= \left(\sum_{i=1}^{|V(G)|} d_{\gamma G}(v_i)\right)^2\\ &= \gamma^2(G)T^2_{m\gamma}(G)\end{aligned}$$

Hence,

$$\gamma M_1(G) \geq \frac{1}{|V(G)|}\gamma^2(G)T^2_{m\gamma}(G).$$

Note that if G is a complete graph, then $d_{\gamma G}(v) = 1$ for all $v \in V(G)$ hence, $\gamma M_1(G) = |V(G)|$. Conversely if $\gamma M_1(G) = \frac{1}{|V(G)|}(\sum_{i=1}^{|V(G)|} d_{\gamma G}(v_i))^2 \Rightarrow \gamma M_1(G)|V(G)| = (\sum_{i=1}^{|V(G)|} d_{\gamma G}(v_i))^2$ and this holds only if G is complete graph. □

Theorem 3.2. *For any graph G,*

$$\gamma M_1(G) \leq \left(\sum_{i=1}^{|V(G)|} \sqrt{d_{\gamma G}(v_i)}\right)^2.$$

Further, this equality is best possible.

Proof. We have,
$\gamma M_1(G) = \sum_{i=1}^{|V(G)|} d^2_{\gamma G}(v_i) = d^2_{\gamma G}(v_1) + d^2_{\gamma G}(v_2) + \cdots + d^2_{\gamma G}(v_{|V(G)|})$
As $d^2_{\gamma G}(v_1), d^2_{\gamma G}(v_2), \cdots, d^2_{\gamma G}(v_{|V(G)|})$ are nonnegative integers so,

$$\gamma M_1(G) \leq \left(\sum_{i=1}^{|V(G)|} \sqrt{d_{\gamma G}(v_i)}\right)^2.$$

□

Proposition 3.3. *Let G be any graph with $D_1, D_2, ...D_t$ as minimum dominating sets. Then*
$\gamma M_1(G) < t^2|V(G)|$, $\gamma M_2(G) < t|E(G)|$, $\gamma M_1^*(G) < 2t|E(G)|$.

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